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Fluctuation theory and stochastic games for spectrally negative Lévy processes

Thesis

Fluctuation Theory and Stochastic Games for Spectrally Negative Lévy Processes

Fluctuatietheorie en Stochastische Spelen
voor Spectraal Negatieve Lévy Processen

(met een samenvatting in het Nederlands)

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Chapter 1

Introduction

1.1 Lévy processes

Lévy processes have stationary, independent increments. This seemingly unassuming (defining) property leads to a surprisingly rich class of processes which appear in a large number of applications including queueing, fragmentation theory, branching processes, dams, risk theory and finance. One may already appreciate the richness of the class of Lévy processes by remarking that it includes both a compound Poisson process (the paths of which are constant in between the jump times) and a Brownian motion (the paths of which are almost surely continuous but nowhere differentiable). Even though there is such a wide variety of Lévy processes, important properties can often be deduced without the need to specify further characteristics of the process over and above the stationarity and independence of its increments. Comprehensive treatises on Lévy processes are the books [18; 41; 63; 104]. For stochastic calculus for Lévy processes we refer to [2]. The recent book [21] illustrates the importance of Lévy processes in the context of fragmentation processes.

Because of the richness of the class of Lévy processes it is not surprising that some of the main results from excursion theory for Lévy processes (such as the Wiener–Hopf factorisation) can be deduced in more detail when we impose some restrictions on the process. A subclass for which many expressions are more explicit is formed by those (real-valued) Lévy processes without positive jumps (if such a process has non-monotone paths, we call it spectrally negative). For example, for a spectrally negative Lévy process we can find the probability that it leaves an interval $[a, b]$ at b, as well as the Laplace transform of the first exit time from an interval. For further details we refer to Chapter 8.2 in [63] and references therein.

A recent surge in interest in Lévy processes is partly due to applications in finance. It has been well documented that there are various flaws in the classical Black–Scholes model ([23; 78]), in which a stock process is modelled by an exponential Brownian motion. For example, the tails of a Brownian motion are too light when compared with market observations, see [51] for a detailed account. Various choices of Lévy processes other than a Brownian motion have numerous advantages; some popular Lévy models for the price of a stock are CGMY [29], variance gamma [76], hyperbolic [44] and normal inverse Gaussian [7]. An overview of applications of Lévy processes in finance can be found in the books [26; 37; 105] and in the collection of research papers [68].

1.2 Optimal stopping problems

When is it optimal to stop? This is the main question in a wide variety of problems, three of which we list here.

1. Quickest detection problem: one needs to decide when to send out an alarm signal for a hazardous event (such as an earthquake, incoming war plane) based on observable data (such as seismic measurements, radar).

2. Sequential testing: an observable process depends on a parameter which has a certain prior distribution. The aim is to find this parameter as quickly and accurately as possible based on a sequence of observations.

3. American option on a stock: the buyer of such an object has the right to terminate the option at any time up to maturity, inducing a pay-off depending on the value of the stock at that time. The stock is modelled by a certain stochastic process.

In all three situations there is an underlying observable random process of which the future value cannot be predicted. The aim is to optimise the expectation of a function of this random process. The respective goal in the above examples can be made more explicit as follows.

1*. Minimise a certain penalty function of false alarm and late response.

2*. Minimise the expectation of a function of the decision error and observation time.

3*. Maximise the expected pay-off.

The optimisation takes place over all stopping strategies which depend only on the history of the observable process up to that time (such strategies are called stopping times). We refer to [49] and [109], where the quickest detection problem was studied for a Poisson process and a Brownian motion, respectively. For sequential testing, see [92] Section 21 for the Brownian case and Section 23 for the Poisson case and the references therein. We also refer to [86] for an overview of these optimal stopping problems. For the general theory of optimal stopping problems and many more examples we refer to the books [92] and [110].

Much attention has been paid to the study of American options with a Brownian motion (or a more general diffusion) as the underlying process. Although for most applications it is more natural for the time horizon to be finite (the decision to stop must be made before a pre-specified deterministic time), we consider optimal stopping problems which have an infinite horizon. We refer to [28] for a method to solve a finite time horizon optimal stopping problem in terms of a sequence of infinite horizon optimal stopping problems for which the underlying process is killed at an exponential rate.

Consider an optimal stopping problem of the form

$$V(x) := \sup_{\tau \in \mathcal{T}} \mathbb{E}_x[e^{-q\tau} f(B_\tau)],$$

where $\mathbb{E}_x$ denotes expectation with respect to the law of a Brownian motion B started from $x \in \mathbb{R}$, where the discount rate q is a non-negative constant and where $\mathcal{T}$ is the set of (possibly infinite) stopping times for B. The general theory of optimal stopping implies that, under some mild conditions, an optimal stopping time is given by

$$\tau^* = \inf\{t \geq 0 : B_t \in D\},$$

where D is the optimal stopping region defined by

$$D = \{x \in \mathbb{R} : f(x) = V(x)\}.$$

Furthermore, it holds that

$$\{e^{-q(t\wedge\tau^*)} V(X_{t\wedge\tau^*})\}_{t\geq 0}$$

is a $\mathbb{P}_x$-martingale and that

$$\{e^{-qt} V(X_t)\}_{t\geq 0}$$

is a $\mathbb{P}_x$-supermartingale. If V is smooth enough, these martingale properties are equivalent (via Itô calculus) to

$$(\mathcal{L} - q)V = 0 \quad \text{on } D^c \tag{1.1}$$

and

$$(\mathcal{L} - q)V \leq 0 \quad \text{on } \mathbb{R}.$$

Here $\mathcal{L}$ denotes the infinitesimal generator of the Brownian motion.

Solving the optimal stopping problem via (1.1) is more challenging for processes with jumps, since in that case the infinitesimal generator $\mathcal{L}$ is non-local. More precisely, the infinitesimal generator of a Lévy process X is given by the integro-differential operator

$$\mathcal{L}f(x) = af'(x) + \frac{\sigma^2}{2} f''(x) + \int_{\mathbb{R}} (f(x+y) - f(x) - yf'(x)\mathbf{1}_{\{|y|\leq 1\}})\Pi(dy),$$

for all functions f for which the right hand side is well defined (for example, for functions f which are twice continuously differentiable and for which f, f' and f'' vanish at $\pm\infty$, see Theorem 31.5 in [104]). Here (a, σ^2, Π) is the Lévy triple of X, i.e. the unique $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and measure Π on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx)$ such that

$$-\log \mathbb{E}[e^{\mathrm{i}\lambda X_1}] = \mathrm{i}a\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_{\mathbb{R}} (1 - e^{\mathrm{i}\lambda x} - \mathrm{i}\lambda x \mathbf{1}_{\{|x|<1\}})\Pi(dx).$$

An additional difficulty over and above the non-locality of the infinitesimal generator is that (unlike in the diffusion case), in general, it is not necessarily known whether V is smooth enough to transform martingale properties into generator equations via Itô calculus, or whether the corresponding free boundary value problem

$$\max((\mathcal{L} - q)V, f - V) = 0$$

has a solution. It is often the case that fluctuation theory for Lévy processes can be used instead and it turns out to be a very useful tool indeed for solving optimal stopping problems for Lévy processes.

Smooth fit at some point $y \in \partial D$ states that when f is a differentiable function, V pastes smoothly onto f at y, by which we mean

$$\frac{dV}{dx} = \frac{df}{dx} \quad \text{at } y \in \partial D. \tag{1.2}$$

This is a well-studied phenomenon in the literature and first appeared in [79]. It holds for a much wider class of processes than just a Brownian motion, see for example [103] and Section 9.1 in [92] (and references therein) for the case of diffusions and also the "reasonable rule of thumb" in [1] for the case of a general strong Markov process (note, however, that in [89] a counterexample is given). Smooth fit turns out to be closely related to regularity. In fact, for almost all optimal stopping problems in the literature (also for those for which the underlying process has jumps), smooth fit at some point $y \in \partial D$ holds precisely when the process started at y hits the set D immediately. When this happens, we say that y is regular for D for the underlying process. Thus, it is easy to come up with optimal stopping problems for which smooth fit breaks down. For example, a bounded variation spectrally negative Lévy process has the property that b is not regular for $(-\infty, b)$, hence for any optimal stopping problem for which the optimal stopping time is of the form $\tau^* = \inf\{t \geq 0 : X_t \leq b\}$ (such as for the American put, see [1; 81]), one would expect smooth fit to break down. When smooth fit breaks down at some point $y \in \partial D$, equation (1.2) is replaced by the continuous fit condition $V(y) = g(y)$. The idea that continuous fit happens as a principle was first noticed in [90] and in [91].

In addition to the American put option mentioned above, various other specific American options have been studied for Lévy processes ([1; 31; 69; 77; 81]). Methods for finding the value of more general American options for Lévy processes can be found in [27] and in [112]. However, it is important to remark that the uniqueness of the martingale measure used for pricing such options in the standard Black–Scholes model breaks down when the underlying process is a general Lévy process and hence the price of the American option depends on the choice of martingale measure. See [30] for a review of various martingale measures.

In [89] an example is given of an optimal stopping problem for a regular diffusion and with an infinitely differentiable pay-off function for which smooth fit does not hold. In that paper it is shown that in some cases the regularity condition is not sufficient. Instead, it may be necessary to impose the stronger condition that the process leaves symmetric intervals upwards with probability one half in the limit as the length of the interval goes to zero. Hence, stronger assumptions than regularity may be necessary to ensure smooth fit holds for optimal stopping problems for processes which are not spatial homogeneous.

Optimal stopping problems for one-dimensional strong Markov processes can be related to obstacle problems (see [100]) which appear in physics. Consider putting a rope over f, whilst pulling its extremities outwards to $\pm\infty$. The shape of the corresponding rope will then coincide with the value of the corresponding optimal stopping problem. When the underlying process is higher dimensional, we can replace the rope by its higher dimensional equivalent and the analogy will still hold. In terms of the rope, smooth fit may then translate to the fact that, when the boundary itself is smooth, the rope does not have kinks. The fact that for certain optimal stopping problems for Lévy processes smooth fit breaks down can be thought of as taking a rope made of a more rigid material, leading to the kinks at the points where the rope moves away from the obstacle. Note that this analogy is not perfect for Lévy processes, since for most examples of optimal stopping problems the value function only has kinks at (certain parts of) the boundary of the stopping region.

1.3 Stochastic games

The stochastic games we consider in this thesis are two-player optimal stopping problems, for which one of the players is a minimiser and the other one a maximiser. Such games first appeared in the literature in [43]. Following the formulation of [47], consider a real-valued strong Markov process $\{X_t\}_{t\geq 0}$ and functions $f, g, h : \mathbb{R} \to \mathbb{R}_+$ satisfying $f \leq h \leq g$ and $\mathbb{E}_x[\sup_t g(X_t)] < \infty$. We say that

$$\sup_{\tau\in\mathcal{T}} \inf_{\sigma\in\mathcal{T}} \mathbb{E}_x[f(X_\tau)\mathbf{1}_{\{\tau<\sigma\}} + g(X_\sigma)\mathbf{1}_{\{\sigma<\tau\}} + h(X_\tau)\mathbf{1}_{\{\tau=\sigma\}}]$$

and

$$\inf_{\sigma\in\mathcal{T}} \sup_{\tau\in\mathcal{T}} \mathbb{E}_x[f(X_\tau)\mathbf{1}_{\{\tau<\sigma\}} + g(X_\sigma)\mathbf{1}_{\{\sigma<\tau\}} + h(X_\tau)\mathbf{1}_{\{\tau=\sigma\}}]$$

are, respectively, the lower and upper value of the game with pay-off functions f, g, h (it is straightforward to check that the first quantity is never greater than the second). A subtlety of a stochastic game (in comparison to a 'standard' optimal stopping problem) is the issue whether we can unambiguously assign a value to it: it can happen that the lower and upper value are different. If they are equal, this common value is called the value $V(x)$ of the game. A pair of stopping times (σ^*, τ^*) is called a saddle point of the game when

$$\begin{aligned}\mathbb{E}_x[f(X_\tau)&\mathbf{1}_{\{\tau<\sigma^*\}} + g(X_{\sigma^*})\mathbf{1}_{\{\sigma^*<\tau\}} + h(X_\tau)\mathbf{1}_{\{\tau=\sigma^*\}}]\\ &\leq \mathbb{E}_x[f(X_{\tau^*})\mathbf{1}_{\{\tau^*<\sigma^*\}} + g(X_{\sigma^*})\mathbf{1}_{\{\sigma^*<\tau^*\}} + h(X_{\tau^*})\mathbf{1}_{\{\tau^*=\sigma^*\}}]\\ &\leq \mathbb{E}_x[f(X_{\tau^*})\mathbf{1}_{\{\tau^*<\sigma\}} + g(X_\sigma)\mathbf{1}_{\{\sigma<\tau^*\}} + h(X_{\tau^*})\mathbf{1}_{\{\tau^*=\sigma\}}]\end{aligned}$$

for all stopping times τ, σ and for all $x \in \mathbb{R}$. This notion is analogous to that of a Nash-equilibrium in game theory, see for example [84]. It is readily seen that when a saddle point (σ^*, τ^*) exists, the stochastic game has a value given by

$$V(x) = \mathbb{E}_x[f(X_{\tau^*})\mathbf{1}_{\{\tau^*<\sigma^*\}} + g(X_{\sigma^*})\mathbf{1}_{\{\sigma^*<\tau^*\}} + h(X_{\tau^*})\mathbf{1}_{\{\tau^*=\sigma^*\}}].$$

In [47] it is shown that for a large class of optimal stopping games and a large class of underlying (strong Markov) processes (not necessarily one-dimensional or real-valued), the stochastic game has a value and that, under a mild additional assumption, a saddle point exists.

Applications of stochastic games in finance come in the form of game options, see [13; 59; 61; 64; 66] and (callable) convertible bonds, see [93; 111].

Finding a saddle point for specific examples is a challenging problem. In general, solving such an optimal stopping game (with infinite horizon) boils down to finding a function V and optimal stopping times τ^* and σ^* such that

$$\begin{aligned}&\sigma^* = \inf\{t \geq 0 : X_t \in D_1\},\\&\tau^* = \inf\{t \geq 0 : X_t \in D_2\},\\&f(x) \leq V(x) \leq g(x),\\&V(X_{t\wedge\sigma^*}) \text{ is a submartingale},\\&V(X_{t\wedge\tau^*}) \text{ is a supermartingale},\end{aligned}$$

where $D_1 = \{x : V(x) = f(x)\}$ and where $D_2 = \{x : V(x) = g(x)\}$. Here we have assumed for simplicity that $\mathbb{P}_x(\max(\tau^*, \sigma^*) < \infty) = 1$ for all x. When the underlying process is a Brownian motion (or more generally, a diffusion), stochastic games have been solved in several cases by finding the solution to the corresponding (via Itô calculus) free boundary value problem, see [13; 46; 61]. In [50] it is shown that under certain conditions on the jump measure, a specific stochastic game for Lévy processes can also be solved directly by considering the corresponding free boundary value problem. As with optimal stopping problems, fluctuation theory can be very useful for solving stochastic games for Lévy processes when it seems difficult to find directly the solution to the corresponding free boundary problem.

Smooth fit (at a point $y_1 \in \partial D_1$ and at $y_2 \in \partial D_2$) in the context of stochastic games means that when f and g are smooth functions, it holds that

$$\frac{dV}{dx} = \frac{df}{dx} \quad \text{at } y_1 \in \partial D_1 \qquad \text{and} \qquad \frac{dV}{dx} = \frac{dg}{dx} \quad \text{at } y_2 \in \partial D_2.$$

Again, it turns out that the issue of smooth fit is closely related to regularity of the underlying process.

In the case of a one-dimensional strong Markov process, there still is an analogy with obstacle problems. Now, consider two obstacles, one above and one below, with the rope sandwiched between them. When one pulls the rope outwards at its extremities, the shape of the rope will coincide with the value function of the corresponding stochastic game. We refer to [88] for a detailed description of this analogy.

1.4 Thesis outline

The outline of this thesis is as follows.

In Chapter 2, we study the last passage problem for a spectrally negative Lévy process. In [36], the Laplace transform was found of the last time a spectrally negative Lévy process, which drifts to infinity, is below a certain level. The main motivation for the study of this random time comes from risk theory: what is the last time the risk process, modelled by a spectrally negative Lévy process drifting to infinity, is (below) zero? We use fluctuation theory to extend this result and find the Laplace transform of the last time before an independent, exponentially distributed time that a spectrally negative Lévy process is below zero. Using similar methods we also find an expression for the Laplace transform of the last passage times above a certain level and of the last hitting time of a point. Furthermore, we extend the result from [39] which states that the event that there exists some $t > 0$ such that a Brownian motion first reaches level t at time t has probability one half.

Chapter 3 focusses on Lévy processes reflected at their minimum. Reflected Lévy processes appear in a wide variety of applications such as the study of the water level in a dam, queueing (see [3; 25; 97]), optimal stopping ([15; 108]) and optimal control ([6; 38; 52; 67]). For many of these applications, it is important to have the overshoot distribution of the reflected process (i.e. the amount by which the process exceeds a level at the time of first passage). We use excursion theory to find the resolvent density for a reflected Lévy process in terms of the two-sided exit problem and the resolvent density of the Lévy process itself. A main ingredient in the proof consists of Lévy processes conditioned to stay positive (see for example [33; 34; 35] for further details on such conditioned processes and, in particular, for how to condition a process on an event which may be of probability zero). As an application of the main result in Chapter 3, we find the potential density of a symmetric (one-dimensional) stable process killed at exceeding a certain level. As a corollary we also find the joint distribution of its undershoot and overshoot.

Chapter 4 deals with an application of a method introduced in [16] to solve optimal stopping problems for diffusions. We show that this method can also be used to solve certain optimal stopping problems for processes with jumps. In particular, we solve an optimal stopping problem for a stable process and a closely related problem for a generalised Ornstein–Uhlenbeck process driven by a spectrally negative Lévy process. We discuss the issue of smooth fit for optimal stopping problems for generalised Ornstein–Uhlenbeck processes driven by a spectrally negative Lévy process with non-zero Gaussian component. The method of proof of smooth fit presented in this chapter can be extended to more general processes with jumps. We also show that for a specific choice of the discount rate, the optimal stopping time for the American put option for a spectrally negative Lévy process is the first hitting time of a point.

