Exploratory activity in the mathematics classroom [1] [2]

João Pedro da Ponte
Instituto de Educação, Universidade de Lisboa

Neusa Branco
Escola Superior de Educação de Santarém, Instituto de Educação da Universidade de Lisboa

Marisa Quaresma
Instituto de Educação, Universidade de Lisboa

Abstract. In this chapter we show that mathematical explorations may be integrated into the core of the daily classroom mathematics activities instead of just being a peripheral activity that is carried out occasionally. Based on two episodes, one on the initial learning of the rational number at grade 5 and the other on the learning of algebraic language at grade 7, we show how teachers may invite students to get involved and interpret such tasks, how they may provide students with significant moments of autonomous work and lead widely participated collective discussions. Thus, we argue that these tasks provide a classroom setting with innovative features in relation to conventional education based on the exposition of concepts and procedures, presentation of examples and practice of exercises and with much more positive results regarding learning.

Key words. Explorations, Teaching practice, Tasks, Classroom communication, Rational numbers, Algebraic thinking.

Introduction

In problems and exploration tasks the students do not have a ready-made procedure to obtain a solution. Therefore, they need to understand the question, formulate a strategy to solve it, carry out this strategy, and review and reflect on the results. Mathematics problems and explorations have much in common – the main difference is that problems indicate in a concise way what is given and what is asked, whereas explorations contain elements of uncertainty or openness, requiring students to undertake a significant work of interpreting the situation and often of re-working the questions (Ponte, 2005).

In this paper, we argue that these tasks may be used to create a productive classroom environment in contrast to the more common classroom based on exposition of concepts and procedures, presentation of examples, and practice of exercises. Our aim is

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to show how such tasks can be presented to the students, providing significant moments of autonomous work and leading to widely participated whole-class or collective discussions, and that such environment has positive implications for students' learning. We illustrate these ideas with two situations based on exploratory work at grade 5 (on rational numbers) and grade 7 (addressing algebraic reasoning).

The exploratory classroom

In an exploratory classroom we identify two key elements: (i) tasks proposed to the students and (ii) ways of working, with associated roles of teacher and students and communication patterns.

Tasks. The tasks are important, not in themselves, but because of the activity of the students in solving them. What students learn in a mathematics classroom mainly results from the activity that they undertake and the reflection that they do on this activity (Christiansen & Walther, 1986). The development of a rich and productive mathematics activity by students may stand on different kinds of tasks such as problems, investigations, explorations, and even exercises (Ponte, 2005). Exercises are tasks with a precise formulation of givens, conditions and questions, aimed at the clarification of concepts and consolidation of procedures that the student already knows. Problems are tasks aimed at the creative application of knowledge also already held by the student. In contrast, explorations are tasks aimed at the construction of new concepts, representations or procedures and investigations are even more challenging tasks aimed at the development of new concepts or at a creative use of concepts already known by the student. The teacher needs to select the tasks according to the objectives set for each class, paying attention to their suitability to the targeted students.

The nature of the context (mathematical or real-life) is a very important aspect of a task. Students may find useful hints to solve tasks in aspects of the context. However, as Skovsmose (2001) points out many supposedly real-life contexts are basically artificial – this author calls them "semi-real" contexts. At the same time, it should be noted that it makes a significant difference whether we work in mathematical contexts with which we have some familiarity or in mathematical contexts that are new to us.

The representations involved are another important aspect on a task. Bruner (1966) distinguishes between enactive (objects, body movements), iconic (pictures) and symbolic representations. Often, alongside the formal representations or even before these representations, students profit from working with informal representations or even from constructing their own representations. Naturally, the teaching materials used, including manipulative materials, daily life objects, digital technologies, etc., determine the representations that students will use to work on a given task to a large extent.

We must note that problems include moments of exploration. We may formulate hypotheses, analyze the given conditions or make conjectures about possible solutions, and test their consequences. In many problems it is possible to generate data and explore regularities in such data. However, exploratory tasks are distinctive in that they always require a careful interpretation of the situation, then the creation or reformulation of more precise questions to investigate, and, often, the construction of new concepts and representations. In this way, besides enabling the application of already learned concepts, explorations may promote the development of new concepts and learning new representations and mathematical procedures.

Problem solving is an important curriculum orientation, especially since the publication of *An agenda for action* (NCTM, 1980). This document proclaimed that “problem solving should be the focus of school mathematics” (p. 1). Later, other curriculum documents kept emphasizing problem solving (e.g., NCTM 2000). As a consequence, problem solving won a positive connotation among textbook authors and mathematics teachers, as corresponding to a necessary and important activity in the mathematics class. However, the place of problem solving in mathematics teachers’ professional practice proved to be problematic. Some teachers hold initiatives such as the “problem of the week” or “problem of the month”. The classroom work went on as before, the only difference being that, from time to time, at a special moment, a problem was proposed, often with little relation to the current topic that the class was studying. On the other hand, particularly in primary school, teachers kept using traditional word problems, sometimes feeling that these tasks were just disguised exercises requiring students to make a simple computation. The difficulties in achieving a productive classroom implementation led to an impasse, prompting mathematics educators to recognize that problem solving as a curriculum orientation was not matching the expectation (Schoenfeld, 1991). Therefore, it becomes necessary to better understand the types of problems that could be useful in the classroom and, most especially, how teachers might use them.

The increasing availability of digital technology tools, such as computers and calculators, is probably the main factor that led to the increasing acceptance of exploration and investigation tasks. These technologies easily allow the simulation of complex situations that would otherwise be difficult to study (Papert, 1972). However, even without digital technology it is possible to explore many situations in a mathematical way. In fact, explorations have much to do with modeling – requiring the creation of representations that may be used to construct a mathematical model of a situation. But they also have important aspects of mathematical work such as using definitions, classifying objects, and relating properties. The two terms, explorations and investigations, are increasingly used, and it is difficult to establish a clear dividing line between them – we talk about "investigations" when the tasks involve mathematical situations with a considerable degree of challenge for most students and talk of "explorations" when the situations allow the easy involvement of most of them.

Ways of working, roles and communication patterns. In the classroom, students may work in many different ways. In collective mode, the teacher interacts with all students at the same time. In group work and in pairs, the students are encouraged to share ideas among themselves. Working individually, the students may find the required concentration to deal with abstract ideas. In all these cases, the students may participate in two kinds of classroom discourse – collective, with all class mates and the teacher, and private, with a few colleagues or directly with the teacher.

An exploratory class usually unfolds in three phases (Ponte, 2005): (i) presentation of the task and its interpretation by students (whole-class); (ii) development of work by students (in groups, pairs, or individually); and (iii) discussion and final synthesis (whole-class). As Bishop and Goffree (1986) indicate, the last phase is the most appropriate occasion to expose connections, allowing students to relate ideas on various topics, and showing how mathematical ideas are naturally intertwined. In addition, discussion moments are opportunities for the negotiation of mathematical meanings and the construction of new knowledge. As the NCTM (2000) indicates, "[l]earning with understanding can be further enhanced by classroom interactions, as students propose mathematical ideas and conjectures, learn to evaluate their own thinking and that of others, and develop mathematical reasoning skills" (p. 21). Therefore, each task always ends with a collective discussion to reflect, contrast ideas, processes, and come to final conclusions.

Classroom discourse is univocal, when it is dominated by the teacher, or dialogic, when the students' contribution is valued as important (Brendefur & Frykholm, 2000). Usually, the role of the teacher is to propose the tasks to carry out, to establish working modes, and to direct the classroom discourse. However, the teacher may assume the role of the single mathematical authority or share it with the students, in which case he/she seeks to stimulate their reasoning and argumentation ability. The role of the students is always working on the proposed tasks. However, this role may vary widely in many respects – for example as the students assume that they must intervene only when they are asked or, on the contrary, that they must intervene, at group or collective level, always when they have a significant contribution to make.

Unlike the conventional collective classroom, strongly controlled by the teacher, and where the students' possibilities of intervention were very limited, in an exploratory classroom students are provided with a significant opportunities for participation. In exploratory classes, Lampert (1990) shows how students are encouraged to present their strategies and solutions as well as to question the solutions and strategies of others, seeking to understand or to refute them. Also in these classes, Wood (1999) highlights the learning potential in valuing justifications and exploring disagreements among students, for the construction of shared meanings.

For Stein, Engle, Smith and Hughes (2008). the starting point for collective discussions must be the students' work. They state that a productive mathematics discussion has two fundamental characteristics: (i) it is based on students' thinking; and (ii) it puts forward important mathematical ideas. These authors underline the complexity of the work of the teacher in conducting a mathematical discussion, pointing out that the students' strategies are often very different from each other and largely unpredictable. They indicate that the teacher needs to give coherence to the diversity of the students' ideas, relating them to established mathematical knowledge, at the same time that students' authority and accountability are enhanced. Focusing on the conduction of classroom collective discussions, Cengiz, Kline, and Grant (2011) identify a set instructional actions through which the teacher may seek to create opportunities to promote pupils' mathematical thinking: eliciting actions that lead students to present their methods, sup porting actions to help children to understand the mathematical ideas and extending actions, to help students to move forward in their thinking.

The next two sections present selected episodes taken from teaching experiments (Branco, 2008; Quaresma, 2010) of students working on tasks and of classroom discussions, indicating how the features of the exploratory work were important for students' learning. Given the aim of this paper, the episodes are analyzed according to four main features of classroom exploratory work: (i) designing tasks and classroom organization;

(ii) promoting the involvement of students in interpreting and carrying out the tasks; (iii) students' work including and negotiation of meanings and (iv) collective discussions.

Exploring representations of rational numbers

Task and classroom organization. The first situation is taken from a study aiming to understand how the work on an exploratory teaching unit using different representations and different meanings of rational numbers, with grade 5 students, may contribute to the understanding of these numbers and of order, comparison, and equivalence of rational numbers (Quaresma, 2010). The task "Folding and folding again" is taken from the supporting materials of the new mathematics curriculum for grades 5-6 (Menezes, Rodrigues, Tavares, & Gomes, 2008) and was proposed in the first lesson to this class dedicated to the study of rational numbers. It should be noted that, in grades 3-4, these students had already studied decimal numbers as well as fractional operators, but did not use the fraction representation.

Folding and folding again

1. Find three paper strips geometrically equivalent. Fold them in equal parts: the first in two; the second in four; the third in eight.
After you fold each strip, represent in different ways the parts that you got.

2. Compare the parts of the three strips that you got by folding. Record your conclusions.

3. In each strip, find the ratio between the length of the parts that you got after folding and the length of the strip. Record your conclusions.