In Chapter 5 and Chapter 6 we study specific examples of stochastic games (which we call the McKean and Shepp–Shiryaev stochastic game, respectively) driven by a spectrally negative Lévy process. These stochastic games were treated for a Brownian motion in [61] by solving the corresponding free boundary problem directly and by making use of the fact that the scale functions are known explicitly for a Brownian motion (they are given in terms of the hyperbolic sine and the hyperbolic cosine). However, when the underlying process has jumps we need to take a different approach. For example, unlike the Brownian case, solving directly the corresponding free boundary problem seems out of reach. For the McKean game based on a Brownian motion, it was found in [61] that the optimal stopping region for the minimiser consists of a single point. A spectrally negative Lévy process without a Gaussian component does not creep downwards (which means that the first passage below a level happens by a jump), and hence one would expect the solution to be different in this case. Indeed, for the McKean game based on a spectrally negative Lévy process without a Gaussian component we observe the phenomenon that the stopping region thickens from a point to an interval. This can also be interpreted intuitively, since one would expect the minimiser to adopt a more conservative strategy to take into account that the process becomes less 'predictable' due to the presence of jumps. For both games we make use of an auxiliary optimal stopping problem: we fix a particular strategy for one of the players and optimise the corresponding pay-off function over the strategies of the other player. We use general results from optimal stopping theory to deduce that there exists an optimal stopping strategy for this auxiliary problem. Furthermore, we deduce some of its properties which allow us to prove that the value function of this auxiliary problem is in fact the value function of the stochastic game. Both in Chapter 5 and in Chapter 6 the scale functions for the spectrally negative Lévy process play an important role. For both stochastic games we observe that the principle of smooth/continuous fit holds at an element a of the stopping region's boundary according to whether a is regular/irregular for the stopping region's interior.

1.5 Publication details

The second chapter of this thesis has been submitted as [11]. A slightly modified version of the third chapter has been accepted for publication in *ALEA, Latin American Journal of Probability and Mathematical Statistics* as [12]:

> Baurdoux, E.J. (2007). Some excursion calculations for reflected Lévy processes.

Chapter 4 appeared in abridged form as [10]:

> Baurdoux, E.J. (2007). Examples of optimal stopping via measure transformation for processes with one-sided jumps, *Stochastics* **79,** 303–307.

The fifth and sixth chapter are joint work with Andreas Kyprianou and have been submitted as [14] and as [15] respectively. A special case of [14] is included in Section 9.6 of the book [63] and a special case of [15] is Exercise 9.2 in that same book.

Chapter 2

Last exit before an exponential time for spectrally negative Lévy processes*

In [36] the Laplace transform was found of the last time a spectrally negative Lévy process, which drifts to infinity, is below a certain level. The main motivation for the study of this random time comes from risk theory: what is the last time the risk process, modelled by a spectrally negative Lévy process drifting to infinity, is zero? We extend the result found in [36] by taking as the time horizon an independent, exponentially distributed random variable $\mathbf{e}_q$ with parameter $q \geq 0$. To be more precise, we find the Laplace transform of the last time before $\mathbf{e}_q$ at which a spectrally negative Lévy process (without any further assumptions) exceeds or hits a certain level. As an application, we extend a result found by Doney in [39].

2.1 Introduction

Consider an insurance company which collects premiums at a constant rate and receives claims from its customers at various (unpredictable) times. The classical risk process $\{X_t\}_{t\geq 0}$ as introduced in [75] consists of a deterministic, strictly positive drift c plus a compound Poisson process which has only negative jumps. We denote by $\lambda > 0$ the rate of the Poisson process and by μ the expected jump size. Of key interest is the moment of ruin, i.e. the first time when the risk process becomes negative. To ensure the moment of ruin is not almost surely finite, the net profit

*Submitted as [11].

condition

$$\frac{\lambda\mu}{c} < 1 \tag{2.1}$$

is imposed. This condition ensures that the risk process drifts to $+\infty$. Recently, various authors ([31], [55], [56], [60]) have replaced the compound Poisson process by general spectrally negative Lévy processes. The latter are Lévy processes which have no positive jumps and which are not the negative of a subordinator. In some cases, the moment of ruin may not be the most important quantity for the risk process. Indeed, consider the following scenario. Instead of going bankrupt when the risk process becomes negative, the firm has other funds which can be used to support the negative surplus for a while. For this reason, in [52], the Laplace transform was found of the last passage time at a certain level for the classical risk process. This was extended to the case of a general spectrally negative Lévy process in [36]. However, a more realistic quantity for study may be the last passage time below zero before a fixed time

$$S_t^- := \sup\{0 \le u \le t : X_u \le 0\} \qquad \text{for } t \ge 0, \tag{2.2}$$

with the convention that $\sup \emptyset = 0$. It turns out that it is easier to replace the fixed, deterministic time horizon by an independent, exponentially distributed time. For $\theta \ge 0$, we denote by $\tilde{\mathbf{e}}_\theta$ an exponentially distributed random variable with parameter θ. Here, we use the convention that an exponential random variable with parameter zero is taken to be infinite with probability one. For a spectrally negative Lévy process $\{X_t\}_{t\ge 0}$ (defined on a probability space $(\Omega, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ satisfying the usual conditions) starting from $x \ne 0$ we write $\mathbb{P}_x$ as its probability measure.

Now, define the random time

$$\sigma_\theta^- = S_{\tilde{\mathbf{e}}_\theta} = \sup\{0 \le t \le \tilde{\mathbf{e}}_\theta : X_t \le 0\}.$$

In Theorem 2.5, the main result of this chapter, we give the Laplace transform of σ_θ^-. Using similar techniques, we also find the Laplace transform of

$$\sigma_\theta^+ = \sup\{0 \le t \le \tilde{\mathbf{e}}_\theta : X_t \ge 0\}$$

and of

$$T_\theta = \sup\{0 \le t \le \tilde{\mathbf{e}}_\theta : X_t = 0\}.$$

For convenience, we suppress the subscript when $\theta = 0$. For spectrally negative Lévy processes drifting to ∞, the Laplace transform of σ^- was found in [36]. Trivially, when X drifts to ∞ it holds that $T = \sigma^-$.

Remark 2.1. The random times introduced above are not stopping times, as they depend on the future of the process $\{X_t\}_{t\ge 0}$.

2.2 Preliminaries

In this section we review some important properties and tools of spectrally negative Lévy processes. For further details we refer to the books [18] and [63]. For a spectrally negative Lévy process $\{X_t\}_{t\geq 0}$ it holds that the Laplace exponent

$$\psi(\lambda) := \log \mathbb{E}[e^{\lambda X_1}] \quad \lambda \geq 0$$

is well defined, convex and infinitely differentiable on $(0, \infty)$ and $\psi(0) = 0$. Also, because X has non-monotone paths, we have that $\lim_{\lambda\to\infty} \psi(\lambda) = \infty$. Furthermore, when X is of bounded variation, we can express the Laplace exponent as

$$\psi(\lambda) = \mathrm{d}\lambda + \int_{(-\infty,0)} (e^{\lambda x} - 1)\, \Pi(dx),$$

where Π is the jump measure of X and $\mathrm{d} > 0$ is called the drift.

The condition in [36] that X drifts to $+\infty$ (i.e. that $\lim_{t\to\infty} X_t = +\infty$) is equivalent to $\psi'(0) > 0$. The case $\psi'(0) < 0$ corresponds to X drifting to $-\infty$ (i.e. that $\lim_{t\to\infty} X_t = -\infty$) and $\psi'(0) = 0$ occurs precisely when X oscillates, i.e. when

$$\limsup_{t\to\infty} X_t = -\liminf_{t\to\infty} X_t = +\infty.$$

For $q \geq 0$, the scale function $W^{(q)}(x)$ is defined as the unique continuous, increasing function on $[0, \infty)$ which satisfies

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x)\, dx = \frac{1}{\psi(\lambda) - q} \quad \text{for any } \lambda > \Phi(q).$$

Here, Φ denotes the right inverse of ψ. See Chapter 8 in [63] for a detailed study of the scale function. We extend $W^{(q)}$ to the negative half line by putting $W^{(q)}(x) = 0$ when $x < 0$. Note that $W^{(q)}$ is not necessarily continuous in 0. In fact, it is not difficult to show that $W^{(q)}(0) = 0$ when X is of unbounded variation and $W^{(q)}(0) = 1/\mathrm{d}$ when X is of bounded variation with drift d. Furthermore, for $q \geq 0$ we define the function $Z^{(q)}$ by

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y)\, dy.$$

Note that $Z^{(q)}(x) = 1$ when $x \leq 0$. Integrating by parts we readily deduce that

$$\int_0^\infty e^{-\lambda x} Z^{(q)}(x)\, dx = \frac{1}{\lambda} + \frac{q}{\lambda} \int_0^\infty e^{-\lambda x} W^{(q)}(x)\, dx = \frac{1}{\lambda} + \frac{q}{\lambda(\psi(\lambda) - q)}. \tag{2.3}$$

For $a, b \in \mathbb{R}$ we denote first passage times by

$$\tau_a^- := \inf\{t > 0 : X_t \leq a\} \tag{2.4}$$

and

$$\tau_b^+ := \inf\{t > 0 : X_t \geq b\}. \tag{2.5}$$

Also, we denote the first hitting time by

$$T(a) := \inf\{t > 0 : X_t = a\}. \tag{2.6}$$

Scale functions play a vital role in exit problems. For example, it holds that

$$\mathbb{E}_x[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}] = Z^{(q)}(x) - W^{(q)}(x)\frac{q}{\Phi(q)}, \tag{2.7}$$

where for the case $q = 0$ the fraction $q/\Phi(q)$ is to be understood in the limiting sense. Expression (2.7) first appeared in the form of its Fourier transform in [48]. For spectrally negative processes, the q-potential measure defined by

$$U^{(q)}(dy) = \int_0^\infty e^{-qt}\,\mathbb{P}(X_t \in dy)\,dt$$

is absolutely continuous with respect to Lebesgue measure and a version of its density is given by

$$u^q(y) = \Phi'(q)e^{-\Phi(q)y} - W^{(q)}(-y) \quad \text{ for } y \in \mathbb{R}, \tag{2.8}$$

see Corollary 1 on page 733 in [22].

Since, for $c \geq 0$, the process $\{e^{cX_t-\psi(c)t}\}$ is a martingale with mean 1, we can introduce the change of measure

$$\left.\frac{d\mathbb{P}^c}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{cX_t-\psi(c)t}.$$

The process $\{X_t\}_{t\geq 0}$ is still a spectrally negative Lévy process under $\mathbb{P}^c$ and we mark the Laplace exponent and scale functions of X under $\mathbb{P}^c$ with the subscript c. It is straightforward to check that

$$\psi_c(\lambda) = \psi(c+\lambda) - \psi(c) \tag{2.9}$$

for $\lambda \geq 0$ and, by taking Laplace transforms, we also find

$$W_c^{(q)}(x) = e^{-cx}W^{(q+\psi(c))}(x) \tag{2.10}$$

for $q \geq 0$. Furthermore, we readily check that for $c, p \geq 0$

$$\begin{aligned}
\Phi_c(p) &= \sup\{x : \psi_c(x) = p\} \\
&= \sup\{x : \psi(x+c) = p + \psi(c)\} \\
&= \Phi(p + \psi(c)) - c.
\end{aligned}$$

For future reference we state the following result.

Lemma 2.2. *For $q > 0$ and $\lambda \geq -\Phi(q)$*

$$\int_{[0,\infty)} e^{-\lambda x}\,W_{\Phi(q)}(dx) = \frac{\lambda}{\psi(\Phi(q)+\lambda) - q}, \tag{2.11}$$

where the right hand side is to be interpreted in the limiting sense as $\Phi'(q)$ for the case $\lambda = 0$.

Note that Lemma 2.2 follows by integration by parts when $\lambda > 0$ (compare with (8.17) in [63]). The proof that (2.11) still holds when $\lambda \in [-\Phi(q), 0]$ is more complicated and we defer it to the appendix of this chapter.

Finally, we collect numerous known expressions for first exit problems, which we will use throughout this chapter.

Lemma 2.3. *For $x > 0$ and $u, v \geq 0$,*

$$\mathbb{E}_x[e^{-u\tau_0^- + vX_{\tau_0^-}} \mathbf{1}_{\{\tau_0^- < \infty\}}] = e^{vx}\left(Z_v^{(p)}(x) - W_v^{(p)}(x)\frac{p}{\Phi_v(p)}\right), \tag{2.12}$$

where $p = u - \psi(v)$. For the case $u = \psi(v)$, $p/\Phi_v(p)$ is to be interpreted in the limiting sense as

$$\lim_{u \to \psi(v)} \frac{u - \psi(v)}{\Phi_v(u - \psi(v))} = \frac{1}{\Phi'(\psi(v))}. \tag{2.13}$$

For $x < 0$ and $q \geq 0$

$$\mathbb{E}_x[e^{-q\tau_0^+}\mathbf{1}_{\{\tau_0^+ < \infty\}}] = e^{\Phi(q)x}. \tag{2.14}$$

Finally, for $x > 0$ and $q \geq 0$

$$\mathbb{E}_x[e^{-qT(0)}\mathbf{1}_{\{T(0) < \infty\}}] = e^{\Phi(q)x} - \psi'(\Phi(q))W^{(q)}(x) \tag{2.15}$$

and the case $x = 0$ is given by $1 - (\mathrm{d}\Phi'(0))^{-1}$ when X has bounded variation with drift d*.*

Expression (2.12) follows after a change of measure and (2.7). From the fact that $\{e^{-qt+\Phi(q)X_t}\}_{t\geq 0}$ is a martingale, one can deduce (2.14). Finally, (2.15) was established in the form of its Laplace transform in Theorem 1 in [39].

2.3 Main result

Not surprisingly, scale functions also appear in expressions for last exit times. The following result is Theorem 3.1 in [36].

Theorem 2.4. *Suppose $\psi'(0) > 0$. Then for $q > 0$ and $x \in \mathbb{R}$*

$$\mathbb{E}_x[e^{-q\sigma^-}\mathbf{1}_{\{\sigma^- > 0\}}] = \Phi'(q)\psi'(0)e^{\Phi(q)x} - \psi'(0)W^{(q)}(x).$$

We extend this result by considering last passage below a certain level before an independent, exponentially distributed time as well as last passage above and last hit of a fixed level before such a time. We state the main result.

Theorem 2.5. *For $q \geq 0, \theta \geq 0$ and $x \in \mathbb{R}$*

$$\begin{aligned}\mathbb{E}_x[e^{-q\sigma_\theta^-}] &= 1 + e^{\Phi(q+\theta)x}\Phi'(q+\theta)\left(\frac{\theta}{\Phi(\theta)} - \frac{\theta}{\Phi(q+\theta)}\right) \\ &\quad + \frac{\theta}{q+\theta}Z^{(q+\theta)}(x) - Z^{(\theta)}(x) \\ &\qquad + \frac{\theta}{\Phi(\theta)}\left(W^{(\theta)}(x) - W^{(q+\theta)}(x)\right). \end{aligned} \tag{2.16}$$

Furthermore,

$$\begin{aligned}\mathbb{E}_x[e^{-q\sigma_\theta^+}] &= \frac{q}{q+\theta}Z^{(q+\theta)}(x) - e^{\Phi(\theta)x}Z^{(q)}_{\Phi(\theta)}(x)\\ &\quad + \frac{\theta}{\theta+q} + e^{\Phi(q+\theta)x}\frac{q\Phi(\theta)\Phi'(q+\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))}. \end{aligned} \tag{2.17}$$

Also,

$$\mathbb{E}_x[e^{-qT_\theta}] = 1 - e^{\Phi(\theta)x} + \frac{1}{\Phi'(\theta)}(W^{(\theta)}(x) - W^{(q+\theta)}(x)) + \frac{\Phi'(q+\theta)}{\Phi'(\theta)}e^{\Phi(q+\theta)x}. \tag{2.18}$$

Combined with the strong Markov property, Theorem 2.5 allows us to readily obtain expressions for the joint Laplace transform of first and last exit times.

Corollary 2.6. *Let $p, q \geq 0$. When X does not oscillate*

$$\mathbb{E}_x[e^{-pT(0)-qT}\mathbf{1}_{\{T(0)<\infty\}}] = \frac{\Phi'(q)}{\Phi'(0)}\left(e^{\Phi(p+q)x} - \frac{1}{\Phi'(p+q)}W^{(p+q)}(x)\right). \tag{2.19}$$

When X drifts to $-\infty$ and $x < 0$

$$\mathbb{E}_x[e^{-p\tau_0^+ - q\sigma^+}\mathbf{1}_{\{\tau_0^+<\infty\}}] = \frac{q\Phi(0)\Phi'(q)}{\Phi(q)(\Phi(q)-\Phi(0))}e^{\Phi(p+q)x}. \tag{2.20}$$

When X drifts to $+\infty$

$$\mathbb{E}_x[e^{-p\tau_0^- - q\sigma^-}\mathbf{1}_{\{\tau_0^-<\infty\}}] = \frac{\Phi'(q)}{\Phi'(0)}\left(e^{\Phi(q)x}Z^{(p)}_{\Phi(q)}(x) - \frac{p}{\Phi(p+q)-\Phi(q)}W^{(p+q)}(x)\right). \tag{2.21}$$

Proof of Corollary 2.6. The third equality was already obtained in [36]. We only prove (2.19), as the proofs of the other claims are similar. Suppose that X drifts to $+\infty$. Then $\Phi(0) = 0$ and from the strong Markov property applied at $T(0)$ and (2.15) we find

$$\begin{aligned}\mathbb{E}_x[e^{-pT(0)-qT}\mathbf{1}_{\{T(0)<\infty\}}] &= \mathbb{E}_x[e^{-(p+q)T(0)}\mathbf{1}_{\{T(0)<\infty\}}\mathbb{E}[e^{-qT}]]\\ &= \frac{\Phi'(q)}{\Phi'(0)}\left(e^{\Phi(p+q)x} - \frac{1}{\Phi'(p+q)}W^{(p+q)}(x)\right).\end{aligned}$$

□

Remark 2.7. Note that Theorem 2.4 follows by taking $p = 0$ in (2.21) (or in (2.19)).

When X is a stable process, we can invert the double Laplace transform in (2.18) when $x = 0$ and retrieve the known result that, for each $t \geq 0$, the random variable defined, analogously to (2.2), by

$$S_t := \sup\{0 \leq u \leq t : X_u = 0\}, \qquad t \geq 0$$

is distributed according to the so-called generalised arcsine law. When $\alpha = 2$, this is the well-known arcsine law for a Brownian motion (see eg. [74]). In fact, using the scaling property of stable processes, the following result can be shown to hold for any stable process with index $\alpha \in (1, 2]$ (i.e. not only in the spectrally negative case). We refer to Theorem VIII.12 in [18] for the proof in the general case.

Corollary 2.8. *Suppose X is a spectrally negative stable process with index $\alpha \in (1, 2]$. Then for $0 \le s \le t$*

$$\mathbb{P}(S_t \in ds) = \frac{\sin(\pi/\alpha)}{\pi} s^{-1/\alpha}(t-s)^{-1+1/\alpha} ds. \tag{2.22}$$

Also, the distribution of S_t^- is given by

$$\mathbb{P}(S_t^- \in ds) = \frac{1}{\alpha}\frac{\sin(\pi/\alpha)}{\pi} s^{-1/\alpha}(t-s)^{-1+1/\alpha} ds + (1 - \frac{1}{\alpha})\delta_t(ds), \tag{2.23}$$

where δ_t is the Dirac measure at t.