The purpose of this task is to introduce the language associated with rational numbers in different representations and meanings. Specifically, the teacher aimed for students to learn: (i) to represent a rational number as a fraction, decimal and percent; (ii) to understand and use rational numbers as a part-whole relations and measures; and (iii) to compare numbers represented in different forms. The task involves the meanings of part-whole, measure, and ratio, with continuous magnitudes (rectangular segments) and is presented in the context of paper strips. The information is given through active representations (paper strips) and the answers may be given in verbal, pictorial, decimal or fraction representations or as percentages. The first question gives the "whole", a strip of paper, and asks the students to represent three different parts of it. Question 2 asks the students to compare the three parts thus obtained. The students may use whichever representation that they wish, however, it is expected that they use the information obtained in the previous question to make the comparisons. Question 3 asks the students to determine the ratio between the length of the strip and the length of each of the parts obtained by folding.

This task provides an opportunity for students to get involved in an exploratory activity. Question 1 begins as an exercise asking students to do several folds, but then assumes a more open nature by asking them to represent it in “different ways”, something that most students have difficulty interpreting as an indication to use fractions, decimals, pictorial, verbal or other representations of rational numbers. Question 2 asks them to compare different strips and to “draw conclusions”, which is a very open statement, allowing for diverse interpretations. Finally, question 3 involves a term whose meaning is not obvious for these students (“reason”), as well as a new indication to draw further conclusions.

The teacher organizes the students in six groups of four or five students gives each of them a sheet of paper with the proposed questions. First, she reads aloud questions 1 and 2, gives about 30 minutes for the students to work and promotes a collective discussion for about 25 minutes. Then, she distributes question 3, gives students about 20 minutes to solve it, and it is then discussed by the whole class for about 15 minutes, after which the class ends. In a subsequent class, the teacher promotes a new discussion to recall important ideas, and to summarize the work carried out. Therefore, there are several cycles with three different moments – presentation and negotiation of the task, students' autonomous work, and collective discussion.

Getting involved and negotiating the task. As the teacher reads aloud questions 1 and 2, the students start working with no difficulty, folding the strips and painting the parts that they get. The situation changes when they face the indication to “represent in different ways” which they have great difficulty in interpreting. This leads the teacher to promote a moment of collective discussion to negotiate the meaning of various terms. In question 1, the teacher holds a strip and folds it into two equal parts so that all students can see, thus making an active representation. Then, she draws the strip on the board, representing pictorially the part to consider, and then asks students to state what part of the strip is painted. Using the verbal representation, many students say that “half of the strip” is painted. Then, the teacher continues to insist on another ways to represent that part, and from the verbal representation “half”, some students suggest the decimal representation “0.5”. The teacher asks for other forms of representation and two students indicate the fraction “one of two”. Finally, as the students do not indicate any further representation, the teacher questions: “What if I wanted to represent it as a percent? It would be possible?” Immediately, most students say that “it is 50%”. This whole-class discussion represents the first collective negotiation of a part of the first question, which allows the continuation of the work.

Students' work. The initial discussion helps the students to get an understanding of the situation. They readily engage in working on the remaining points of question 1. The teacher moves around the room, observing the work of the different groups, paying attention to the new discoveries and questions that arise. Taking into account the previous situation, the students do not show difficulty in indicating various representations of $\frac{1}{4}$ of the strip and conclude that $\frac{1}{4} = 0.25$. However, they have noticeable difficulty in finding a decimal representation for $\frac{1}{8}$, because they have trouble finding half of 0.25, showing some insecurity in the use of the decimal number system. After the students solved question 1, the teacher asks them to stick their responses on the board to present their discoveries to the whole-class. Thus, active representations (paper strips) become pictorial representations, and are the basis for solving questions 2 and 3. Solving question 2 also requires an explanation of the work to carry out, as students show difficulty understanding what “compare the parts of the three strips that you got by folding” means. The teacher begins by showing the first two strips ($\frac{1}{2}$ and $\frac{1}{4}$) and asks the students to compare them. The visualization leads the students to conclude that $\frac{1}{4}$ is half of $\frac{1}{2}$ and this the starting point for the group work. Later, in question 3, the teacher also feels the need to help students to understand the statement. As the paper strips have different measures, she chooses to ask them to consider that all measure 20 cm and, from this, the students easily come to recognize some relationships.

Collective discussions. At the beginning of the whole-class discussion of question 1, to support the participation of students, the teacher asks each group to post their work on the board. Then she asks the first group to present their work to the class. Diana, the spokeswoman, says “In figure B we wrote: fourth-part, 1 by 4 [$\frac{1}{4}$]; 1 divided by 4, 25% and 0.4” The students do not realize the error of her colleague when she says “0.4”. The teacher decides to go on to the presentation of another group (Figure 1):

Tiago: So we have: fourth-part, one of four [$\frac{1}{4}$], 1 divided by 4, 25 and 0.25%.

Teacher: (...) Do you agree Diana?

Diana: Yes...?

Class: No! That is wrong...

Teacher: What is wrong?

Rui: It's 0.25...

Teacher: Why?

Rui: Because is the fourth part.

Daniel: It is 0.25 because it is a half of the first. The first was 50, if we make half is 25.

André: Oh teacher! I think it is because 0.25 is the fourth part of 100. Because 25 times 4 gives 100.

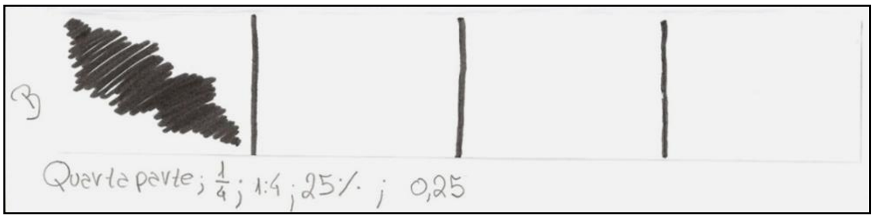


Figure 1 – Solution of question 1a) by Leonor, Rui, Henrique and Tiago.