Proof of Corollary 2.8. When X is a spectrally negative stable process of index α, it holds that (without loss of generality) $\psi(\lambda) = \lambda^\alpha$ for $\lambda \ge 0$ and thus $\Phi(q) = q^{1/\alpha}$ for $q \ge 0$. It is straightforward to check that

$$\int_0^\infty \int_s^\infty e^{-qs-\theta t} s^{-1/\alpha}(t-s)^{-1+1/\alpha}\, dt\, ds = \Gamma(1/\alpha)\Gamma(1-1/\alpha)\theta^{-1/\alpha}(\theta+q)^{-1+1/\alpha}.$$

From (2.18) we deduce (2.22) and (2.23) follows in a similar way from (2.16). □

2.4 Proof of Theorem 2.5

For $q \ge 0$, we denote by $\mathbf{e}_q$ an exponentially distributed random variable with parameter q which is independent of X and $\tilde{\mathbf{e}}_\theta$. We split the proof of Theorem 2.5 in several parts.

Proof of (2.16). Let

$$A^+ = \{\tilde{\mathbf{e}}_\theta \ge \mathbf{e}_q, X_s > 0 \text{ for all } s \in [\mathbf{e}_q, \tilde{\mathbf{e}}_\theta]\}.$$

We can then write the event $\{\sigma_\theta^- < \mathbf{e}_q\}$ as a disjoint union

$$\{\sigma_\theta^- < \mathbf{e}_q\} = \{\tilde{\mathbf{e}}_\theta < \mathbf{e}_q\} \cup A^+. \tag{2.24}$$

We thus have for $x \in \mathbb{R}$,

$$\mathbb{E}_x[e^{-q\sigma_\theta^-}] = \mathbb{P}_x(\sigma_\theta^- < \mathbf{e}_q) = \mathbb{P}(\tilde{\mathbf{e}}_\theta < \mathbf{e}_q) + \mathbb{P}_x(A^+) = \frac{\theta}{\theta+q} + \mathbb{P}_x(A^+).$$

We find for $x \le 0$

$$\begin{aligned}
\mathbb{P}_x(A^+) &= q\mathbb{E}_x\left[\int_0^\infty e^{-qt}\mathbf{1}_{\{\tilde{\mathbf{e}}_\theta \ge t\}}\mathbf{1}_{\{X_s>0 \text{ for all } s\in[t,\tilde{\mathbf{e}}_\theta]\}}\,dt\right]\\
&= q\int_0^\infty e^{-(q+\theta)t}\int_{(0,\infty)}\mathbb{P}_x(X_t\in dy)\mathbb{P}_y(\tau_0^- > \tilde{\mathbf{e}}_\theta)\,dt\\
&= q\int_0^\infty u^{(q+\theta)}(y-x)(1-\mathbb{E}_y[e^{-\theta\tau_0^-}])\,dy\\
&= q\int_0^\infty \Phi'(q+\theta)e^{-\Phi(q+\theta)(y-x)}(1-Z^{(\theta)}(y)+W^{(\theta)}(y)\frac{\theta}{\Phi(\theta)})\,dy\\
&= q\Phi'(q+\theta)e^{\Phi(q+\theta)x}\left(\frac{1}{\Phi(q+\theta)}-\frac{1}{\Phi(q+\theta)}-\frac{\theta}{q\Phi(q+\theta)}+\frac{\theta}{q\Phi(\theta)}\right)\\
&= \Phi'(q+\theta)e^{\Phi(q+\theta)x}\left(\frac{\theta}{\Phi(\theta)}-\frac{\theta}{\Phi(q+\theta)}\right), \qquad (2.25)
\end{aligned}$$

where the second line follows from the Markov property and lack of memory of the exponential distribution, the fourth line from (2.8) and (2.12) and the fifth line from (2.3) and the definition of $W^{(q)}$. Hence,

$$\mathbb{E}_x[e^{-q\sigma_\theta^-}] = \frac{\theta}{\theta+q} + \Phi'(q+\theta)e^{\Phi(q+\theta)x}\left(\frac{\theta}{\Phi(\theta)}-\frac{\theta}{\Phi(q+\theta)}\right) \quad \text{for } x \le 0.$$

Next, let $x > 0$. In this case, σ_θ^- is equal to zero whenever X does not become negative before $\tilde{\mathbf{e}}_\theta$. Taking this into account, we refine (2.24) and write the event $\{\sigma_\theta^- < \mathbf{e}_q\}$ as a disjoint union

$$\begin{aligned}
\{\sigma_\theta^- < \mathbf{e}_q\} &= \{\tilde{\mathbf{e}}_\theta < \mathbf{e}_q\}\cup\{\sigma_\theta^- = 0, \tilde{\mathbf{e}}_\theta \ge \mathbf{e}_q\}\cup\{\sigma_\theta^- \in (0,\mathbf{e}_q), \tilde{\mathbf{e}}_\theta \ge \mathbf{e}_q\}\\
&= \{\tilde{\mathbf{e}}_\theta < \mathbf{e}_q\}\cup\{\tau_0^- > \tilde{\mathbf{e}}_\theta, \tilde{\mathbf{e}}_\theta \ge \mathbf{e}_q\}\cup(\{\tau_0^- < \tilde{\mathbf{e}}_\theta\}\cap A^+).
\end{aligned}$$

We thus have that

$$\mathbb{E}_x[e^{-q\sigma_\theta^-}] = \frac{\theta}{\theta+q} + \mathbb{P}_x(\tau_0^- > \tilde{\mathbf{e}}_\theta \ge \mathbf{e}_q) + \mathbb{P}_x(\tau_0^- < \tilde{\mathbf{e}}_\theta, A^+).$$

We deduce

$$\begin{aligned}
\mathbb{P}_x(\tau_0^- > \tilde{\mathbf{e}}_\theta \ge \mathbf{e}_q) &= \mathbb{E}_x\left[\int_0^\infty \theta e^{-\theta y}\mathbf{1}_{\{\tau_0^- > y > \mathbf{e}_q\}}\,dy\right]\\
&= \mathbb{E}_x[(e^{-\theta\mathbf{e}_q}-e^{-\theta\tau_0^-})\mathbf{1}_{\{\tau_0^- > \mathbf{e}_q\}}]\\
&= \mathbb{E}_x\left[\int_0^\infty (e^{-\theta z}-e^{-\theta\tau_0^-})qe^{-qz}\mathbf{1}_{\{\tau_0^- > z\}}\,dz\right]\\
&= \frac{q}{q+\theta}+\frac{\theta}{q+\theta}\mathbb{E}_x[e^{-(q+\theta)\tau_0^-}]-\mathbb{E}_x[e^{-\theta\tau_0^-}]\\
&= \frac{q}{q+\theta}+\frac{\theta}{q+\theta}\left(Z^{(q+\theta)}(x)-\frac{q+\theta}{\Phi(q+\theta)}W^{(q+\theta)}(x)\right)\\
&\quad -Z^{(\theta)}(x)+\frac{\theta}{\Phi(\theta)}W^{(\theta)}(x),
\end{aligned}$$

where (2.7) was used for the last equality. Denote for $\theta, q \geq 0$,

$$\lambda(\theta, q) = \Phi'(q+\theta)\left(\frac{\theta}{\Phi(\theta)} - \frac{\theta}{\Phi(q+\theta)}\right).$$

From the strong Markov property, the memoryless property of the exponential distribution, (2.12) and (2.25) we deduce that

$$\begin{aligned}
\mathbb{P}_x(\tau_0^- < \tilde{\mathbf{e}}_\theta, A^+) &= \mathbb{E}_x[\mathbf{1}_{\{\tau_0^- < \mathbf{e}_q \wedge \tilde{\mathbf{e}}_\theta\}} \mathbb{P}_{X_{\tau_0^-}}(A^+)] \\
&= \lambda(\theta, q)\mathbb{E}_x[e^{\Phi(q+\theta)X_{\tau_0^-}} \mathbf{1}_{\{\tau_0^- < \mathbf{e}_q \wedge \tilde{\mathbf{e}}_\theta\}}] \\
&= \lambda(\theta, q)\mathbb{E}_x[\, e^{-(q+\theta)\tau_0^- + \Phi(q+\theta)X_{\tau_0^-}} \mathbf{1}_{\{\tau_0^- < \infty\}}] \\
&= \lambda(\theta, q) e^{\Phi(q+\theta)x}\left(1 - W_{\Phi(q+\theta)}(x)\frac{1}{\Phi'(q+\theta)}\right),
\end{aligned}$$

since $\mathbf{e}_q \wedge \tilde{\mathbf{e}}_\theta$ is exponentially distributed with parameter $q+\theta$. From (2.10) we know that $e^{\Phi(p+q)x} W_{\Phi(p+q)}(x) = W^{(p+q)}(x)$ and thus (2.16) follows. □

Proof of (2.17). We can write the event $\{\sigma_\theta^+ < \mathbf{e}_q\}$ as a disjoint union

$$\{\tilde{\mathbf{e}}_\theta < \mathbf{e}_q\} \cup A^-,$$

where

$$A^- = \{\tilde{\mathbf{e}}_\theta > \mathbf{e}_q, X_s < 0 \text{ for all } s \in [\mathbf{e}_q, \tilde{\mathbf{e}}_\theta]\}.$$

We thus have

$$\mathbb{E}_x[e^{-q\sigma_\theta^+}] = \mathbb{P}_x(\sigma_\theta^+ < \mathbf{e}_q) = \frac{\theta}{\theta+q} + \mathbb{P}_x(A^-).$$

Let $x \geq 0$. Then

$$\begin{aligned}
\mathbb{P}_x(A^-) &= q\mathbb{E}_x\left[\int_0^\infty e^{-qt}\mathbf{1}_{\{\tilde{\mathbf{e}}_\theta > t\}}\mathbf{1}_{\{X_s<0 \text{ for all } s\in[t,\tilde{\mathbf{e}}_\theta]\}}\, dt\right] \\
&= q\int_0^\infty e^{-(q+\theta)t}\int_{(-\infty,0)} \mathbb{P}_x(X_t \in dy)\mathbb{P}_y(\tau_0^+ > \tilde{\mathbf{e}}_\theta)\, dt \\
&= q\int_{(-\infty,0)} u^{(q+\theta)}(y-x)\mathbb{P}_y(\tau_0^+ > \tilde{\mathbf{e}}_\theta)\, dy \\
&= q\int_{(-\infty,0)} \left(\Phi'(q+\theta)e^{\Phi(q+\theta)(x-y)} - W^{(q+\theta)}(x-y)\right)(1 - e^{\Phi(\theta)y})\, dy \\
&= q\int_{(0,\infty)} \left(\Phi'(q+\theta) - W_{\Phi(q+\theta)}(x+y)\right) e^{\Phi(q+\theta)(x+y)}(1 - e^{-\Phi(\theta)y})\, dy \\
&= qe^{\Phi(q+\theta)x}\int_{(0,\infty)}\int_{x+y}^\infty W'_{\Phi(q+\theta)}(z)(e^{\Phi(q+\theta)y} - e^{(\Phi(q+\theta)-\Phi(\theta))y})\, dz\, dy \\
&= qe^{\Phi(q+\theta)x}\int_{(x,\infty)}\int_0^{z-x} W'_{\Phi(q+\theta)}(z)(e^{\Phi(q+\theta)y} - e^{(\Phi(q+\theta)-\Phi(\theta))y})\, dy\, dz,
\end{aligned}$$

where the second line follows from the Markov property and the lack of memory property of the exponential distribution, the fourth line from (2.8) and (2.14), the penultimate equality from $W_{\Phi(q+\theta)}(\infty) = \Phi'(q+\theta)$, see (2.34) in the appendix of this chapter and the last equality from an application of Fubini's theorem. Denote

$$f(x,z) := \frac{e^{\Phi(q+\theta)(z-x)}}{\Phi(q+\theta)} - \frac{e^{(\Phi(q+\theta)-\Phi(\theta))(z-x)}}{\Phi(q+\theta)-\Phi(\theta)} + \frac{\Phi(\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))}.$$

Integration by parts yields

$$\begin{aligned}
&\int_{(x,\infty)} \int_0^{z-x} W'_{\Phi(q+\theta)}(z)(e^{\Phi(q+\theta)y} - e^{(\Phi(q+\theta)-\Phi(\theta))y})\,dy\,dz \\
&\quad= \int_{[0,\infty)} f(x,z)\,W_{\Phi(q+\theta)}(dz) - \int_{[0,x]} f(x,z)\,W_{\Phi(q+\theta)}(dz) \\
&\quad= \int_{[0,\infty)} f(x,z)\,W_{\Phi(q+\theta)}(dz) \\
&\qquad + \int_0^x W_{\Phi(q+\theta)}(z)(e^{\Phi(q+\theta)(z-x)} - e^{(\Phi(q+\theta)-\Phi(\theta))(z-x)})\,dz \\
&\quad= \int_{[0,\infty)} f(x,z)\,W_{\Phi(q+\theta)}(dz) \\
&\qquad + e^{-\Phi(q+\theta)x}\frac{Z^{(q+\theta)}(x)-1}{q+\theta} - e^{(\Phi(\theta)-\Phi(q+\theta))x}\frac{Z^{(q)}_{\Phi(\theta)}(x)-1}{q}.
\end{aligned}$$

We find that

$$\begin{aligned}
\mathbb{E}_x[e^{-q\sigma_\theta^+}] &= \frac{\theta}{q+\theta} + \mathbb{P}_x(A^-) \\
&= \frac{\theta}{q+\theta} + qe^{\Phi(q+\theta)x}\int_{[0,\infty)} f(x,z)\,W_{\Phi(q+\theta)}(dz) \\
&\quad + \frac{q}{q+\theta}\left(Z^{(q+\theta)}(x)-1\right) - e^{\Phi(\theta)x}\left(Z^{(q)}_{\Phi(\theta)}(x)-1\right) \\
&= \frac{\theta}{q+\theta} + \frac{q}{q+\theta} - \frac{q}{\Phi(q+\theta)-\Phi(\theta)}\frac{\Phi(\theta)-\Phi(q+\theta)}{\psi(\Phi(\theta))-(q+\theta)}e^{\Phi(\theta)x} \\
&\quad + \frac{q\Phi(\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))}W_{\Phi(q+\theta)}(\infty)e^{\Phi(q+\theta)x} \\
&\quad + \frac{q}{q+\theta}\left(Z^{(q+\theta)}(x)-1\right) - e^{\Phi(\theta)x}\left(Z^{(q)}_{\Phi(\theta)}(x)-1\right) \\
&= \frac{q}{q+\theta}Z^{(q+\theta)}(x) - e^{\Phi(\theta)x}Z^{(q)}_{\Phi(\theta)}(x) \\
&\quad + \frac{\theta}{\theta+q} + e^{\Phi(q+\theta)x}\frac{q\Phi(\theta)\Phi'(q+\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))},
\end{aligned} \tag{2.26}$$

where the third equality follows from an application of Lemma 2.2 and where the last equality is again a consequence of $W_{\Phi(q+\theta)}(\infty) = \Phi'(q+\theta)$. Note that (2.26) implies that

$$\mathbb{P}(A^-) = \frac{q\Phi(\theta)\Phi'(q+\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))} - \frac{\theta}{\theta+q} \tag{2.27}$$

Next, let $x<0$. We decompose $\{\mathbf{e}_q > \sigma_\theta^+\}$ as

$$\{\sigma_\theta^+ < \mathbf{e}_q\} \quad = \quad \{\tilde{\mathbf{e}}_\theta < \mathbf{e}_q\} \cup \{\tau_0^+ > \tilde{\mathbf{e}}_\theta, \tilde{\mathbf{e}}_\theta \geq \mathbf{e}_q\} \cup (\{\tau_0^+ < \tilde{\mathbf{e}}_\theta\} \cap A^-).$$

As before, we deduce from the strong Markov property, the memoryless property of the exponential distribution, (2.14) and (2.27) that

$$\begin{aligned}
\mathbb{E}_x[e^{-q\sigma_\theta^+}] &= \frac{\theta}{\theta+q} + \mathbb{P}_x(\tau_0^+ > \tilde{\mathbf{e}}_\theta > \mathbf{e}_q) + \mathbb{P}_x(\tau_0^+ < \tilde{\mathbf{e}}_\theta, A^-) \\
&= \frac{\theta}{\theta+q} + \mathbb{E}_x\left[\int_0^\infty \theta e^{-\theta y}\mathbf{1}_{\{\tau_0^+ > y > \mathbf{e}_q\}}\,dy\right] + \mathbb{E}_x\left[\mathbf{1}_{\{\tau_0^+ < \mathbf{e}_q \wedge \tilde{\mathbf{e}}_\theta\}}\mathbb{P}(A^-)\right] \\
&= \frac{\theta}{\theta+q} + \mathbb{E}_x[(e^{-\theta\mathbf{e}_q} - e^{-\theta\tau_0^+})\mathbf{1}_{\{\tau_0^+ > \mathbf{e}_q\}}] \\
&\quad + \left(\frac{q\Phi(\theta)\Phi'(q+\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))} - \frac{\theta}{\theta+q}\right)\mathbb{P}_x(\tau_0^+ < \mathbf{e}_q \wedge \tilde{\mathbf{e}}_\theta) \\
&= \frac{\theta}{\theta+q} + \mathbb{E}_x\left[\int_0^\infty q e^{-qt}(e^{-\theta t} - e^{-\theta\tau_0^+})\mathbf{1}_{\{\tau_0^+ > t\}}\,dt\right] \\
&\quad + \left(\frac{q\Phi(\theta)\Phi'(q+\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))} - \frac{\theta}{\theta+q}\right)\mathbb{E}_x[e^{-(q+\theta)\tau_0^+}] \\
&= 1 + \frac{\theta}{q+\theta}\mathbb{E}_x[e^{-(q+\theta)\tau_0^+}] - \mathbb{E}_x[e^{-\theta\tau_0^+}] \\
&\quad + \left(\frac{q\Phi(\theta)\Phi'(q+\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))} - \frac{\theta}{\theta+q}\right)e^{\Phi(q+\theta)x} \\
&= 1 - e^{\Phi(\theta)x} + \frac{q\Phi(\theta)\Phi'(q+\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))}e^{\Phi(q+\theta)x},
\end{aligned}$$

which is (2.17), since $Z_\nu^{(r)}(x) = 1$ for all $x \leq 0$ and $\nu, r \geq 0$. □

Proof of (2.18). We can write the event $\{T_\theta < \mathbf{e}_q\}$ as a disjoint union

$$\{\tilde{\mathbf{e}}_\theta < \mathbf{e}_q\} \cup A^- \cup A,$$

where

$$A^- = \{\tilde{\mathbf{e}}_\theta > \mathbf{e}_q, X_s < 0 \text{ for all } s \in [\mathbf{e}_q, \tilde{\mathbf{e}}_\theta]\}$$

and where

$$A = \{\tilde{\mathbf{e}}_\theta > \mathbf{e}_q, X_{\mathbf{e}_q} > 0, X_s \neq 0 \text{ for all } s \in [\mathbf{e}_q, \tilde{\mathbf{e}}_\theta]\}.$$

Since we already have an expression for $\mathbb{P}_x(A^-)$ we need only to consider A.