One must note that, after the teacher's question "Why?", several students (Rui, Daniel, André) present successively more refined explanations. Out of the six groups only that of Diana makes the mistake of "transforming" the denominator of the fraction into a decimal number. The remaining groups obtained the decimal number by comparing the previous value and 100%.

In the third strip, all groups use correctly verbal representations, fractions, quotients and percent. However, in general, they show difficulties in the decimal representation. There are essentially two types of errors. One, as we saw, appeared in Diana's group that writes the decimal numeral by transforming the denominator of the fraction. Another error made by some groups originates in the difficulty in determining one half of 0.25. The students begin by seeking to determine half of 25% and get 12.5%, but they have difficulty in obtaining half of 0.25. The students know that $\frac{1}{4}$ is 0.25, but when making half 0.25, they get 12.5 (Figure 2). This result creates a conflict because they believe that it does not make sense, as the half of 0.25 should be a smaller number and, in this case, 12.5 is greater. The students also show difficultly in understanding the decimal number system and do not remember that they may add a zero to get 0.250 and, from there, easily find 0.125.

Figure 2 – Solution of the question 1c) by Leonor, Rui, Henrique and Tiago.

However, during the discussion of the task, the students get the correct answer:

Daniel: It is 12,5% because C is half of B.
Teacher: If B is 25%, is C…
Daniel: It is the half, that is 12.
Teacher: It is 12%?
Luís: No teacher, it is 12.5%. Because 12.5+12.5 is 25.
Teacher: So how is it in decimals?
Tiago: It is 0.125.
(…)
André: It is 0.125.
Teacher: Why?
André: It is 0.125 because 0.125 × 8 gives a whole unit.

Tiago indicates the right answer, but it is André that justifies it by establishing the relationship with the unit.

In question 2, all groups establish some relationships between the parts, but only a few compare all the strips. All groups use only verbal language to express these relationships. Here is an example (Figure 3):

b) is the half of a).
c) is the fourth "half" of a)
a) is twice b).
a) is four times c).
c) is half of b)
b) is twice c).

Figure 3 – Solution of question 4 by André, Francisco, Rodrigo and Miguel.

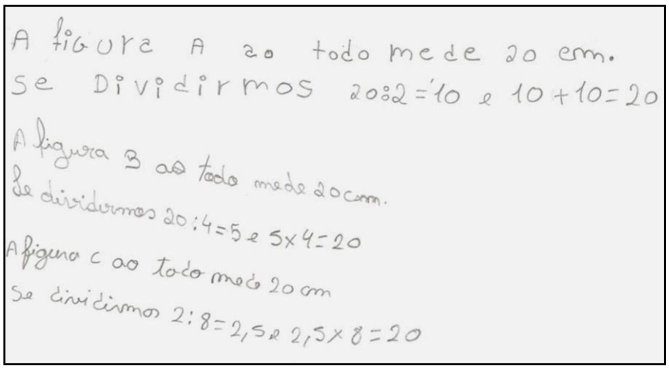
André‘s group , besides the simple relationships of “half” and “double”, establishes more complex relationships such as “quadruple” (based on the “double of the double”) and “fourth part” (“fourth half”, as they say, to mean “half of half "). Identical formulations were presented by the Mariana’s group. In discussing this question, the teacher asks each group to indicate the relationships that they found. Since the students only use the verbal representation, the teacher asks them to use the mathematics language:

> *Daniel*: The relationship between the first and second, is that the second is a half of the first.
> *Teacher*: How can I write that using numbers? How do I make a half?
> André: Divide by 2.
> *Rui*: “One of four” is equal to a half divided by two.
> *André*: b is the double of c.
> *Teacher*: How do I write that?
> *André*: One of four is the double.
> *Teacher*: How is it the double?
> *André*: Two times...
> *Teacher*: Two times what?
> *André*: One dash eight.
> *Teacher*: One-eighth. One fourth is the double of one-eighth.
> *Alexandre*: The first is the double of the second.
> *Teacher*: How do I write that?
> *Alexandre*: One half is the double of... One over four.

One must note that the students use a spontaneous language to speak of fractions (“one of four”, “one mark eight”), language that the teacher tries to improve whenever is possible.

Notwithstanding several difficulties, the students are able to find the main relationships among $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$ essentially using the strips as active representations. The students are able to compare the three fractions presented by themselves, which aids them in understanding rational numbers, especially in regards to the meaning of part-whole and understanding the magnitude of a rational number. They express these relationships in verbal language and show difficulties in using mathematical language. This was the first class where they met this topic in a formal way, so it is natural that they show difficulties with the language of fractions.

Question 3 aims to develop the students' understanding of ratio. To facilitate the solution the teacher provides a “friendly” size for the strip, 20 cm. The students build upon the relationships among the parts of the strip, discussed in the previous question, to find the length of each part, which they represent as follows (Figure 4):


A figura A ao todo mede 20 cm.
Se Dividirmos 20:2=10 e 10+10=20
A figura B ao todo mede 20cm.
Se dividirmos 20:4=5 e 5x4=20
A figura C ao todo mede 20 cm
Se dividirmos 2:8=2,5 e 2,5x8=20

Picture A is 20 cm in all.
If we divide 20:2=10 and 10+10=20.

Picture B is 20 cm in all.
If we divide 20:4=5 and 5x4=20.

The picture C is 20 cm in all.

If we divide 20:8=2.5 and 2.5x8=20

Figure 4 – Solution of question 3 by Carolina, Diana and Filipe.