First, assume that $x \leq 0$. We use (2.8), (2.15), the lack of memory property of the exponential distribution and the Markov property in a similar fashion as we did to show (2.25) and deduce that

$$\begin{aligned}
\mathbb{P}_x(A) &= q\mathbb{E}_x\left[\int_0^\infty e^{-qt}\mathbf{1}_{\{\tilde{\mathbf{e}}_\theta \geq t\}}\mathbf{1}_{\{X_t>0, X_s\neq 0 \text{ for all } s\in[t,\tilde{\mathbf{e}}_\theta]\}}\,dt\right] \\
&= q\int_0^\infty e^{-(q+\theta)t}\int_0^\infty \mathbb{P}_x(X_t\in dy)\mathbb{P}_y(T(0)>\tilde{\mathbf{e}}_\theta)\,dt \\
&= q\int_0^\infty u^{(q+\theta)}(y-x)\mathbb{P}_y(T(0)>\tilde{\mathbf{e}}_\theta)\,dy \\
&= q\int_0^\infty \Phi'(q+\theta)e^{-\Phi(q+\theta)(y-x)}(1-e^{\Phi(\theta)y}+\psi'(\Phi(\theta))W^{(\theta)}(y))\,dy \\
&= q\Phi'(q+\theta)e^{\Phi(q+\theta)x}\left(\frac{1}{\Phi(q+\theta)}-\frac{1}{\Phi(q+\theta)-\Phi(\theta)}+\frac{1}{q\Phi'(\theta)}\right).
\end{aligned}$$

We find that

$$\begin{aligned}
\mathbb{E}_x[e^{-qT_\theta}] &= \mathbb{P}_x(A)+\mathbb{P}_x(A^-)+\frac{\theta}{\theta+q} \\
&= q\Phi'(q+\theta)e^{\Phi(q+\theta)x}\left(\frac{1}{\Phi(q+\theta)}-\frac{1}{\Phi(q+\theta)-\Phi(\theta)}+\frac{1}{q\Phi'(\theta)}\right) \\
&\quad +1-e^{\Phi(\theta)x}+e^{\Phi(q+\theta)x}\frac{q\Phi(\theta)\Phi'(q+\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))} \\
&= 1-e^{\Phi(\theta)x}+\frac{\Phi'(q+\theta)}{\Phi'(\theta)}e^{\Phi(q+\theta)x}.
\end{aligned}$$

Finally, let $x>0$. As before, we find

$$\{T_\theta<\mathbf{e}_q\} = \{\tilde{\mathbf{e}}_\theta<\mathbf{e}_q\}\cup\{T(0)>\tilde{\mathbf{e}}_\theta,\tilde{\mathbf{e}}_\theta\geq\mathbf{e}_q\}\cup\left(\{T(0)<\tilde{\mathbf{e}}_\theta\}\cap(A\cup A^-)\right).$$

An application of the strong Markov property at $T(0)$, the memoryless property of the exponential distribution, (2.12), (2.13) and (2.15) imply that

$$\begin{aligned}
\mathbb{E}_x[e^{-qT_\theta}] &= \frac{\theta}{\theta+q}+\mathbb{E}_x[(e^{-\theta\mathbf{e}_q}-e^{-\theta T(0)})\mathbf{1}_{\{T(0)>\mathbf{e}_q\}}] \\
&\quad +\mathbb{P}_x(T(0)<\mathbf{e}_q\wedge\tilde{\mathbf{e}}_\theta)\mathbb{P}(T_\theta<\mathbf{e}_q<\tilde{\mathbf{e}}_\theta) \\
&= 1+\frac{\theta}{\theta+q}\mathbb{E}_x[e^{-(q+\theta)T(0)}]-\mathbb{E}_x[e^{-\theta T(0)}] \\
&\quad +\mathbb{E}_x[e^{-(q+\theta)T(0)}]\left(\frac{\Phi'(q+\theta)}{\Phi'(\theta)}-\frac{\theta}{\theta+q}\right) \\
&= 1-e^{\Phi(\theta)x}+\frac{1}{\Phi'(\theta)}(W^{(\theta)}(x)-W^{(q+\theta)}(x))+\frac{\Phi'(q+\theta)}{\Phi'(\theta)}e^{\Phi(q+\theta)x},
\end{aligned}$$

which completes the proof of Theorem 2.5. □

Remark 2.9. Two of the main ingredients in the proof of Theorem 2.5 are the q-potential measure of X and the Laplace transform of the first passage time above or below a given level. These quantities are also known for certain Lévy processes which do have positive jumps. Proposition 2 in [4] indicates that results similar to (2.16) and (2.17) can be obtained for so-called phase-type Lévy processes. Similarly, as mentioned before in Corollary 2.8, we can use the scaling property to find the Laplace exponent of the last hitting time of zero for any stable process with index $\alpha > 1$. See Lemma VIII.13 in [18] for details.

Remark 2.10. As mentioned in the introduction, result (2.16) could be useful in risk theory, since it gives information about the last time when the risk process is negative before an independent, exponentially distributed time. Indeed, the last passage of X below zero before a fixed time horizon can be found by inverting the double Laplace transform in (2.16). Unfortunately, this seems to be tractable analytically only in very specific cases. An additional complication is the fact that only in few cases the scale functions are known explicitly. For example, the scale functions of a spectrally negative stable process with index $\alpha \in (1, 2]$ were found in [19] in terms of the so-called Mittag–Leffler functions. To be precise, in this case the scale function $Z^{(q)}(x)$ is given by

$$Z^{(q)}(x) = \sum_n \frac{q^n}{\Gamma(1+\alpha n)} x^{\alpha n} \quad \text{for } q, x \geq 0,$$

where Γ is the gamma function, see [71]. For a spectrally negative Lévy process of bounded variation with drift d it holds that (see [63])

$$W(dx) = \frac{1}{\mathrm{d}} \sum_n \nu^{*n}(dx) \quad x \geq 0, \tag{2.28}$$

where $\nu(dx) = \mathrm{d}^{-1}\Pi(-\infty, -x)dx$ and where ν^{*n} denotes the n-fold convolution of ν, with $\nu^{*0} = \delta_0(dx)$ the Dirac measure at zero. From (2.9) and (2.10) we can find a similar expression for the scale function $W^{(q)}(x)$. Indeed, we have that

$$W^{(q)}(x) = e^{\Phi(q)x} W_{\Phi(q)}(x).$$

Also, under $\mathbb{P}^{\Phi(q)}$ the process X is still of bounded variation and

$$\begin{aligned}
\psi_{\Phi(q)}(\lambda) &= \psi(\lambda + \Phi(q)) - q \\
&= \mathrm{d}(\lambda + \Phi(q)) + \int_{(-\infty,0)} (e^{(\lambda+\Phi(q))x} - 1)\,\Pi(dx) \\
&\quad - \mathrm{d}\Phi(q) - \int_{(-\infty,0)} (e^{\Phi(q)x} - 1)\,\Pi(dx) \\
&= \mathrm{d}\lambda + \int_{(-\infty,0)} (e^{\lambda x} - 1)e^{\Phi(q)x}\,\Pi(dx),
\end{aligned}$$

which shows that X has drift d and jump measure $e^{\Phi(q)x}\Pi(dx)$ under $\mathbb{P}^{\Phi(q)}$.
In the general case, scale functions can be evaluated numerically. We refer to [101] and [113] for such numerical schemes.

2.5 An extension of a result of Doney

In Corollary 3 in [39] it was shown that for a spectrally negative stable process $\{X_t\}_{t\geq 0}$ with index α, it holds that

$$\mathbb{P}(X_t = \overline{X}_t = t \text{ for some } 0 < t < \infty) = \frac{1}{\alpha},$$

where $\overline{X}_t$ is the running supremum of X, i.e. $\overline{X}_t = \sup_{0\leq s\leq t} X_s$. In this section we extend this result and, in particular, we find (for a general spectrally negative Lévy process) the Laplace exponent of the random time τ_1 defined by

$$\tau_1 := \sup\{t \geq 0 : X_t = \overline{X}_t = t\},$$

recalling the convention that $\sup\emptyset = 0$. Similarly, we define

$$\begin{aligned} \tau_2 &= \sup\{t \geq 0 : X_t = t\}, \\ \tau_3 &= \sup\{t \geq 0 : X_t \geq t\}, \\ \tau_4 &= \sup\{t \geq 0 : \overline{X}_t \geq t\}. \end{aligned}$$

Since

$$\{t \geq 0 : X_t = \overline{X}_t = t\} \subseteq \{t \geq 0 : X_t = t\} \subseteq \{t \geq 0 : X_t \geq t\} \subseteq \{t \geq 0 : \overline{X}_t \geq t\},$$

it holds that

$$\tau_1 \leq \tau_2 \leq \tau_3 \leq \tau_4.$$

These random times are trivial when X is of bounded variation with drift $\mathrm{d} \leq 1$ (since they are all equal to the first jump time when $\mathrm{d} = 1$ and all equal to zero when $\mathrm{d} < 1$) and hence we assume that $\mathrm{d} > 1$ whenever X is of bounded variation. Let $q > 0$. Since

$$\lim_{\lambda\to\infty} \frac{\psi(\lambda)}{\lambda} = \begin{cases} \infty & \text{when } X \text{ is of unbounded variation,} \\ \mathrm{d} & \text{when } X \text{ is of bounded variation with drift d,} \end{cases}$$

we see that there exists a unique $y_q > 0$ such that

$$\psi(y_q) = q + y_q.$$

Now let $z_q := \psi(y_q)$. Then $\Phi(z_q) = \Phi(\psi(y_q)) = y_q = \psi(y_q) - q = z_q - q$. Finally, define

$$y_0 := \begin{cases} 0 & \text{when } \psi'(0) \geq 1, \\ y & \text{when } \psi'(0) < 1, \end{cases}$$

where y is the unique solution on $(0,\infty)$ of $\psi(\lambda) = \lambda$ when $\psi'(0) < 1$. We can use Theorem 2.5 to establish the following result.

Corollary 2.11. *Let $q > 0$ and suppose X is a spectrally negative process which is of unbounded variation or of bounded variation with drift* $\mathrm{d} > 1$*. Then*

$$\mathbb{E}[e^{-q\tau_1}] = \frac{\psi'(y_q)}{\psi'(y_0)} \frac{\psi'(y_0) - 1}{\psi'(y_q) - 1}, \tag{2.29}$$

$$\mathbb{E}[e^{-q\tau_2}] = \frac{\psi'(y_0) - 1}{\psi'(y_q) - 1} \quad \textit{and} \tag{2.30}$$

$$\mathbb{E}[e^{-q\tau_3}] = \frac{qy_0}{y_q(y_q - y_0)(\psi'(y_q) - 1)}. \tag{2.31}$$

Finally,

$$\mathbb{E}[e^{-q\tau_4}] = \frac{qy_0\psi'(y_q)}{\psi(y_q)(\psi(y_q) - y_0)(\psi'(y_q) - 1)}. \tag{2.32}$$

Proof. First, suppose that X does not drift to $-\infty$. Introduce the processes $Y_t = X_t - t$ and $Z_t = t - \tau_t^+$, which are both spectrally negative Lévy processes. The assumption that X does not drift to $-\infty$ is used here to ensure that $\mathbb{P}(\tau_t^+ < \infty) = 1$. Note that, since

$$\{t \geq 0 : X_t = \overline{X}_t = t\} = \{t \geq 0 : \tau_t^+ = t\},$$

it holds that τ_1 and τ_2 are, respectively, the last hitting times T of zero for Z and Y, and that τ_3 and τ_4 are the last passage times above 0 of Y and Z, respectively. Using obvious notation, it holds that $\psi^Y(\lambda) = \psi(\lambda) - \lambda$ and $\psi^Z(\lambda) = \lambda - \Phi(\lambda)$ and thus

$$\Phi^Y(q) = y_q \quad \text{and } \Phi^Z(q) = z_q.$$

From the implicit function theorem we find that

$$\frac{d}{dq} y_q = \frac{1}{\psi'(y_q) - 1}$$

and that

$$\frac{d}{dq} z_q = \frac{1}{1 - \Phi'(z_q)} = \frac{\psi'(y_q)}{\psi'(y_q) - 1}.$$

The result now follows from Theorem 2.5 by taking $\theta = 0$.

When X does drift to $-\infty$, equations (2.30) and (2.31) still hold, but in this case τ_t^+ is a subordinator killed at exponential rate $\Phi(0)$, which is strictly positive as $\psi'(0) < 0$. Hence, we are now looking for the last passage times before $\mathbf{e}_{\Phi(0)}$ of a Lévy process with Laplace exponent given by $\lambda - \Phi(\lambda) + \Phi(0)$. Statements (2.29) and (2.32) thus follow from an application of Theorem 2.5 with $\theta = \Phi(0)$. □

Define for $s \geq 0$

$$A_s := \{\text{there exists some } t > s : X_t = \overline{X}_t = t\}$$

and denote $A = A_0$. In [39] it was shown that for a spectrally negative stable process of index α, $\mathbb{P}(A) = 1/\alpha$. We show how such a result can be obtained from Corollary 2.11.

Corollary 2.12. *For a spectrally negative Lévy process we have*

$$\mathbb{P}(A) = \begin{cases} \frac{1}{\psi'(y_0)} & \textit{when} \quad X \textit{ is of unbounded variation,} \\ \frac{\mathrm{d}-\psi'(y_0)}{\mathrm{d}\psi'(y_0)-\psi'(y_0)} & \textit{when} \quad X \textit{ is of bounded variation with drift } \mathrm{d} > 1. \end{cases}$$

In particular, $\mathbb{P}(A) = \frac{1}{\alpha}$ *for a spectrally negative stable process of index* $\alpha \in (1, 2]$. *Also*

$$\mathbb{P}(A) = 1 \Leftrightarrow \psi'(0) = 1.$$

In fact, when $\psi'(0) = 1$,

$$\mathbb{P}(A_s) = 1$$

for all $s \geq 0$.

Proof. Since

$$\lim_{\lambda \to \infty} \psi'(\lambda) = \begin{cases} \infty & \text{when } X \text{ is of unbounded variation,} \\ \mathrm{d} & \text{when } X \text{ is of bounded variation with drift d,} \end{cases}$$

it follows from Corollary 2.11 that

$$\begin{aligned} \mathbb{P}(A) &= 1 - \mathbb{P}(\tau_1 = 0) \\ &= 1 - \lim_{q \to \infty} \mathbb{E}[e^{-q\tau_1}] \\ &= 1 - \lim_{q \to \infty} \frac{\psi'(y_q)}{\psi'(y_0)} \frac{\psi'(y_0) - 1}{\psi'(y_q) - 1} \\ &= \begin{cases} \frac{1}{\psi'(y_0)} & \text{when} \quad X \text{ is of unbounded variation,} \\ \frac{\mathrm{d}-\psi'(y_0)}{\mathrm{d}\psi'(y_0)-\psi'(y_0)} & \text{when} \quad X \text{ is of bounded variation with drift } \mathrm{d} > 1. \end{cases} \end{aligned}$$

When X is a stable process of index α, we have $y_0 = 1$ and thus

$$\mathbb{P}(A) = \frac{1}{\psi'(1)} = 1/\alpha.$$

We also see that $\psi'(0) = 1$ implies $y_0 = 1$ and hence $\mathbb{P}(A) = 1$.

For the other direction we remark that when $\psi'(0) > 1$ it holds that $\psi'(y_0) = \psi'(0) > 1$ and when $\psi'(0) < 1$ we have that $\psi'(y_0) > 1$, because in the latter case y_0 is the unique solution to $\psi(y) = y$ on $(0, \infty)$ and because ψ is a strictly convex function on $[0, \infty)$. We conclude that whenever $\psi'(0) \neq 1$ we have that $\psi'(y_0) > 1$ from which it follows that $\mathbb{P}(A) < 1$.
From (2.29) we see that $\psi'(0) = 1$ implies that $\mathbb{E}[e^{-q\tau_1}] = 0$ for any $q > 0$. The final statement in Corollary 2.12 now follows. □

Remark 2.13. As an example, we consider a standard Brownian motion. Of course, by continuity, $\tau_2 = \tau_3$. We have $\psi(\lambda) = \lambda^2/2$, so $\psi_Y(\lambda) = \lambda^2/2 - \lambda$ and $\psi_Z(\lambda) = \lambda - \sqrt{2\lambda}$, hence

$$y_q = 1 + \sqrt{2q+1} \quad \text{and } z_q = 1 + q + \sqrt{2q+1}.$$

From Corollary 2.11 we readily deduce that

$$\begin{aligned}
\mathbb{E}[e^{-q\tau_1}] &= \frac{a_q+1}{2a_q}, \\
\mathbb{E}[e^{-q\tau_2}] &= E[e^{-q\tau_3}] = \frac{1}{a_q}, \\
\mathbb{E}[e^{-q\tau_4}] &= \frac{2a_q+2}{(q+2)a_q+4q+2},
\end{aligned}$$

where $a_q = \sqrt{2q+1}$.

Appendix: proof of Lemma 2.2

Here, we prove Lemma 2.2.

Proof. Suppose $q > 0$. First, let $\lambda > 0$. Then (2.11) follows by integration by parts. Indeed, in this case

$$\begin{aligned}
\int_{[0,\infty)} e^{-\lambda x} W_{\Phi(q)}(dx) &= \lambda \int_0^\infty e^{-\lambda x} W_{\Phi(q)}(x)dx \\
&= \frac{\lambda}{\psi_{\Phi(q)}(\lambda)} \\
&= \frac{\lambda}{\psi(\Phi(q)+\lambda)-q}. \qquad (2.33)
\end{aligned}$$

Under $\mathbb{P}_{\Phi(q)}$, the process $\{X_t\}_{t\geq 0}$ drifts to ∞ and now from equation (8.15) in [63] we deduce that

$$W_{\Phi(q)}(x) = \frac{1}{\psi'_{\Phi(q)}(0+)} \mathbb{P}_x(\inf_{t\geq 0} X_t \geq 0).$$

It follows that

$$\lim_{x\to\infty} W_{\Phi(q)}(x) = \Phi'(q) \qquad (2.34)$$

and hence (2.11) follows in this case as well.

Next, we show that (2.11) holds for $\lambda = -\Phi(q)$. We make use of an expression for the resolvent measure for the reflected process $\{Y_t\}_{t\geq 0}$ defined by

$$Y_t = \sup_{0\leq s\leq t} (X_s \vee 0) - X_s.$$

In Theorem 1 (ii) in [94], the resolvent measure

$$R_a^q(x,dy) = \int_0^\infty e^{-qt}\, \mathbb{P}_x(Y_t \in dy, \sup_{0\leq s\leq t} Y_s \leq a)$$

of Y killed at exceeding a certain level $a > 0$ was found. In particular, for $x = 0$ it holds that

$$R_a^q(0,dy) = \left(W^{(q)}(a) \frac{W^{(q)\prime}(y+)}{W^{(q)\prime}(a+)} - W^{(q)}(y) \right) dy \quad \text{for } y \in (0,a]$$

and $R_a^q(0,\{0\}) = W^{(q)}(a)W^{(q)}(0)/W^{(q)\prime}(a+)$. Using the fact that $W_{\Phi(q)}(\infty) < \infty$ and (2.10), we can take $a \to \infty$ and deduce that the resolvent measure

$$R^q(0,dy) = \int_0^\infty e^{-qt}\,\mathbb{P}_0(Y_t \in dy)$$

of the unkilled reflected process is given by

$$R^q(0,dy) = \left(\frac{1}{\Phi(q)}W^{(q)'}(y+) - W^{(q)}(y)\right)dy = \frac{1}{\Phi(q)}e^{\Phi(q)y}W_{\Phi(q)}(dy) \quad \text{for } y \geq 0.$$

We can use Fubini's theorem

$$\begin{aligned}
\int_{[0,\infty)} e^{\Phi(q)x}\,W_{\Phi(q)}(dx) &= \Phi(q)\int_{[0,\infty)} R^{(q)}(0,dx) \\
&= \Phi(q)\int_{[0,\infty)}\int_0^\infty e^{-qt}\,\mathbb{P}(Y_t \in dx)\,dt \\
&= \Phi(q)\int_0^\infty e^{-qt}\mathbb{P}(Y_t \in [0,\infty))\,dt \\
&= \frac{\Phi(q)}{q}, \qquad (2.35)
\end{aligned}$$

which is (2.11) for $\lambda = -\Phi(q)$.

Finally, for the case $-\Phi(q) < \lambda < 0$ we make use of analytic extension. We can extend the function ψ to those $z \in \mathbb{C}$ for which $\Re(z) > 0$ and we denote this extension by Ψ. Define the function $g : A \to \mathbb{C}$ by

$$g(z) = \begin{cases} \frac{z}{\Psi(z+\Phi(q))-q} & \text{when } z \neq 0 \text{ and } \Re(z) > -\Phi(q), \\ \Phi'(q) & \text{when } z = 0, \end{cases}$$

where A is an open set in $\mathbb{C}$ containing $\{z \in \mathbb{C} : \Re(z) \in (-\Phi(q),\infty), \Im(z) = 0\}$ such that $\Psi(z+\Phi(q)) \neq q$ on $A\backslash\{0\}$. We prove that g is analytic on A.
Since the Laplace exponent Ψ is analytic when $\Re(z) > 0$, we can write

$$\Psi(z+\Phi(q)) = q + \sum_{k=1}^{\infty}\frac{z^k}{k!}\Psi^{(k)}(\Phi(q)) \quad \text{when } \Re(z) > -\Phi(q),$$

where $\Psi^{(k)}$ denotes the kth derivative of Ψ. The fact that $\psi'(\Phi(q)) > 0$ implies that Ψ is bounded in some (complex) neighbourhood of 0, and we can use the Riemann removable singularity theorem to deduce that $g(\lambda)$ is real analytic for $\lambda > -\Phi(q)$. The coefficients c_n in the power series of g are given in terms of the nth (right) derivative at zero of the left hand side of (2.33). Specifically, because of (2.35),

$$c_n = \int_{[0,\infty)} \frac{(-x)^n}{n!}\,W_{\Phi(q)}(dx) \quad \text{for } n \in \mathbb{N}.$$

In particular, for $\lambda \in (-\Phi(q),0)$

$$g(\lambda) = \sum_{n=0}^{\infty}\lambda^n\int_{[0,\infty)}\frac{(-x)^n}{n!}\,W_{\Phi(q)}(dx) = \sum_{n=0}^{\infty}\int_{[0,\infty)}\frac{|\lambda x|^n}{n!}\,W_{\Phi(q)}(dx),$$

which allows us to use Fubini's theorem to find for any $|\lambda| < \Phi(q)$ that

$$\begin{aligned} g(\lambda) &= \sum_{n=0}^{\infty} c_n \lambda^n \\ &= \sum_{n=0}^{\infty} \lambda^n \int_{[0,\infty)} \frac{(-x)^n}{n!} W_{\Phi(q)}(dx) \\ &= \int_{[0,\infty)} \sum_{n=0}^{\infty} \frac{(-\lambda x)^n}{n!} W_{\Phi(q)}(dx) \\ &= \int_{[0,\infty)} e^{-\lambda x} W_{\Phi(q)}(dx). \end{aligned}$$

This completes the proof of Lemma 2.2. $\square$

Chapter 3

Some excursion calculations for reflected Lévy processes*

Using methods analogous to those introduced in [40], we express the resolvent density of a (killed) reflected Lévy process in terms of the resolvent density of the (killed) Lévy process. As an application we find a previously unknown identity for the potential density for killed reflected symmetric stable processes.

3.1 Introduction

Lévy processes reflected at their maximum or at their minimum appear in a wide variety of applications, such as the study of the water level in a dam, queueing (see [3; 25; 97]), optimal stopping ([15; 108]) and optimal control ([6; 38; 52; 67]). For example, in [108] it was shown that finding the value of the so-called Russian option (on a Brownian motion B) is equivalent to solving an optimal stopping problem of the form

$$\sup_{\tau} \mathbb{E}[e^{-\alpha\tau+Y_\tau^x}], \tag{3.1}$$

where α is some constant, where Y is the process B reflected at its infimum and where the supremum is taken over all stopping times with respect to the filtration generated by B. Recently, there have been various studies on (3.1) with the Brownian motion B replaced by a more general Lévy process X, see for example [4; 5] and also [15] for a two-player version of (3.1). It was found that for a broad class of Lévy processes, an optimal stopping time τ^* in (3.1) is given by the first time the reflected Lévy process exceeds a certain level, i.e.

$$\tau^* = \inf\{t \geq 0 : Y_t \geq b\}, \tag{3.2}$$

*This chapter is an extended version of [12], which has been accepted for publication in *ALEA Lat. Am. J. Probab. Math. Stat.*

for a specific choice of b and where Y now is the process X reflected at its infimum. A similar strategy was proved to be optimal (under some conditions) for the optimal control problem considered for a Lévy process without positive jumps in [38]. Hence, a further understanding of reflected Lévy processes killed at exceeding a certain level could be helpful for the study of certain optimal stopping and optimal control problems.

In Theorem 3.5 we express the resolvent density of a (killed) reflected Lévy process in terms of the resolvent density of the (killed) Lévy process. The proof of Corollary 3.10 indicates that the compensation formula allows us to find the joint law of the undershoot and the overshoot of Y in terms of the resolvent density of Y and the jump measure of the Lévy process, which, in turn, gives us information about the expressions involving the first passage time (3.2). As an application of Theorem 3.5 we find the potential density of a killed, reflected symmetric stable process.

In [40] Doney introduced a new method based on excursion theory to find an expression for the resolvent density for reflected spectrally negative Lévy processes killed at exceeding a certain level. Previously, this density had been obtained in [94] using excursion theory, Itô calculus and martingale techniques (see also [82]). We extend the method introduced in [40] to general reflected Lévy processes. As a new result and an application of Theorem 3.5, we find in Section 3.5 the potential density for the killed reflected symmetric process. Possibly, the result in the symmetric stable case could lead to proving similar results for a broader class of reflected Lévy processes.

3.2 Preliminaries

Let $X = \{X_t\}_{t\geq 0}$ be a Lévy process, starting at 0, with respect to some probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. To avoid trivialities, we exclude the case when X has monotone paths and the case when X is a compound Poisson process. We refer to the books [18] and [63] for a detailed description of Lévy processes. We denote by $\mathbb{P}_x$ the law of the Lévy process starting at x. Define the process $Y = \{Y_t\}_{t\geq 0}$ by

$$Y_t = X_t - \underline{X}_t,$$

where $\underline{X}_t = \inf_{0\leq s\leq t} X_s \wedge 0$. Denote by $L(t)$ a local time of Y at zero (note that the definition of L depends on the nature of the zero set of Y) and let n be the measure of excursions of Y away from zero, defined on the excursion space $\mathcal{E}$ (see chapter 4 of [18]). Since a positive multiple of a local time is again a local time, most expressions concerning local time also involve a multiplicative constant. However, this constant does not play a role in the results in this chapter and henceforth we omit it. Define the inverse local time of Y by

$$L^{-1}(t) = \begin{cases} \inf\{s > 0 : L(s) > t\} & \text{when } t < L(\infty), \\ \infty & \text{otherwise.} \end{cases}$$

Furthermore, denote by $H = \{H_t\}_{t\geq 0}$ the downward ladder height process of X, i.e. $H_t = X_{L^{-1}(t)}$ when $0 \leq t < L(\infty)$ and $H_t = -\infty$ otherwise. We denote the exit times for excursions ε by

$$\rho_a = \inf\{t \geq 0 : \varepsilon(t) \geq a\}$$

and by ζ the length of an excursion. The renewal function $h : [0,\infty) \to [0,\infty)$ of H is defined by

$$h(x) = \int_0^\infty \mathbb{P}(H_t \geq -x)\, dt = \mathbb{E}\left[\int_0^\infty \mathbf{1}_{\{\underline{X}_t \geq -x\}}\, dL(t)\right].$$

Denote the first passage times for X by

$$\tau_b^- = \inf\{t \geq 0 : X_t \leq b\} \quad \text{and} \quad \tau_a^+ = \inf\{t \geq 0 : X_t \geq a\}$$

and by

$$T_a^+ = \inf\{t \geq 0 : Y_t \geq a\}$$

the first passage time of Y. For $q > 0$, let $\mathbf{e}_q$ be an exponentially distributed random variable with parameter q, independent of X. The function h can also be expressed in terms of the excursion measure as

$$h(x) = \lim_{q \downarrow 0} \frac{\mathbb{P}_x(\tau_0^- > \mathbf{e}_q)}{\eta q + n(\mathbf{e}_q < \zeta)},$$

where $\eta \geq 0$ is the drift of $L^{-1}(t)$. This is a consequence of the following result which we will use later.

Lemma 3.1. *Let $q > 0$. Then*

$$\mathbb{P}_x(\mathbf{e}_q < \tau_0^-) = \mathbb{E}\left[\int_{[0,\infty)} e^{-qt} \mathbf{1}_{\{\underline{X}_t \geq -x\}}\, dL(t)\right](\eta q + n(\mathbf{e}_q < \zeta)) \tag{3.3}$$

Proof. Note that

$$\mathbb{P}_x(\tau_0^- > \mathbf{e}_q) = \mathbb{E}\left[\int_0^\infty q e^{-qt} \mathbf{1}_{\{\underline{X}_t \geq -x\}}\, dt\right].$$

By distinguishing between those times t for which $X_t = \underline{X}_t$ and those which lie in an excursion interval of the process $\{X_s - \underline{X}_s\}_{s \geq 0}$ we find that

$$\mathbb{P}_x(\tau_0^- > \mathbf{e}_q) = \mathbb{E}\left[\int_0^\infty q e^{-qt} \mathbf{1}_{\{\underline{X}_t \geq -x,\, \underline{X}_t = X_t\}}\, dt\right] + \mathbb{E}\left[\sum_g \mathbf{1}_{\{\underline{X}_g \geq -x\}} \int_g^d q e^{-qt}\, dt\right], \tag{3.4}$$

where the sum is taken over all left end points g of excursion intervals (g,d). Since $\mathbf{1}_{\{\underline{X}_t = X_t\}} dt = \eta dL(t)$ (see Theorem 6.8 in [63]), the first term on the right hand side of (3.4) is equal to

$$\eta q \mathbb{E}\left[\int_{[0,\infty)} e^{-qt} \mathbf{1}_{\{\underline{X}_t \geq -x\}}\, dL(t)\right].$$

From an application of the compensation formula it follows that the second term on the right hand side of (3.4) is equal to

$$\mathbb{E}\left[\sum_g e^{-qg}\mathbf{1}_{\{\underline{X}_g\geq -x\}}\mathbf{1}_{\{\mathbf{e}_q<d-g\}}\right] = n(\mathbf{e}_q<\zeta)\mathbb{E}\left[\int_{[0,\infty)} e^{-qt}\mathbf{1}_{\{\underline{X}_t\geq -x\}}\,dL(t)\right],$$

which completes the proof. □

We say that X drifts to ∞ ($-\infty$) when $\lim_{t\to\infty} X_t = \infty$ ($-\infty$). Whenever there exists some $\nu > 0$ (which is then called the Lundberg exponent of X) such that

$$\mathbb{E}[e^{\nu X_1}] = 1,$$

we can then define the Laplace exponent ψ of X by

$$\psi(\lambda) = \log(\mathbb{E}[e^{\lambda X_1}]), \quad \lambda \in [0,\nu].$$

The function ψ is strictly convex on $[0,\nu]$ and $\psi(0) = \psi(\nu) = 0$, so we find that $\psi'(0+) < 0$, which implies that X drifts to $-\infty$. Furthermore, we can change measure by defining

$$\left.\frac{d\mathbb{P}^\nu}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\nu X_t}.$$

Trivially, the Laplace exponent ψ_ν of X under $\mathbb{P}^\nu$ is given by

$$\psi_\nu(\lambda) = \log(\mathbb{E}^\nu[e^{\lambda X_1}]) = \psi(\lambda+\nu) \quad \text{for } \lambda \in [-\nu, 0].$$

In particular, $\psi_\nu'(0-) = \psi'(\nu-) > 0$ and thus X drifts to $+\infty$ under $\mathbb{P}^\nu$.

3.3 Excursion measure in terms of renewal function

In this section we show that for a large class of Lévy processes, the excursion measure n can be expressed in terms of the renewal function h. We make use of various results obtained in [35] concerning Lévy processes conditioned to stay positive. The Lévy processes we consider are given in the following definition.

Definition 3.2. *Let $\mathcal{H}$ be the class of those Lévy processes X such that X is not a compound Poisson process and X does not have monotone paths, and X has a Lundberg exponent if it drifts to $-\infty$.*

Remark 3.3. For future reference we remark that $\mathcal{H}$ contains any Lévy process for which its Lévy measure has support bounded from above. Indeed, when the support of the Lévy measure of X is bounded from above we know (e.g. Theorem 25.3 in [104]) that the Laplace exponent $\psi(\lambda)$ is finite for $\lambda \geq 0$. Furthermore, it is not difficult to check that ψ is strictly convex and that $\lim_{\lambda\to\infty}\psi(\lambda) = \infty$. When X drifts to $-\infty$ it holds that $\psi'(0) < 0$ and thus the Lundberg exponent exists.

The following result indicates how, for processes in $\mathcal{H}$, the excursion measure n is related to the renewal function h.

Lemma 3.4. *Let $X \in \mathcal{H}$ and let A be a Borel subset of $\mathbb{R}_+$ satisfying* $\inf A > 0$ *and $n(\partial A) = 0$ (here, ∂A denotes the boundary of A with respect to the Skorokhod topology). We then have for $q \geq 0$*

$$\int_0^\infty e^{-qt} n(\varepsilon(t) \in A, t < \zeta \wedge \rho_a)\, dt = \lim_{z \downarrow 0} \frac{\int_0^\infty e^{-qt} \mathbb{P}_z(X_t \in A, t < \tau_a^+ \wedge \tau_0^-)\, dt}{h(z)}. \tag{3.5}$$

Furthermore,

$$n(\rho_a \leq \zeta \wedge \mathbf{e}_q) = \lim_{x \downarrow 0} \frac{1}{h(x)} \mathbb{P}_x(\tau_a^+ < \tau_0^- \wedge \mathbf{e}_q), \tag{3.6}$$

and, when $q > 0$,

$$n(\mathbf{e}_q < \rho_a \wedge \zeta) = \lim_{z \downarrow 0} \frac{\mathbb{P}_z(\mathbf{e}_q < \tau_a^+ \wedge \tau_0^-)}{h(z)}. \tag{3.7}$$

Proof. Let $X \in \mathcal{H}$ and suppose for the moment that X does not drift to $-\infty$. According to Lemma 1 in [35] we can then introduce the family of probability measures by

$$\mathbb{P}_x^\uparrow(X_t \in dy) = \frac{h(y)}{h(x)} \mathbb{P}_x(X_t \in dy, t < \tau_0^-) \quad \text{for } x, y > 0.$$

Proposition 1 in [35] provides the justification for calling $\mathbb{P}_x^\uparrow$ the law of X conditioned to stay positive. When X is regular upwards we know from Theorem 2 in [35] that the laws $\mathbb{P}_x^\uparrow$ converge in the Skorokhod topology as $x \downarrow 0$ to a probability measure denoted by $\mathbb{P}^\uparrow$. For the case when X is irregular upwards, it can be checked that, at time zero, the process with law $\mathbb{P}_x^\uparrow$ does not converge in probability towards zero. However, Theorem 2 in [35] states that in this case $(X \circ \theta_\delta, \mathbb{P}_x^\uparrow)$ converges in probability to $(X \circ \theta_\delta, \mathbb{P}^\uparrow)$ for any $\delta > 0$, where θ denotes the shift operator and $\mathbb{P}^\uparrow$ is some probability measure.