Although the students are able to establish relationships between the total length of the strips and the length of the parts, they do not use the symbolic representation of a ratio as a fraction. Instead, they express it in verbal language:

> *Teacher*: So let's see what conclusions you reached. What relationship did you found between the length of the strip and the length of the first part?
>
> *Luís*: The half measures 10 cm. That is, if the strip is 20, the half is 10. That is 20 divided by 2.
>
> *Teacher*: So, what is the relationship between the length of part and whole?
>
> *Several students*: Is the half.

We must note the teacher's questioning style in the collective discussion, marked by open questions ("come to explain..." "agree…", "what is wrong?", "why?", "and then what happens to the decimal?"). Note, also, that the classroom culture integrates the notion that the students may contribute with different responses as well as disagree and argue with each other.

Synthesis. In the final synthesis, the teacher poses several questions to summarize the aspects where students showed more difficulty. Thus, referring to question 1, she points to the decimal number system, so that students understand why 025 : 2 = 0.125. It is concluded that that fraction bars correspond to the operation of division, in this case, $\frac{1}{8} = 1{:}\,8 = 0{,}125$. Based on the students' work, the teacher asks them to state a "rule" to convert a decimal into a percent. The students, analyzing the examples discussed, conclude that they may "move" the decimal two "places" to the right, which the teacher explains as related to a multiplication by 100.

Returning to question 2, some equivalent fractions are analyzed, and it is concluded that a given part may be represented by an infinite number of fractions. The representation of the unit is also discussed and it is concluded that when the numerator and the denominator are the same, there is a unit which we may represent, for example, by $\frac{4}{4} = 1$. A student also verifies that $\frac{4}{8} = \frac{1}{2}$ because 4 is half of 8, leading the class to conclude that there are several fractions that represent the same as $\frac{1}{2}$, all of them with a numerator that is half of the denominator. Although the equivalence of fractions was not explicitly addressed, this was a first move towards this notion. The terms related to fractions (numerator, denominator) and their relationship where also explained, especially in the part-whole meaning. To confirm students' understanding, the teacher asks them to order the three fractions obtained, and they easily indicate that $\frac{1}{2} > \frac{1}{4} > \frac{1}{8}$ and conclude that "as we fold the strip, the parts become increasingly small".

The work on Question 3 confirmed that students did not know about fractions, but had an intuition for fraction relationships. Therefore, the teacher used this mainly to collect information and later prepare the approach of the concept of ratio.

Developing algebraic thinking

Task and classroom organization. The second situation arises from a study aiming to understand how a teaching unit for grade 7, based on the study of patterns, contributes to the development of students' algebraic thinking, in particular for their understanding of variables and equations (Branco, 2008). This is consistent with the curriculum recommendations that stress the development of algebraic thinking through promoting students' generalizations and representations (Blanton & Kaput, 2005), working with pictorial and numerical sequences, and emphasizing a structural interpretation of equations (Kieran, 1992). The task "Crossing the river" (adapted from Herbert & Brown, 1999) was proposed to students after they had already undertaken some work with pictorial sequences, supporting a first contact with the algebraic language, and before the formal study of equations.

Crossing the river

1. Six adults and two children want to cross a river. The small boat available may only take an adult or one or two children (that is, there are three possibilities: 1 adult in the boat; 1 child in the boat; 2 children in the boat). Any person may conduct the boat. How many

trips the boat needs to make, crossing the river, so that everybody is on the order bank?

2. What happens if the river is to be crossed by:
   - 8 adults and 2 children;
   - 15 adults and 2 children;
   - 3 adults and 2 children.

3. Describe in words how you solve the problem if the group of people is constituted by two children and an unknown number of adults? Verify if your rule works for 100 adults?

4. Write a formula for a number of *A* adults and two children.

5. A group of adults and two children made 27 trips to cross the river. How many adults were in this group?

6. What happens if the number of children changes? In the following examples, verify what changes in your formula:
   - 6 adults and 3 children;
   - 6 adults and 4 children;
   - 8 adults and 4 children;
   - *A* adults and 7 children.

The aims of this task are to promote the students' ability to: (i) find regularities and generalize them; (ii) use and interpret the algebraic language; and (iii) analyze how different problem conditions yield different solutions. This section describes and analyzes several episodes of students working on the task and classroom discussion, indicating how the features of the exploratory class were important to promoting their learning.

The students had worked previously with pictorial sequences, formulating generalizations about the underlying rules and representing them algebraically. However, this task involves a new kind of situation. In fact, the task is formulated in natural language and its solution depends on finding an appropriate representation to be able to design a suitable strategy and interpret the results obtained. So, this task provides an opportunity for an exploration activity. First, it requires a careful interpretation of the situation and to carry out simulations satisfying the given conditions (How may an adult pass to the other bank? And a child?). During this exploration, the regularities may be expressed in different representations. The identification of regularities in the movements of adults and children allows students to generalize the situation and to represent it algebraically, as they have done in previous classes. This situation also requires the students to do a careful interpretation of the results that they obtained. The consideration of a fixed number of children and a variable number of adults leads to the exploration of a sequence (1 adult, 2 adults, and so on). The consideration of a variable number of children and a variable number of adults gives rise to a family of sequences, making this exploration even more complex.

As in the previous case, the work on this task involves different patterns of class-room work – with presentation of the task and negotiation of ideas, students' autono-mous work (in pairs) and collective discussions. Every moment of autonomous work is followed by a whole-class discussion, in successive cycles. The realization of the task ends with a final synthesis.