In [34] Theorem 3, under the assumptions that 0 is regular downwards for X, X does not drift to $-\infty$ and its semigroup is absolutely continuous, it is shown that this measure is related to the excursion measure n in the following way:

$$n(B, t < \zeta) = \mathbb{E}^\uparrow[(h(X_t))^{-1} \mathbf{1}_B] \quad \text{for any } B \in \mathcal{F}_t \text{ such that } n(\partial B) = 0, \tag{3.8}$$

where ∂B denotes the boundary of B with respect to the Skorokhod topology. However, from Theorem 1 in [35] it follows that (3.8) still holds whenever X is not a Poisson process. By Fubini's theorem and (3.8) we then find for any Borel subset A of $\mathbb{R}_+$ satisfying $\inf A > 0$ that

$$n\left(\int_0^{\zeta \wedge \rho_a} e^{-qt} \mathbf{1}_{\{\varepsilon_t \in A\}}\, dt\right) = \mathbb{E}^\uparrow\left[\int_0^{\tau_a^+} e^{-qt} \mathbf{1}_{\{X_t \in A\}} (h(X_t))^{-1}\, dt\right].$$

Let $T \in \mathbb{R}_+$ We show that for any bounded and continuous F, it holds that $\omega \to \int_0^T F(\omega_t)\, dt$ is a continuous functional of ω in the set D of paths which are right-continuous and have a left limit. Denote by d a metric which induces the Skorokhod topology and let $\omega^n \in D$ and $\omega \in D$ be such that $d(\omega^n, \omega) \to 0$ as $n \to \infty$. Define the countable set

$$C := \cup_{n \in \mathbb{N}} \{t : \omega^n_t \neq \omega^n_{t-}\} \cup \{t : \omega_t \neq \omega_{t-}\}.$$

Since Skorokhod convergence implies pointwise convergence at points of continuity we can use bounded convergence to deduce that

$$\int_0^T (F(\omega^n_t) - F(\omega_t))\, dt \quad = \quad \int_0^T (F(\omega^n_t) - F(\omega_t)) \mathbf{1}_{\{C^c\}}\, dt \to 0$$

as $n \to \infty$. Since h is an increasing function and since $\inf A > 0$, we deduce by a monotone class argument and (3.8) that

$$\begin{aligned} n\left(\int_0^{\zeta \wedge \rho_a} e^{-qt} \mathbf{1}_{\{\varepsilon_t \in A\}}\, dt\right) &= \lim_{x \downarrow 0} \mathbb{E}^{\uparrow}_x \left[\int_0^\infty e^{-qt} \mathbf{1}_{\{X_t \in A, t < \tau_a^+\}} (h(X_t))^{-1}\, dt\right] \\ &= \lim_{x \downarrow 0} \frac{1}{h(x)} \mathbb{E}_x \left[\int_0^\infty e^{-qt} \mathbf{1}_{\{X_t \in A, t < \tau_a^+ \wedge \tau_0^-\}}\, dt\right], \end{aligned}$$

which is (3.5). The proof of (3.6) is similar, since $h(X_{\tau_a^+})$ is bounded away from zero. Next, we show (3.7) (still under the assumption that X drifts to $+\infty$). Let $q > 0$ and suppose X is regular upwards. Similar to the reasoning above, we can show that for any $\delta > 0$ it holds that

$$\lim_{x \downarrow 0} \mathbb{E}^{\uparrow}_x \left[\int_0^\infty e^{-qt} (h(X_t))^{-1} \mathbf{1}_{\{X_t > \delta\}}\, dt\right] = \mathbb{E}^{\uparrow} \left[\int_0^\infty e^{-qt} (h(X_t))^{-1} \mathbf{1}_{\{X_t > \delta\}}\, dt\right]. \tag{3.9}$$

Also, as $q > 0$, it follows from the definition of h that

$$\mathbb{E}\left[\int_0^\infty e^{-qt} \mathbf{1}_{\{\underline{X}_t \geq -x\}}\, dL(t)\right] \leq h(x).$$

Since X is regular upwards, the drift η of $L^{-1}(t)$ is equal to zero and thus we deduce from (3.3) that for $\delta, x > 0$

$$\frac{1}{h(x)} \mathbb{E}_x \left[\int_0^{\tau_0^-} e^{-qt} \mathbf{1}_{\{X_t \leq \delta\}}\, dt\right] + \frac{1}{h(x)} \mathbb{E}_x \left[\int_0^{\tau_0^-} e^{-qt} \mathbf{1}_{\{X_t > \delta\}}\, dt\right] \tag{3.10}$$

$$\begin{aligned} &= \frac{1}{h(x)} \mathbb{E}_x \left[\int_0^{\tau_0^-} e^{-qt}\, dt\right] \\ &= \frac{1}{h(x)} \mathbb{E}\left[\int_{[0,\infty)} e^{-qt} \mathbf{1}_{\{\underline{X}_t \geq -x\}}\, dL(t)\right] n\left(\int_0^\zeta e^{-qt}\, dt\right) \\ &\leq n\left(\int_0^\zeta e^{-qt}\, dt\right) \\ &= n\left(\int_0^\zeta e^{-qt} \mathbf{1}_{\{\varepsilon(t) \leq \delta\}}\, dt\right) + n\left(\int_0^\zeta e^{-qt} \mathbf{1}_{\{\varepsilon(t) > \delta\}}\, dt\right). \end{aligned} \tag{3.11}$$

It follows from (3.9) that the second term in (3.10) converges to the second term in (3.11) as $x \downarrow 0$. The equation

$$n\left(\int_0^\zeta \mathbf{1}_{\{\varepsilon(t)=0\}}\,dt\right) = 0$$

implies that for any $\xi > 0$ there exists some $\delta_0 > 0$ such that for $0 < \delta < \delta_0$

$$\limsup_{x\downarrow 0} \frac{1}{h(x)}\mathbb{E}_x\left[\int_0^{\tau_0^-} e^{-qt}\mathbf{1}_{\{X_t\le\delta\}}\,dt\right] \le n\left(\int_0^\zeta e^{-qt}\mathbf{1}_{\{\varepsilon(t)\le\delta\}}\,dt\right) \le \xi.$$

It now readily follows that

$$\begin{aligned}\lim_{x\downarrow 0}\frac{1}{h(x)}\mathbb{E}_x\left[\int_0^{\tau_a^+\wedge\tau_0^-} e^{-qt}\,dt\right] &= \lim_{\delta\downarrow 0}\lim_{x\downarrow 0}\frac{1}{h(x)}\mathbb{E}_x\left[\int_0^{\tau_a^+\wedge\tau_0^-} e^{-qt}\mathbf{1}_{\{X_t>\delta\}}\,dt\right]\\ &= \lim_{\delta\downarrow 0} n\left(\int_0^{\rho_a\wedge\zeta} e^{-qt}\mathbf{1}_{\{\varepsilon(t)>\delta\}}\,dt\right)\\ &= n\left(\int_0^{\rho_a\wedge\zeta} e^{-qt}\,dt\right),\end{aligned}$$

which is (3.7).

When X is irregular upwards (and still does not drift to $-\infty$), the drift η of $L^{-1}(t)$ is strictly positive. We now deduce from (3.3) and reasoning similar to the above that for $0 < r \le q$ and $\delta > 0$

$$\begin{aligned}\limsup_{x\downarrow 0}\frac{1}{h(x)}\mathbb{E}_x\left[\int_0^{\tau_0^-} e^{-qt}\mathbf{1}_{\{X_t\le\delta\}}\,dt\right] &\le \limsup_{x\downarrow 0}\frac{1}{h(x)}\mathbb{E}_x\left[\int_0^{\tau_0^-} e^{-rt}\mathbf{1}_{\{X_t\le\delta\}}\,dt\right]\\ &\le n\left(\int_0^{\zeta} e^{-rt}\mathbf{1}_{\{\varepsilon(t)\le\delta\}}\,dt\right) + r\eta,\end{aligned}$$

which can be made arbitrarily small by taking r and δ close to zero. The proof of (3.7) now follows similarly to the regular case above.

Finally, let X be a process in $\mathcal{H}$ which drifts to $-\infty$. Since $X \in \mathcal{H}$, its Lundberg exponent exists and we denote it by ν. We denote by n^ν the excursion measure of $X_t - \underline{X}_t$ under $\mathbb{P}^\nu$. Then we claim that the excursion measure n^ν can be expressed in terms of n by

$$n^\nu(\varepsilon(t)\in dy, t<\zeta) = e^{\nu y} n(\varepsilon(t)\in dy, t<\zeta). \tag{3.12}$$

In order to prove (3.12), we show that the left hand side and the right hand side of (3.12) have the same double Laplace transform in t and y. Denote by κ ($\hat\kappa$) the Laplace exponent of the downward (upward) ladder height. We use the obvious notation κ_ν and $\hat\kappa_\nu$ for the analogous objects under $\mathbb{P}^\nu$. Similar to $\underline{X}$, we denote the running supremum of X by $\overline{X}_t = \sup_{0\leq s\leq t} X_s$. Then, for any $q, \lambda \geq 0$

$$\begin{aligned} n^\nu\left(\int_0^\zeta \int_0^\infty e^{-qt-(\nu+\lambda)y}\mathbf{1}_{\{\varepsilon(t)\in dy\}}\,dt\right) &= \frac{\kappa_\nu(q,0)}{q}\mathbb{E}^\nu[e^{-(\nu+\lambda)(X_{\mathbf{e}_q}-\underline{X}_{\mathbf{e}_q})}] \\ &= \frac{\kappa_\nu(q,0)}{q}\mathbb{E}^\nu[e^{-(\nu+\lambda)\overline{X}_{\mathbf{e}_q}}] \\ &= \frac{\kappa_\nu(q,0)}{q}\frac{\hat\kappa_\nu(q,0)}{\hat\kappa_\nu(q,\nu+\lambda)} \\ &= c\frac{1}{\hat\kappa_\nu(q,\nu+\lambda)} \end{aligned}$$

for some constant $c > 0$. Here, we have used equation (7) on p. 120 in [18] for the first equality, the duality principle (which implies that $X_{\mathbf{e}_q} - \underline{X}_{\mathbf{e}_q}$ has the same distribution as $\overline{X}_{\mathbf{e}_q}$) for the second equality and the Wiener–Hopf factorisation for the third and fourth equalities. We also have that (with $(\hat L_t, \hat H_t)$ the upward ladder process)

$$\begin{aligned} \hat\kappa_\nu(q,\nu+\lambda) &= -\log\left(\mathbb{E}^\nu[e^{-q\hat L_1^{-1}-(\nu+\lambda)\hat H_1}\mathbf{1}_{\{\hat L(\infty)>1\}}]\right) \\ &= -\log\left(\mathbb{E}[e^{-q\hat L_1^{-1}-(\nu+\lambda)\hat H_1+\nu X_{\hat L_1^{-1}}}\mathbf{1}_{\{\hat L(\infty)>1\}}]\right) \\ &= \hat\kappa(q,\lambda). \end{aligned}$$

This implies (3.12).

By denoting h^ν the renewal function under $\mathbb{P}^\nu$, we use the fact that X drifts to $+\infty$ under $\mathbb{P}^\nu$ to deduce that for any $t, y > 0$

$$\begin{aligned} n(\varepsilon(t)\in dy, t<\zeta) &= e^{\nu y}n^\nu(\varepsilon(t)\in dy, t<\zeta) \\ &= e^{\nu y}k\lim_{x\downarrow 0}\frac{\mathbb{P}^\nu_x(X_t\in dy, t<\tau_0^-)}{h^\nu(x)} \\ &= \lim_{x\downarrow 0}\frac{\mathbb{P}_x(X_t\in dy, t<\tau_0^-)}{h(x)}. \end{aligned}$$

The last equality is implied by

$$\lim_{x\downarrow 0}\frac{h^\nu(x)}{h(x)} = 1,$$

which is a consequence of

$$e^{-\nu x}\mathbb{P}(X_{L^{-1}(t)} \geq -x) \leq \mathbb{E}[e^{\nu X_{L^{-1}(t)}}\mathbf{1}_{\{X_{L^{-1}(t)}\geq -x\}}] \leq \mathbb{P}(X_{L^{-1}(t)} \geq -x).$$

The results in Lemma 3.4 now follow. $\square$

3.4 Resolvent measure of the killed reflected process

The q-resolvent measure of X killed at exiting $[0, a]$ is given by

$$U^{(q)}(x, dy) = \int_0^\infty e^{-qt} \mathbb{P}_x(X_t \in dy, t < \tau_0^- \wedge \tau_a^+)\, dt.$$

We assume throughout that $U^{(q)}(x, dy)$ is absolutely continuous with respect to Lebesgue measure and we denote a version of its density by $u^{(q)}(x, y)$. We also assume that X is regular upwards, which means that the first hitting time of $(0, \infty)$ is almost surely equal to zero. These assumptions are not strictly necessary, but suffice for the application we consider in Section 3.5. We refer to Remark 3.12 for a discussion about how we can weaken the assumptions.

Similarly, denote by $R^{(q)}(x, dy)$ the q-resolvent measure of the process $\{Y_t\}_{t\geq 0}$ killed at exceeding a, i.e.

$$R^{(q)}(x, dy) = \int_0^\infty e^{-qt} \mathbb{P}_x(Y_t \in dy, t < T_a^+)\, dt.$$

In this section we show that (under the conditions above) $R^{(q)}(x, dy)$ is absolutely continuous with respect to Lebesgue measure and we find an expression for its density.

By the strong Markov property applied at τ_0^-, we have for any $y \geq 0$

$$R^{(q)}(x, dy) = u^{(q)}(x, y)dy + \mathbb{E}_x[e^{-q\tau_0^-} \mathbf{1}_{\{\tau_0^- < \tau_a^+\}}] R^{(q)}(0, dy) \tag{3.13}$$

and thus the problem of finding $R^{(q)}(x, dy)$ reduces to finding an expression for $R^{(q)}(0, dy)$, provided of course that we have an expression for the two-sided exit problem. The following result shows that $R^{(q)}(x, dy)$ is absolutely continuous with respect to Lebesgue measure and that a density is given in terms of $u^{(q)}(x, y)$ and the two-sided exit problem.

Theorem 3.5. *Let $0 < x \leq a$, $0 \leq y \leq a$ and $q \geq 0$. The resolvent measure of the killed reflected process has a density, which can be expressed in terms of $u^{(q)}$ and the two sided exit problem as*

$$r^{(q)}(x, y) = u^{(q)}(x, y) + \mathbb{E}_x[e^{-q\tau_0^-} \mathbf{1}_{\{\tau_0^- < \tau_a^+\}}] r^{(q)}(0, y), \tag{3.14}$$

where

$$r^{(q)}(0, y) = \lim_{z \downarrow 0} \frac{u^{(q)}(z, y)}{1 - \mathbb{E}_z[e^{-q\tau_0^-} \mathbf{1}_{\{\tau_0^- < \tau_a^+\}}]}. \tag{3.15}$$

Similarly,

$$r(x, y) := r^{(0)}(x, y) = u(x, y) + \mathbb{P}_x(\tau_0^- < \tau_a^+) r(0, y), \tag{3.16}$$

where

$$r(0, y) := r^{(0)}(0, y) = \lim_{z \downarrow 0} \frac{u(z, y)}{\mathbb{P}_z(\tau_a^+ < \tau_0^-)}. \tag{3.17}$$

Before proving Theorem 3.5, we obtain a couple of auxiliary results. Since $R^{(q)}$ depends only on the behavior of Y until the first time Y exceeds the level a, we can replace all jumps of X greater than a by jumps of size a without affecting $R^{(q)}$. Hence, recalling Remark 3.3, we may assume without loss of generality that $X \in \mathcal{H}$.

Denote by $\overline{\varepsilon}$ the height of a generic excursion ε, i.e.

$$\overline{\varepsilon} = \sup\{\varepsilon(s) : 0 \leq s \leq \zeta\}.$$

Recall that ρ_a is the first time an excursion exceeds the level a. Now, for any $q > 0$, define the event $A_q = B_q \cup C_q$, where

$$B_q = \{\varepsilon \in \mathcal{E} : \rho_a(\varepsilon) \leq \zeta(\varepsilon) \wedge \mathbf{e}_q\} \text{ and } C_q = \{\varepsilon \in \mathcal{E} : \mathbf{e}_q < \rho_a(\varepsilon) \wedge \zeta(\varepsilon)\}.$$

Hence, an excursion is in A_q if and only if its height is at least a or if its length is at least $\mathbf{e}_q$. Similarly, we define

$$A := \{\varepsilon \in \mathcal{E} : \rho_a(\varepsilon) \leq \zeta(\varepsilon)\}. \tag{3.18}$$

In the following lemma we find an expression for the excursion measure of the set A_q.

Lemma 3.6. *For $q > 0$*

$$n(A_q) = \lim_{z\downarrow 0} \frac{1 - \mathbb{E}_z[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^- < \tau_a^+\}}]}{h(z)}$$

and

$$n(A) = \lim_{z\downarrow 0} \frac{\mathbb{P}_z(\tau_a^+ \leq \tau_0^-)}{h(z)}.$$

Proof of Lemma 3.6. Let $q > 0$. Conditional on $\rho_a < \infty$, $\{\varepsilon(t+\rho_a)\}_{t\geq 0}$ is equal in law to the process $\{X_t\}_{t\geq 0}$, started at $\varepsilon(\rho_a)$ and killed at entering $(-\infty, 0]$. Using this observation in combination with an application of the strong Markov property at time ρ_a and the assumption that X is regular upwards allows us to deduce that $n(\overline{\varepsilon} = a) = 0$. From the definition of A_q and B_q it then follows that $n(\partial A_q) = n(\partial B_q) = 0$ and thus we can apply Lemma 3.4 to deduce that

$$\begin{aligned}
n(A_q) &= n(B_q) + n(C_q) \\
&= n(\rho_a \leq \zeta \wedge \mathbf{e}_q) + n(\mathbf{e}_q < \rho_a \wedge \zeta) \\
&= \lim_{z\downarrow 0} \frac{1}{h(z)} \left(\mathbb{E}_z\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+ < \tau_0^-\}}\right] + \mathbb{P}_z(\mathbf{e}_q < \tau_0^- \wedge \tau_a^+) \right) \\
&= \lim_{z\downarrow 0} \frac{1}{h(z)} \left(1 - \mathbb{E}_z[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^- < \tau_a^+\}}] \right)
\end{aligned}$$

The expression for $n(A)$ follows similarly. □

Next, we show that $R^{(q)}(0, dy)$ can be expressed as a quotient involving excursion measures.

Lemma 3.7. *For $q > 0$*

$$R^{(q)}(0, dy) = \frac{n(\varepsilon \in \mathcal{E} : \mathbf{e}_q < \zeta, \varepsilon(\mathbf{e}_q) \in dy, \overline{\varepsilon}(\mathbf{e}_q) \le a)}{qn(A_q)}. \tag{3.19}$$

Also,

$$R(0, dy) = \frac{\int_0^\infty n(\varepsilon \in \mathcal{E} : \varepsilon(t) \in dy, t < \rho_a \wedge \zeta)\, dt}{n(A)}. \tag{3.20}$$

Proof of Lemma 3.7. Let $q > 0$. We have

$$R^{(q)}(0, dy) = \int_0^\infty e^{-qt}\mathbb{P}(Y_t \in dy, \overline{Y}_t \le a)\, dt = \frac{\mathbb{P}(Y_{\mathbf{e}_q} \in dy, \overline{Y}_{\mathbf{e}_q} \le a)}{q}.$$

We denote by $\mathcal{T}$ the (countable) set of times t such that $L^{-1}(t-) < L^{-1}(t)$ and note that excursions away from zero of Y always start at time $L^{-1}(t-)$ for some $t \in \mathcal{T}$. We introduce the family $\{\mathbf{e}_q^t\}_{t\in\mathcal{T}}$ of independent copies of the exponential random variable $\mathbf{e}_q$ and we assume that this family is independent of X as well. Since $\{\varepsilon_t\}_{t\in\mathcal{T}}$ is a Poisson point process with characteristic measure n, the random variable σ_q defined by

$$\sigma_q = \inf\{t \in \mathcal{T} : \varepsilon_t \in A_q\}$$

has an exponential distribution with parameter $n(A_q)$. The memoryless property of the exponential distribution allows us to use the compensation formula in excursion theory to deduce that

$$\begin{aligned}
&\mathbb{P}(Y_{\mathbf{e}_q} \in dy, \overline{Y}_{\mathbf{e}_q} \le a)\\
&= \mathbb{E}\left[\sum_{t\in\mathcal{T}} \mathbf{1}_{\{\varepsilon_t(\mathbf{e}_q^t)\in dy, \mathbf{e}_q^t \in (L^{-1}(t-), L^{-1}(t)), \mathbf{e}_q^t < \rho_a(\varepsilon_t), \sup_{s<t, s\in\mathcal{T}} \overline{\varepsilon}_s \le a\}}\right]\\
&= \mathbb{E}[\sigma_q] n(\varepsilon \in \mathcal{E} : \mathbf{e}_q < \zeta, \varepsilon(\mathbf{e}_q) \in dy, \overline{\varepsilon}(\mathbf{e}_q) \le a),
\end{aligned} \tag{3.21}$$

from which (3.19) follows. For (3.20) we use similar reasoning to deduce that

$$\int_0^\infty \mathbb{P}(Y_t \in dy, \overline{Y}_t \le a)\, dt = \mathbb{E}[\sigma] \int_0^\infty n(\varepsilon \in \mathcal{E} : \varepsilon(t) \in dy, t < \rho_a \wedge \zeta)\, dt, \tag{3.22}$$

where the random variable σ defined by

$$\sigma = \inf\{t \in \mathcal{T} : \varepsilon_t \in A\}$$

has an exponential distribution with parameter $n(A)$. □

Proof of Theorem 3.5. By the strong Markov property, it suffices to show (3.15) and (3.17). For (3.15) we use (3.19) and Lemmas 3.4 and 3.6 to find

$$\begin{aligned}
R^{(q)}(0, dy) &= \frac{n(\mathbf{e}_q < \zeta, \varepsilon(\mathbf{e}_q) \in dy, \overline{\varepsilon}(\mathbf{e}_q) \le a)}{qn(A_q)}\\
&= \lim_{z\downarrow 0} \frac{u^{(q)}(z, y)}{1 - \mathbb{E}_z[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^- < \tau_a^+\}}]} dy,
\end{aligned}$$

where the limit is understood in the weak sense. For (3.17) we use Lemma 3.4 and (3.20) to find

$$\begin{aligned} R(0,dy) &= \frac{\int_0^\infty n(\varepsilon \in \mathcal{E} : \varepsilon(t) \in dy, t < T_a(\varepsilon) \wedge \zeta)\, dt}{n(A)} \\ &= \lim_{z \downarrow 0} \frac{u(z,y)}{\mathbb{P}(\tau_a^+ \leq \tau_0^-)}\, dy. \end{aligned}$$

This completes the proof of Theorem (3.5). □

3.5 Resolvent density for reflected symmetric stable process killed at exceeding a

In this section, as an application of Theorem 3.5, we find the resolvent density for reflected symmetric stable processes killed at exceeding a. We say that a process is symmetric when X and $-X$ have the same law. A Lévy process $\{X_t\}_{t\geq 0}$ is called stable if X_1 has a stable distribution, i.e. when for all $n \geq 1$ there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that $X_1^{(1)} + \ldots + X_1^{(n)}$ has the same distribution as $a_n X_1 + b_n$, where $X_1^{(i)}$ are independent copies of X_1. It turns out that a_n is then of the form $a_n = n^{1/\alpha}$ for some $\alpha \in (0,2]$, where α is referred to as the index of the stable process. When $b_n = 0$ for all $n \in \mathbb{N}$ we say that X is strictly stable. It then holds that for each $k > 0$ the process $\{k^{-1/\alpha} X_{kt}\}_{t\geq 0}$ has the same finite-dimensional distributions as $\{X_t\}_{t\geq 0}$. This is called the scaling property. The characteristic exponent of a strictly stable process is of the form

$$\Psi(\theta) = \begin{cases} c|\theta|^\alpha (1 - \mathrm{i}\beta \tan \frac{\pi\alpha}{2} \operatorname{sgn} \theta) & \text{when } \alpha \neq 1, \\ c|\theta| + \mathrm{i}\eta\theta & \text{when } \alpha = 1, \end{cases}$$

where $\beta \in [-1,1]$, $c > 0$, $\eta \in \mathbb{R}$ and $\operatorname{sgn}\theta = \mathbf{1}_{\{\theta>0\}} - \mathbf{1}_{\{\theta<0\}}$. It is not difficult to see that a symmetric stable process is also strictly stable and that in terms of the characteristic exponent this means that either $\alpha = 1$ and $\eta = 0$ **or** $\alpha \neq 1$ and $\beta = 0$. We refer to [104] and [115] for further details.

For a killed symmetric stable process we have the following expression for the potential density, which follows after rescaling the formula in Corollary 4 in [24].

Theorem 3.8. *The potential measure for a symmetric stable process killed at exiting* $[0,a]$ *has a density given by*

$$u^{(0)}(x,y) = \frac{1}{2^\alpha \Gamma^2(\alpha/2)} \mid x - y \mid^{\alpha-1} \int_0^{s(x,y)} \frac{u^{\alpha/2-1}}{\sqrt{u+1}}\, du$$

where

$$s(x,y) = \frac{4xy(a-x)(a-y)}{a^2(x-y)^2}. \tag{3.23}$$

Furthermore,

$$\mathbb{P}_x(\tau_a^+ < \tau_0^-) = \frac{2^{1-\alpha}\Gamma(\alpha)}{\Gamma^2(\alpha/2)} \int_{-1}^{-1+2x/a} (1-u^2)^{\alpha/2-1}\, du.$$

We can apply Theorem 3.5 to deduce the following result.

Theorem 3.9. *The potential measure for a reflected symmetric stable process killed at exceeding a has a density given by*

$$r(0,y) = \frac{y^{\alpha/2-1}(a-y)^{\alpha/2}}{\Gamma(\alpha)} \quad \textit{for } y \in [0,a] \tag{3.24}$$

and thus for any $x, y \in [0,a]$

$$\begin{aligned} r(x,y) &= \frac{1}{2^\alpha \Gamma^2(\alpha/2)} \mid x-y \mid^{\alpha-1} \int_0^{4xy(a-x)(a-y)/(a(x-y))^2} \frac{u^{\alpha/2-1}}{\sqrt{u+1}}\, du \\ &+ \frac{y^{\alpha/2-1}(a-y)^{\alpha/2}}{\Gamma(\alpha)} \left(1 - \frac{2^{1-\alpha}\Gamma(\alpha)}{\Gamma^2(\alpha/2)} \int_{-1}^{-1+x/2a} (1-u^2)^{A/2-1}\, du\right). \end{aligned}$$

Proof. Any non-monotone stable process is regular upwards (in the case of bounded variation, this follows from the fact that the Lévy measure of such a process satisfies the integral test in [20]) and thus we are within the scope of Theorem 3.5. Let s be defined as in (3.23). A quick calculation shows that

$$\lim_{z \downarrow 0} \frac{\partial s(z,y)}{\partial z} = \frac{4(a-y)}{ay}.$$

For (3.24) we deduce from Theorem 3.5 and 3.8 that

$$\begin{aligned} r(0,y) &= \lim_{z\downarrow 0} \frac{u(z,y)}{\mathbb{P}_z(\tau_a^+ < \tau_0^-)} \\ &= \frac{1}{2\Gamma(\alpha)} \lim_{z\downarrow 0} \frac{\mid z-y \mid^{\alpha-1} \int_0^{s(z,y)} u^{\alpha/2-1}(u+1)^{-1/2}\, du}{\int_{-1}^{-1+2z/a} (1-u^2)^{\alpha/2-1}\, du} \\ &= \frac{y^{\alpha-1}}{2\Gamma(\alpha)} \lim_{z\downarrow 0} \frac{s(z,y)^{\alpha/2-1}(s(z,y)+1)^{-1/2} \frac{\partial s(z,y)}{\partial z}}{(1-(2z/a-1)^2)^{\alpha/2-1} 2/a} \\ &= \frac{y^{\alpha/2-1}(a-y)^{\alpha/2}}{\Gamma(\alpha)}. \end{aligned}$$

The second part of Theorem 3.9 now follows directly from (3.13). □

As a corollary we find the joint law of the undershoot and the overshoot at level a of the reflected symmetric stable process Y.

Corollary 3.10. *For $0 \le z \le a \le y$*

$$\mathbb{P}(Y_{T_a^+ -} \in dz, Y_{T_a^+} \in dy) = \frac{\alpha \sin(\alpha\pi/2)}{\pi} (y-z)^{-\alpha-1} z^{\alpha/2-1}(a-z)^{\alpha/2} dy\, dz.$$

Proof. The ladder height process of a stable process is again stable and hence it has no drift. It follows that X does not creep upwards, which implies $\mathbb{P}(Y_{T_a}^+ = a) = 0$, and thus Y exceeds the level a by a jump. By the compensation formula we find that for any $0 \leq z \leq a \leq y$

$$\mathbb{P}(Y_{T_a^+ -} \in dz, Y_{T_a^+} \in dy) = r(0,z)\Pi(dy - dz). \tag{3.25}$$

The result now follows from (3.24) and from taking into account that the right hand side of (3.25) has unit mass on $[0,a] \times [a,\infty)$. □

Remark 3.11. When we integrate both sides of the equation in Corollary 3.10 over z, we deduce the result in Theorem 2 in [62] for the special case when the stable process is symmetric.

3.6 Bounded variation strictly stable process

Let X now be a strictly stable process with index $0 < \alpha < 1$ and which is not monotone. Note that this implies that X is of bounded variation. For any $t > 0$, denote by $q = \mathbb{P}(X_t > 0)$ the positivity parameter, which indeed does not depend on t since the scaling property gives us

$$\mathbb{P}(X_t > 0) = \mathbb{P}(t^{1/\alpha} X_1 > 0) = \mathbb{P}(X_1 > 0).$$

This parameter can be expressed as

$$q = \frac{1}{2} + \frac{1}{\pi\alpha} \arctan(\beta \tan(\pi\alpha/2)). \tag{3.26}$$

We can prove (3.26) by using the expression for the density of a stable process (equation (2.6.4) in [115])

$$\frac{\mathbb{P}(X_1 \in dx)}{dx} = \frac{1}{\pi} \Im \int_0^\infty \exp\left(-xue^{\mathrm{i}\pi\rho/2} - u^\alpha e^{-\mathrm{i}\pi\rho\alpha/2} + \mathrm{i}\pi\rho/2\right) du,$$

where, from page 17 of [115], $\rho = 1/2 + (\pi\alpha)^{-1} \arctan(\beta \tan(\pi\alpha/2))$. After some analysis (see p. 113 in [115]) it can be deduced that for any $s \geq 0$

$$\int_0^\infty e^{-sx} \mathbb{P}(X_1 \in dx) = \frac{1}{\pi} \int_0^\infty e^{-(su)^\alpha} \frac{\sin(\pi\rho)}{u^2 + 2u\cos(\pi\rho) + 1} du.$$

By taking $s = 0$ and by substituting $u = z\sin(\pi\rho) - \cos(\pi\rho)$ we find that

$$\begin{aligned}
q &= \frac{\sin(\pi\rho)}{\pi} \int_0^\infty \frac{1}{u^2 + 2u\cos(\pi\rho) + 1} du \\
&= \frac{\sin(\pi\rho)}{\pi} \int_{-\cot(\pi(\rho))}^\infty \frac{1}{\sin(\pi\rho)} \frac{1}{z^2 + 1} dz \\
&= \frac{1}{\pi}(\arctan\infty - \arctan(-\cot(\pi\rho))) \\
&= \frac{1}{2} - \frac{1}{\pi} \arctan(\sin(\pi/2 - \pi\rho)/\cos(\pi/2 - \pi\rho)) \\
&= \rho.
\end{aligned}$$

In Exercise 39.2 in [104] the potential density of X is given by

$$p(x) = C(1 + \beta \operatorname{sgn} x)|x|^{\alpha-1},$$

where

$$C = \left(2c\Gamma(\alpha)\cos(\pi\alpha/2)\left(1 + \beta^2 \tan^2(\pi\alpha/2)\right)\right)^{-1}.$$

See pages 440 and 441 in [104] for a proof. Denote by τ the first exit of X from $[0,1]$, i.e.

$$\tau := \inf\{t \geq 0 : X_t \notin [0,1]\} = \tau_0^- \wedge \tau_1^+.$$

By the strong Markov property applied at time τ we have

$$p(y-x)\,dy = U(x,dy) + \int_{z \notin [0,1]} \mathbb{P}_x(X_\tau \in dz)p(y-z)\,dy.$$

In Theorem 1 in [102] the law of X_τ was found to be for $x \in [0,1]$ and $y > 0$

$$\mathbb{P}_x(1 \leq X_\tau \leq 1+y) = f_1(x,y)$$

and

$$\mathbb{P}_x(-y \leq X_\tau \leq 0) = f_2(x,y),$$

where

$$f_1(x,y) = \frac{\sin(\pi\alpha q)}{\pi}(1-x)^{\alpha q}x^{\alpha(1-q)} \int_0^y t^{-\alpha q}(t+1)^{-\alpha(1-q)}(t+1-x)^{-1}\,dt \quad (3.27)$$

and

$$f_2(x,y) = \frac{\sin(\pi\alpha(1-q))}{\pi}(1-x)^{\alpha q}x^{\alpha(1-q)} \int_0^y t^{-\alpha(1-q)}(t+1)^{-\alpha q}(t+x)^{-1}\,dt.$$

From (3.27) it follows that

$$\mathbb{P}_x(\tau_1^+ < \tau_0^-) = f_1(x,\infty) = \frac{1}{B(\alpha q, \alpha(1-q))}\int_0^x u^{\alpha q-1}(1-u)^{\alpha(1-q)-1}\,du,$$

where B is the beta function. For $x, y \in [0,1)$, we denote by $R(x,dy)$ the potential measure of X reflected at its infimum and killed at exceeding 1. Then Theorem 3.5 implies that R has a density, which can in principle be deduced from

$$r(0,y) = \lim_{x \downarrow 0} \frac{p(y-x) - \int_{z\notin[0,1]} \mathbb{P}_x(X_\tau \in dz)p(y-z)}{\mathbb{P}_x(\tau_1^+ < \tau_0^-)} \quad \text{for } y \in (0,1),$$

using the functions f_1, f_2 and the expression for p above.

3.7 Concluding remarks

Remark 3.12. As mentioned before, the assumptions that X is regular upwards and that the resolvent measure $U^{(q)}(x,dy)$ has a density can be relaxed. Reconsidering the proof of Theorem 3.5 and Lemma 3.6 in particular, we find that $R^{(q)}(x,dy)$ is still given as in Theorem 3.5 for $x>0$ when the regularity condition is replaced by $n(\bar{\varepsilon}=a)=0$. The latter holds if Lévy measure Π of X does not have an atom at a. When X is irregular upwards, the time Y spends at zero has positive Lebesgue measure and hence in this case $R(x,dy)$ has an atom at zero. We use the strong Markov property to derive

$$R^{(q)}(x,\{0\}) = \mathbb{E}_z[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}]R^{(q)}(0,\{0\}).$$

Next we remark that

$$qR^{(q)}(0,\{0\}) = \mathbb{P}(\mathbf{e}_q < T_a^+) - q\int_0^a r^{(q)}(0,y)\,dy$$

and it now follows from Lemma 3.4, Theorem 3.5, Lemma 3.6 and (3.21) that

$$R^{(q)}(0,\{0\}) = \frac{1}{q}\lim_{z\downarrow 0}\frac{1-\mathbb{E}_z[e^{-q(\tau_0^-\wedge\tau_a^+)}]}{1-\mathbb{E}_z[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}]} - \int_0^a \lim_{z\downarrow 0}\frac{u^{(q)}(z,y)}{1-\mathbb{E}_z[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}]}\,dy. \tag{3.28}$$

When $U^{(q)}(x,dy)$ is not absolutely continuous with respect to Lebesgue measure, a version of Theorem 3.5 can be obtained in terms of measures.
When X is spectrally one-sided, $u^{(q)}(x,y)$ and the two-sided exit problem are given in terms of the so-called scale function and we find Theorem 1 of [94] (note that a bounded variation spectrally positive Lévy process is irregular upwards and thus the atom at zero of $R^{(q)}(x,dy)$ is given by (3.28)). This essentially is the method of proof as introduced in [40].

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Chapter 4

Examples of optimal stopping via measure transformation for processes with one-sided jumps*

We show that the method introduced by Beibel and Lerche in [16] for solving certain optimal stopping problems for Brownian motion can be applied to some optimal stopping problems involving processes with one-sided jumps as well.

4.1 Introduction

In [16] Beibel and Lerche proposed a method for solving certain optimal stopping problems for a Brownian motion B (see also [72] and, for the case of regular diffusions, [17]). They used a change of measure to reduce the optimal stopping problem to the problem of finding the maximum of a (deterministic) function. One example solved in [16] is

$$\sup_{\tau} \mathbb{E}\left[\frac{B_\tau}{\tau+1}\right]. \tag{4.1}$$

This problem was first solved in ([106], Theorem 1) and, independently, in ([114], Example 2). In section 10 of [106] it was suggested that it is of interest to replace B in (4.1) by a stable process of index $\alpha \in (1,2)$. We show that in some cases, the method proposed in [16] can be used for processes with one-sided jumps as well. In particular, for a spectrally negative strictly stable process of index $\alpha \in (1,2)$ we solve the problem (4.1) in two ways: firstly by a change of measure similar to

*Appeared in abridged version as [10].

the one used in Problem 3 in [16] and secondly by using results from [83] about generalised Ornstein–Uhlenbeck processes.