Getting involved and negotiating the task. The teacher organizes the students in-to pairs to discuss the situation among them and to formulate strategies to solve it. As she presents the task, she highlights the conditions given for the trips. The students begin simulating the first trip and discussing various possibilities among their groups. However, the situation appears to be quite confusing and they pose many questions to the teacher regarding the given conditions and presenting hypotheses. This leads to a somewhat agitated environment.

At this point the teacher realizes that a more extended collective discussion is necessary to help the students to understand the conditions of the situation. So, she re-quests the students' attention and, following the suggestion of a student pair, she asks what happens if, in the first trip, the boat is driven by an adult. The students realize that the first trip must be done by two children in order to allow the boat to return to the starting point, driven by a single child. The teacher asks students to think about who can go on the boat on the subsequent trips. She alerts students that it is sought the minimum number of trips, and thus trips back by adults returning with the boat must be avoided.

Students' work. The students try several possibilities. Some of them create situa-tions where the adult returns with the boat without an adult passing effectively to the other bank. Some student pairs try to put an adult and two children on a trip or an adult and a child. The following dialogue between Diana and the teacher shows how the stu-dents at this stage are still struggling to understand the conditions of the problem:

Diana: If the two children go to there. Then another comes.

Teacher: Yes. Two go to there and then one stay there and the other comes.

Diana: But an adult cannot go with a child??

Teacher: No. An adult has to go it alone, isn't it?

Diana: I get it. Go two children there, then one comes. Then she stays there and an adult comes.

Teacher: Right.

Diana: Then, the other child comes and goes another adult.

Teacher: Yes. And will stay here two adults. Who will take the boat to there?

Diana: So!?

Diana and Mariana simulate the trips and, like other student pairs they end up with a correct indication of the first four trips. However, they suggest that the fifth trip is made by an adult, leading to the situation in which an adult returns with the boat. For some time, the students continue their exploration in an autonomous way, closely observed by the teacher. After some time, all student pairs find that the fifth trip must be done again by the two children and keep thinking about the trips required until everyone is on the other bank. The students use different representations. Some describe all trips in natural language (Figure 5) while others produce schemes or combine iconic and symbolic representations (Figure 6):

1 – first crossing 2 children
2 → 1 child returns
3 → 1 adult goes
4 → 1 child return
5 → 2 children go
6 → 1 child returns
7 → 1 adult goes
8 → 1 child return
9 → 2 children go
10 → 1 child returns
11 → 1 adult goes
12 → 1 child return
13 → 2 children go
14 → 1 child returns
15 → 1 adult goes
16 → 1 child return
17 → 2 children go
18 → 1 child returns
19 → 1 adult goes
20 → 1 child return
21 → 2 children go
22 → 1 child returns
23 → 1 adult goes
24 → 1 child return
25 → In the end 2 children go

Figure 5 – Representation of Beatriz and Andreia in Question 1.

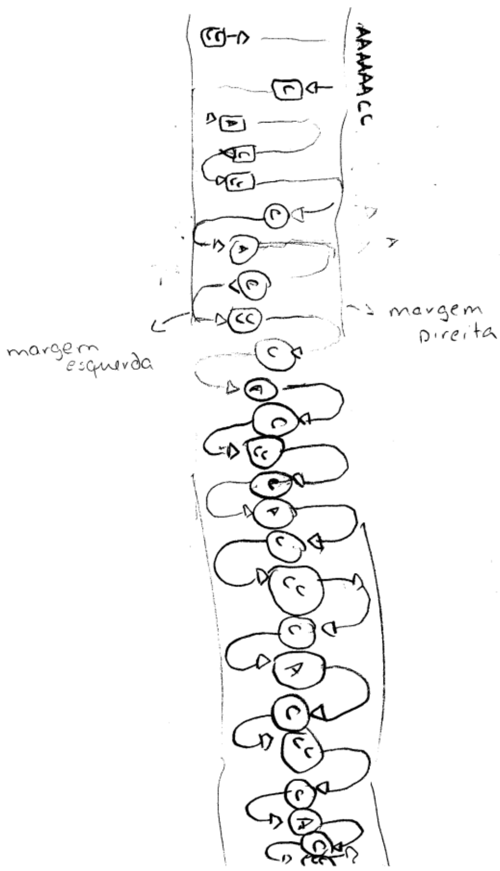


Figure 6 – Representation of Joana and Catarina in Question 1.

Diana and Mariana identify the regularity in the trips, but they do not indicate the total number of trips and they do not say what happens at the end. These students do not represent 25 trips but only the 4 trips necessary for each adult to cross the river (Figure 7):

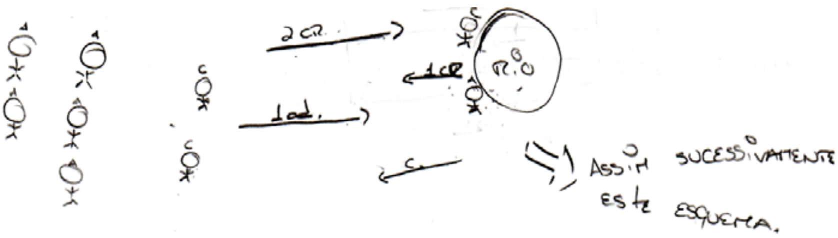


Figure 7 – Representation of Diana and Mariana in Question 1 [They write: “Thus this scheme successively”].

Discussing with the teacher, Diana and Mariana conclude that this set of four trips must be repeated 6 times, with two children staying on the first bank. They find that in addition to the 24 trips, a last trip is necessary to move the two children to the place where the adults are, making a total of 25 trips. They write this in a rather abstract language (Figure 8):

$$\boxed{4 \times 6 + 1 = 25}$$

Figure 8 – Representation of the number of trips by Diana and Mariana in Question 1.

As the student pairs complete the solution of this first question, the teacher, moving around the room, finds that, although all found the correct number of trips, some did not identify the pattern of trips required for an adult to change to the other bank.

The students move on to Question 2, which involves a constant number of children and a varying number of adults. Having found the regularity in the previous question, the students easily respond to the three points. With 8 adults and 2 children, for example, some students represent the total number of trips by the expression $8 \times 4 + 1 = 33$ showing that they understood the regularity and are able to apply it to new situations. Other students, like Joana and Catarina give an answer in a mixture of symbolic and natural language: “If there are 8 adults and 2 children, we add 2 sets of 4 trips, so we add 8 trips to 25, 25+8=33, therefore 33 trips are required". These students arrive at their answer starting from the previous situation with 6 adults and 2 children.

Collective discussions. When most students finish Question 2, the teacher promotes a moment of collective discussion to assess their understanding of the situation and contrast their different representations. Some representations enable a quick identification of the pattern of trips, such as those indicated in Figures 5, 6 and 7. The presentation and discussion of strategies is important for students to understand the situation in a deeper way, clarify meanings, and realize the importance of the efficient use of representations. This discussion also creates the conditions for students to make further generalizations.

In the case of 8 adults and 2 children, Susana shows a rather abstract reasoning when she writes the symbolic expression $8 \times 4 + 1 = 33$. Going to the board she explains its meaning to her colleagues:

> We make eight adults times the number of trips that they have to do in order to take an adult to the other bank. And we do one more, that is the number that the children do in returning.

Students' further work and discussions. Based on this discussion and on the conclusions that they reached, the students continue to work autonomously on Questions 3, 4 and 5, which they solve quickly. Question 3 asks that they describe what happens for any number of adults. Most students give their answer in natural language. But some associate natural and symbolic language like Joana and Catarina who say "It is the number of adults x 4+1". During the collective discussion of this question, Susana presents her rule to determine the total number of trips, using her own words: "It is the number of adults times the four trips plus a trip of the children". She indicates once more the meaning that she ascribes to the different elements of the expression.

In Question 4, when the students write the required expression, they give very concrete meanings to its terms. In the discussion, they present several algebraic expressions, such as $A \times 4 + 1$, $4 \times A + 1$ and $4A + 1$, promoting a discussion about the commutative property of multiplication and the omission of the signal "×". The meanings of terms and coefficients are also discussed – the coefficient 4 is the sequence of 4 trips that is repeated, the term 4*A* indicates the number of trips required for an adult to change to the other bank and the term 1 represents the last trip made by two children. The teacher asks the students how they can calculate the number of trips for different numbers of adults, using the algebraic expression and taking into account the meaning of the terms and correlation coefficients:

Teacher: If A is equal to 26 what does it mean?

Joana: That there are 26 adults.

Teacher: If A is equal to 26, I say that there are 26 adults. How must I do?

Susana: I tis 26 times 4 plus 1.

Question 5 indicates the number of trips and asks the number of adults in the group, without adults repeating trip. The students suggest carrying out the inverse operations, but not everyone is clear about what operation to do first. They use an arithmetic approach, which is natural since they had not yet started the study of equations. Some divide 27 by 4 but verify that in such case it does not make sense to remove the last trip. During the collective discussion, Joana indicates her response “From 27 one subtracts 1 and then divides by 4” and identifies that this cannot be possible because the value of 6.5 is not a natural number. This question allows the discussion of the adequacy of the result and of the response to give taking into account the context and promoting the interpretation of the mathematical result obtained. The teacher proposes the analysis of a new situation that is not indicated in the task: trying to verify whether the students understand and solve it and interpret their results. She questions how many adults are in a group that makes 81 trips, keeping the 2 children. The students show that they understand the Joana’s strategy and can use it in new situations.

Question 6 introduces a new issue. The students are asked to explore the influence of variation in the number of children in the number of trips. Analyzing this new situation, they note that this does not change the set of 4 trips needed for an adult to change to the other bank. With two children, beyond the set of four trips per adult, there is a trip at the end to carry a child. If there are 3 children, they realize that only changes the number of trips at the end but note that it is necessary to make two more trips for the third child to also change the bank. They answer to each situation based on the particular schemes, without establishing a generalization. As some students show difficulty in understanding, the teacher promotes a collective discussion to solve this last question:

Teacher: When I have 2 children it is this here [$8 \times 4 + 1$]. How many children are missing?

Batista: Two.

Teacher: And how many trips I have to do to get them?

Susana: Four more.

Teacher: Every time I get one more child it is two trips [conclusion of the previous questions]. If there are 8 adults and 4 children… Say, Filipe.

Xico: Eight times four plus one plus four.

Teacher: And now, if you have *A* adults e 7 children? [No one answers] For 2 children the expression is that [4*A* +1]. But now I don't have two, I have how many more?

Andreia: Five.

Teacher: I'll have five more. How many trips do I have to do for each one?

Diana: Two.

Andreia: Five times two.

Teacher: So, how is it? How do I simplify the expression?

Susana: 4*A* plus…

Diana: Two times five gives ten.

Batista: Eleven.

In the discussion of this last question, the students analyze the influence of changing the number of children in the expression they initially wrote. They note that the sequence of four trips required to put an adult on the other bank is the same and that, besides at the end of the trip carried out by the two children, there are two trips for each additional child.

Synthesis. At the end of the lesson, the pattern identified in the first question is made explicit and the meaning of the algebraic expression that generalizes the situation whatever the number of adults holding two children in the group is revised. The impossibility of simplifying the expression $4A+1$ is recognized. The students find that this expression is not equivalent to 5*A*, both based on the context and in the use of the distributive property. The interpretation of the terms of the expression according to the context is thus remembered by students that identify its importance for a proper analysis of the results obtained in the last two questions. In question 5, the students realize that a given number may be the number of trips made by a group of adults and two children if, after subtracting one to the number obtained it is divisible by 4. In the last question, the fact one may use the answer to question 1 to understand what happens when the number of children increases, as well as the use of the expression 4*A*+1 to determine the number of trips to *A* adults and different number of children is highlighted.

Discussion and conclusion

The classes that we described based on exploration tasks aimed at promoting significant learning. In solving the task “folding and folding again” the students use strategies based on visualization and supported on active and pictorial representations. Doing this task, the students develop their ability to recognize and use various types of representations of rational numbers, especially fractions and associated verbal language and recall the percent representation. They also use one half as a reference point (Post, Behr & Lesh, 1986) to relate the different parts of the strip, showing they understand the pattern in question and they use that knowledge to reach other representations without always starting from the unit. Further, they conclude that, as the number of parts increases, the sections become smaller and smaller. Thus, they establish various multiplicative relationships between $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$ which supports them in developing rational number sense. The students recognize multiple representations of rational numbers and state rules to convert decimal numerals into percents, although sometimes they do not apply them in the subsequent questions. They compare rational numbers in various active and pictorial representations and establish simple multiplicative relationships (double, half) and more complex relationships (four times, fourth). They show some sense of equivalent fractions and compare the three presented fractions, although, as expected, they did not come to represent the ratio as a fraction.

The overall results of this teaching experiment (Quaresma, 2010) show that students improved their understanding of fractions and percents, as well as of decimals. In addition, they developed their understanding of comparing and ordering rational numbers, using mainly the decimal representation. The understanding that students show of rational numbers, realizing that a rational number can be represented in different ways and showing flexibility in choosing the most appropriate representation with which they can solve the proposed tasks supports the teaching and learning hypothesis.

Doing the task “Crossing the River” also reaches the aims set by the teacher. The initial exploration carried out by the students and collective discussions were critical for solving all the questions of the task. This exploitation that arises in an informal way, provides the emergence of different representations and leads them to formulate a generalization and, subsequently, to study the situation more formally. In this way, the students rebuild their representations of the situation and make a natural use of letters as variables. The presentation of the findings by the students, using their own words, diagrams and symbols, allows a better understanding of the situation and the ascribing of meaning to the generalization that they later express in algebraic language. The students also develop the capacity to interpret the algebraic language from their analysis of the different expressions, in particular as regards the use of properties of operations. This interpretation of language algebraic allows them to reason backwards to determine the number of adults given the total number of trips. Based on the algebraic expression that gives the total number of trips for a group with A adults and 2 children, $4A +1$, identify the inverse operations to carry out and the correct order to undertake them. Finally, they analyze the effect of varying the number of adults and children in the number of required number of trips. They generalize this situation by using natural language and mathematical symbols, identifying the need for two trips per child in addition to a group of A adults and 2 children. The moments of autonomous work of student pairs allow them to progress in the interpretation of the situation and in searching for answers and in discussing in detail several opportunities. In the interactions with the students, the teacher poses questions to ascertain students' understanding of the concepts and their ability to use them, as well as to help students' further understanding and mastery of concepts. The collective discussions in an early stage of solving the task help the students to understand the situation, at an intermediate stage allow for the sharing of representations and the identification of regularities so that everyone can get along on the following questions and at the end favor the systematization of the results and conclusions obtained and the analysis of more complex situations.

The overall results of this teaching experiment (Branco, 2008) show that the students developed some aspects of algebraic thinking, including the ability to generalize and use algebraic language to express their generalizations, also supporting the teaching and learning hypothesis. However, the evolution of the students is not equally significant in all domains considered. In problem solving involving equations, hey favor arithmetic strategies that do not always prove effective, and exhibit some difficulty in using the algebraic language to represent the proposed situations. They show development in the understanding of the algebraic language concerning the different meanings of the symbols in various contexts and the meaning and manipulation of expressions, but in many aspects specific this understanding is still fragile, suggesting that they have a long way to go in developing their algebraic thinking.

It should be noted that, in addition to the open nature of exploration tasks that require extensive students' effort in interpreting, representing, and simulating cases, such learning also results from the structure of the class and the communication style promoted by the teachers. Both classes were held in cycles composed of moments for presentation and interpretation of tasks, moments of autonomous work in student pairs or groups, and moments of collective discussion. The work was completed with a summary of the main ideas. The communication style promoted by the teachers sought to value the contributions of students, highlighting the arguments and counter arguments they provided. This kind of exploratory class has been increasingly used in Portugal, under the new basic education mathematics curriculum, with a positive influence on students' learning (Ponte, 2012).

The situations presented show that the exploration tasks may form the basis of everyday work in the classroom, providing a suitable environment for learning the concepts, representations, and procedures that constitute the core of the mathematics curriculum. Such tasks also constitute a favorable ground for the development of transversal skills such as mathematical reasoning and communication. Unlike other kinds of tasks that tend to assume a peripheral role in teachers' practice, we find that explorations can be naturally integrated into teaching and learning of various mathematical topics. The development of the suitable conditions for implementing them at different educational levels poses interesting challenges to teachers and mathematics educators.

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