4.2 Preliminaries

In this section we review some properties of (spectrally negative) Lévy processes and generalised Ornstein–Uhlenbeck processes. For further details about Lévy processes we refer to [18] and [63]. For generalised Ornstein–Uhlenbeck processes we refer to [53], [83] and [85]

Let Z be a spectrally negative Lévy process (i.e. a Lévy process with no positive jumps and non-monotone paths) defined on a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. We denote by (a, σ, Π) its Lévy triple, where $a \in \mathbb{R}$, $\sigma \geq 0$ (called the Gaussian component) and where Π is a measure with mass zero on the positive halfline (due to the assumption that Z is spectrally negative) satisfying the integral condition

$$\int_{(-\infty,0)} (1 \wedge x^2)\, \Pi(dx) < \infty. \tag{4.2}$$

If we denote by B_t a standard Brownian motion and by N a Poisson random measure on $([0,\infty) \times \mathbb{R}, \mathcal{B}[0,\infty) \times \mathcal{B}(\mathbb{R}))$ with intensity $dt \times \Pi(dx)$ we write Z as a sum of three independent Lévy processes

$$Z_t = X_t^{(1)} + X_t^{(2)} + X_t^{(3)}, \tag{4.3}$$

where $X_t^{(1)} = at + \sigma B_t$, where

$$X_t^{(2)} = \int_{[0,t]} \int_{\{x \leq -1\}} x\, N(ds \times dx)$$

and where

$$X_t^{(3)} = \int_{[0,t]} \int_{\{-1 < x < 0\}} x\, \left(N(ds \times dx) - \Pi(dx)\, ds\right).$$

Note that $X_t^{(2)}$ is a compound Poisson process and that $X_t^{(3)}$ is a martingale. This decomposition is known as the Lévy–Itô decomposition and is attributed to [57] and [73]. The Laplace exponent ψ of Z is given by

$$\psi(\lambda) = \frac{\sigma^2}{2}\lambda^2 + a\lambda + \int_{(-\infty,0)} \left(e^{\lambda x} - 1 - \lambda x \mathbf{1}_{\{x \geq -1\}}\right) \Pi(dx), \quad \lambda \geq 0. \tag{4.4}$$

A Lévy process A is said to be of bounded variation if its paths are of bounded variation on each compact interval of $\mathbb{R}_+$ (almost surely), which means that for any $t \geq 0$

$$\sup_{\mathcal{S}} \sum_{k=1}^{m} |A_{t_k} - A_{t_{k-1}}| < \infty \quad \mathbb{P}\text{-a.s.},$$

where the supremum is taken over all partitions $\mathcal{S} = \{t_0, \ldots ..., t_m\}$ of $[0, t]$. Otherwise, A is said to be of unbounded variation. For the Lévy triple of a bounded variation Lévy process this means that $\sigma = 0$ and that the jump measure Π must satisfy

$$\int_{(-\infty,0)} (1 \wedge |x|) \, \Pi(dx) < \infty,$$

see p. 15 of [18]. This implies that whenever Z is of bounded variation, we can rewrite its Laplace exponent ψ as

$$\psi(\lambda) = \left(a - \int_{(-\infty,0)} x \, \Pi(dx) \right) \lambda + \int_{(-\infty,0)} \left(e^{\lambda x} - 1 \right) \Pi(dx)$$

and Z may then be written as

$$Z_t = \left(a - \int_{(-\infty,0)} x \, \Pi(dx) \right) t + \int_{[0,t]} \int_{(-\infty,0)} x \, N(ds \times dx).$$

Since we have excluded the case when Z has monotone paths, it follows that

$$a - \int_{(-\infty,0)} x \, \Pi(dx) > 0. \tag{4.5}$$

In the rest of this section we discuss some properties of spectrally negative strictly stable Lévy processes and generalised Ornstein–Uhlenbeck processes driven by spectrally negative Lévy processes.

Denote by $\{X_t\}_{t \geq 0}$ a spectrally negative strictly stable process of index $\alpha \in (1, 2)$ defined on a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ which satisfies the usual conditions. We denote by $\mathbb{P}_x$ the translation of $\mathbb{P}$ under which $X_0 = x$. The Laplace exponent of X is given by $\psi(\lambda) = C\lambda^\alpha$, where $C > 0$ is a constant. We exclude the case when $\alpha = 2$ as this corresponds to $X = \sqrt{2C} B_t$, with B a Brownian motion. The jump measure of a spectrally negative strictly stable Lévy process is given by

$$\Pi(dx) = c(-x)^{-1-\alpha} dx \quad \text{for } x < 0,$$

where $c > 0$ is some constant (see Theorem C.1 in [115]). Naturally, the case $\alpha = 1$ is excluded as well, as this just corresponds to X being a deterministic drift. Furthermore, we have that $\alpha < 2$ since the jump measure Π needs to satisfy the integral condition (4.2). When $0 < \alpha < 1$, it holds that X is of bounded variation and that $a = \int_{(-\infty,0)} x \, \Pi(dx)$, which implies that the paths of X are monotone decreasing. Summarising, for a spectrally negative strictly stable process with jumps it must hold that $\alpha \in (1, 2)$. In particular, a spectrally negative strictly stable Lévy process is always of unbounded variation. We refer to [115], Chapter VIII in [18] and Chapter 3 in [104] for further details about stable processes.
In the rest of this section, we recall some properties of generalised Ornstein–Uhlenbeck processes. Most of the results are from [83], but for completeness we include them here. The (standard) Ornstein–Uhlenbeck process $\{V_t\}_{t \geq 0}$ is the solution to the stochastic differential equation

$$dY_t = -\gamma Y_t dt + dB_t, \tag{4.6}$$

where $\gamma, \sigma > 0$ and where $\{B_t\}_{t\geq 0}$ is a standard Brownian motion (see p. 358 in [58]). Equation (4.6) is known as Langevin's equation and originates from physics where it was used to model (the speed of) a particle which experiences friction. Since this friction increases as the speed of the particle increases, the constant γ is taken to be strictly positive. Using Itô's formula (see [58]) it can be checked that

$$Y_t = Y_0 e^{-\gamma t} + e^{-\gamma t} \int_0^t e^{\gamma s}\, dB_s$$

solves (4.6). In fact, this is the unique solution to (4.6). If we replace the Brownian motion in (4.6) by a spectrally negative Lévy process $\{Z_t\}_{t\geq 0}$, i.e.

$$dY_t = -\gamma Y_t dt + dZ_t, \quad Y_0 = y \quad \text{under } \mathbb{P}_y \tag{4.7}$$

we can use an extended version of Itô's formula (see [2]) to deduce that the solution is now given by

$$Y_t = Y_0 e^{-\gamma t} + e^{-\gamma t} \int_0^t e^{\gamma s}\, dZ_s. \tag{4.8}$$

An application of such non-Gaussian Ornstein–Uhlenbeck processes to financial economics can be found in [8] and in [9]. See also Theorem 17.5 in [104] for the close link between generalised Ornstein–Uhlenbeck processes and so-called selfdecomposable distributions. We give a review of those elements of [83] which will be useful for Section 4.4 below, where we study optimal stopping problems for Y. The main ingredient we need for these problems is

$$\text{a function } G \text{ such that } \{e^{-rt}G(r, Y_t)\}_{t\geq 0} \text{ is a martingale.} \tag{4.9}$$

As we will study an optimal stopping problem for a generalised Ornstein–Uhlenbeck process, we want to exclude the case when Y has monotone paths. It turns out that this happens whenever X is of bounded variation or when

$$a - \int_{(-1,0)} x\, \Pi(dx) \leq \gamma Y_0.$$

Indeed, if this is the case, we deduce from (4.3) that

$$\begin{aligned}
Y_t &= Y_0 e^{-\gamma t} + a e^{-\gamma t} \int_0^t e^{\gamma s}\, ds + e^{-\gamma t} \int_0^t \int_{(-1,0)} e^{\gamma s} x\, (N(ds,dx) - \Pi(dx)ds) \\
&\quad + e^{-\gamma t} \int_0^t \int_{(-\infty,-1]} x e^{\gamma s}\, N(ds,dx) \\
&\leq Y_0 e^{-\gamma t} + \frac{a}{\gamma}(1 - e^{-\gamma t}) - e^{-\gamma t} \int_0^t \int_{(-1,0)} x e^{\gamma s}\, \Pi(dx)\, ds \\
&= e^{-\gamma t} Y_0 + \frac{1}{\gamma}\left(a - \int_{(-1,0)} x\, \Pi(dx)\right)(1 - e^{-\gamma t}) \\
&\leq Y_0.
\end{aligned}$$

On the other hand it turns out that whenever Y is of unbounded variation or when

$$a - \int_{(-1,0)} x\,\Pi(dx) > \gamma b \tag{4.10}$$

it holds that $\mathbb{P}_y(\sigma_b < \infty) = 1$ for all $y < b$. We remark that, because of (4.5), the latter always holds when $b \le 0$.

Next, let $(t_k^{(n)})_{k,n\in\mathbb{N}}$ be a sequence such that for all $n \in \mathbb{N}$

$$0 = t_0^n < t_1^n \ldots < t_k^{(n)} < t_{k+1}^{(n)} < \ldots t_\infty^n = t$$

and such that for all $k \in \mathbb{N}$

$$t_k^{(n)} - t_k^{(n+1)} \to 0 \quad \text{as } n \to \infty.$$

By dominated convergence and using the fact that Z has stationary, independent increments we find that

$$\begin{aligned}
\mathbb{E}[e^{\lambda Y_t}] &= e^{\lambda Y_0 e^{-\gamma t}} \mathbb{E}\left[\exp\left(\lambda e^{-\gamma t}\int_0^t e^{-\gamma s}\,dZ_s\right)\right] \\
&= e^{\lambda Y_0 e^{-\gamma t}} \mathbb{E}\left[\exp\left(\lambda \int_0^t e^{-\gamma u}\,dZ_u\right)\right] \\
&= e^{\lambda Y_0 e^{-\gamma t}} \mathbb{E}\left[\exp\left(\lambda \int_0^t \lim_{n\to\infty}\sum_k e^{-\gamma t_k^{(n)}} \mathbf{1}_{[t_k^{(n)},t_{k+1}^{(n)})}(u)\,dZ_u\right)\right] \\
&= e^{\lambda Y_0 e^{-\gamma t}} \lim_{n\to\infty}\prod_k \mathbb{E}\left[\exp\left(\lambda e^{-\gamma t_k^{(n)}}(Z_{t_{k+1}^{(n)}} - Z_{t_k^{(n)}})\right)\right] \\
&= e^{\lambda Y_0 e^{-\gamma t}} \lim_{n\to\infty}\prod_k \exp\left(\psi(\lambda e^{-\gamma t_k^{(n)}})(t_{k+1}^{(n)} - t_k^{(n)})\right) \\
&= \exp\left(\lambda Y_0 e^{-\gamma t} + \int_0^t \psi(\lambda e^{-\gamma s})\,ds\right),
\end{aligned}$$

from which it readily follows that the process $\{C_t\}_{t\ge 0}$ defined by

$$C_t(\lambda) = \exp\left(\lambda e^{\gamma t} Y_t - \int_0^t \psi(\lambda e^{\gamma s})\,ds\right)$$

is a martingale for any $\lambda \ge 0$.
We now make the assumption that the driving spectrally negative Lévy process Z satisfies

$$\mathbb{E}[\log(1 + (-Z_1)^+)] < \infty, \tag{4.11}$$

where $y^+ := \max(y, 0)$. This assumption implies that the function ϕ defined by

$$\phi(s) = \frac{1}{\gamma}\int_0^s v^{-1}\psi(v)\,dv \quad s \ge 0,$$

is finite. To see this, note that (4.2) and (4.4) imply that an infinite value of $\phi(s)$ can only be caused by the term

$$\int_0^u \frac{1}{v}\int_{(-\infty,-1)} (e^{vx}-1)\,\Pi(dx)\,dv$$

being infinite. However, for any $x<0$ and $u>0$,

$$\int_0^u v^{-1}(e^{vx}-1)\,dv = \int_0^1 y^{-1}(e^{-y}-1)\,dy - \log(-x) - \log(u) - \int_{-ux}^1 y^{-1}e^{-y}\,dy.$$

It now follows from condition (4.11) and $\Pi(-\infty,-1)<\infty$ (due to (4.2)) that for any $u>0$

$$\int_{-\infty}^{-1}\left(\int_0^1 y^{-1}(e^{-y}-1)\,dy - \log(-x) - \log(u) - \int_{-ux}^1 y^{-1}e^{-y}dy\right)\Pi(dx) > -\infty. \tag{4.12}$$

We conclude that ϕ is well defined.
For $r>0$, integrating $C_t(\lambda)$ over $[0,\infty)$ with respect to the measure

$$\mu(d\lambda) = e^{-\phi(\lambda)}\lambda^{r-1}\,d\lambda,$$

allows us to deduce that the process $\{D_t(r)\}_{t\geq 0}$ defined by

$$\begin{aligned}
D_t(r) &= \int_0^\infty C_t(\lambda)\,\mu(d\lambda)\\
&= \int_0^\infty \exp\left(\lambda e^{\gamma t}Y_t - \int_0^t \psi(\lambda e^{\gamma s})ds - \phi(\lambda)\right)\lambda^{r-1}\,d\lambda\\
&= \int_0^\infty e^{\lambda e^{\gamma t}Y_t - \phi(\lambda e^{\gamma t})}\lambda^{r-1}\,d\lambda\\
&= e^{-r\gamma t}\int_0^\infty e^{uY_t-\phi(u)}u^{r-1}\,du
\end{aligned}$$

is a martingale for any $r>0$ as well (see Lemma 1 in [83] for the finiteness of $D_t(r)$). We see that the function G alluded to in (4.9) can be taken as

$$G(r,x) = \int_0^\infty e^{ux-\phi(u)du}u^{-1+r/\gamma}\,du.$$

4.3 Alphabolic boundaries

Recall that X is a spectrally negative strictly stable process of index $\alpha\in(1,2)$. Let $\beta>0$ and define the function

$$H(\beta,x) = \int_0^\infty e^{ux-u^\alpha}u^{\alpha\beta-1}\,du,$$

which is finite since $\alpha > 1$. Suppose h is a function on $\mathbb{R}$ such that there exists some x^* satisfying

$$x^* = \arg\max_x \frac{h(x)}{H(\beta,x)}. \tag{4.13}$$

Denote by $\mathcal{T}$ the set of stopping times with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. In this section we study optimal stopping problems of the form

$$V(x) := \sup_{\tau\in\mathcal{T}} \mathbb{E}_x \left[\frac{h\left((\tau+1)^{-1/\alpha}X_\tau\right)}{(\tau+1)^\beta} \mathbf{1}_{\{\tau<\infty\}} \right]. \tag{4.14}$$

We have the following result.

Theorem 4.1. *Let h be a function on $\mathbb{R}$ such that x^* in (4.13) exists. Suppose $x < x^*$. The optimal stopping time in (4.14) is given by*

$$\tau^* = \inf\{t \geq 0 : X_t = (t+1)^{1/\alpha}x^*\}.$$

Furthermore,

$$V(x) = \frac{h(x^*)}{H(\beta,x^*)} H(\beta,x).$$

Proof. By changing variables $y = u(t+1)^{-1/\alpha}$ we find that

$$\begin{aligned} H(\beta,(t+1)^{-1/\alpha}X_t) &= \int_0^\infty e^{u(t+1)^{-1/\alpha}X_t - u^\alpha} u^{\alpha\beta-1}\,du \\ &= (t+1)^\beta \int_0^\infty e^{yX_t - y^\alpha t - y^\alpha} y^{\alpha\beta-1}\,dy. \end{aligned}$$

Since $\mathbb{E}[e^{yX_t}] = e^{\psi(y)t}$, it holds that $\{A_t(y)\}_{t\geq 0} := \{e^{yX_t - y^\alpha t}\}_{t\geq 0}$ is a martingale for any $y \geq 0$. It follows that $\{M_t\}_{t\geq 0}$ defined by

$$\begin{aligned} M_t &= \int_0^\infty A_t(y) e^{-y^\alpha} y^{\alpha\beta-1}\,dy \\ &= \frac{H(\beta,(t+1)^{-1/\alpha}X_t)}{H(\beta,x)(t+1)^\beta} \end{aligned}$$

is a mean one martingale under $\mathbb{P}_x$. Hence, for any $\mathbb{P}_x$ stopping time τ we have that

$$\begin{aligned} &\mathbb{E}_x \left[\frac{h((\tau+1)^{-1/\alpha}X_\tau)}{(\tau+1)^\beta} \mathbf{1}_{\{\tau<\infty\}} \right] \\ &\qquad = \mathbb{E}_x \left[H(\beta,x) \frac{h((\tau+1)^{-1/\alpha}X_\tau)}{H(\beta,(\tau+1)^{-1/\alpha}X_\tau)} M_\tau \mathbf{1}_{\{\tau<\infty\}} \right] \\ &\qquad \leq H(\beta,x) \frac{h(x^*)}{H(\beta,x^*)} \mathbb{E}_x[M_\tau \mathbf{1}_{\{\tau<\infty\}}] \\ &\qquad \leq H(\beta,x) \frac{h(x^*)}{H(\beta,x^*)} \end{aligned}$$

and thus

$$\tau^* := \inf\{t \geq 0 : (t+1)^{-1/\alpha} X_t = x^*\}$$

is the optimal stopping time if we can show that

$$\mathbb{P}_x(\tau^* < \infty) = 1 \quad \text{and } \mathbb{E}_x[M_{\tau^*}] = 1.$$

By the law of iterated logarithm for spectrally negative stable processes (see Theorem 5 (ii) in [18]) we deduce that for any $x < x^*$

$$\mathbb{P}_x(\tau^* < \infty) = 1.$$

Also, since H is an increasing function and since $(\tau^* + 1)^{-1/\alpha} X_{\tau^*} \leq x^*$ we deduce that for $x < x^*$ and any $n \in \mathbb{N}$

$$M_{\tau^* \wedge n} \leq \frac{H(\beta, x^*)}{H(\beta, x)} \quad \text{under } \mathbb{P}_x.$$

We use the optional sampling theorem and bounded convergence to deduce that

$$\begin{aligned} 1 &= \lim_{n\to\infty} \mathbb{E}_x[M_{\tau^* \wedge n}] \\ &= \mathbb{E}_x[M_{\tau^*}]. \end{aligned}$$

This completes the proof. □

4.4 Optimal stopping problems for a generalised Ornstein–Uhlenbeck process

In this section we consider optimal stopping problems for generalised Ornstein–Uhlenbeck processes driven by spectrally negative Lévy processes. When the driving spectrally negative process is strictly stable, then Remark 4.3 below indicates how these optimal stopping problems are related to those treated in the previous section.

Denote by Z the spectrally negative Lévy process which drives the generalised Ornstein–Uhlenbeck process Y defined in (4.15). Let $r > 0$. In this section we consider optimal stopping problems of the form

$$U(y) := \sup_{\tau \in \mathcal{T}} \mathbb{E}_y[e^{-r\tau} g(Y_\tau) \mathbf{1}_{\{\tau < \infty\}}], \tag{4.15}$$

where g belongs to a class of functions which is yet to be specified and where now, with some abuse of notation, $\mathbb{E}_y$ denotes the expectation when the process Y starts from y, i.e. when $Y_0 = y$. Assume that

$$\sigma > 0 \quad \text{or} \quad a - \int_{-1}^{0} z\,\Pi(dz) > \gamma y, \tag{4.16}$$

since otherwise the generalised Ornstein–Uhlenbeck process never hits points $b > y$ with probability one. Clearly, (4.16) is satisfied when Z is of unbounded variation.

To simplify, we also assume (but see Remark 4.4 below) that condition (4.11) holds. Recall that for $u \geq 0$

$$\phi(u) = \frac{1}{\gamma} \int_0^u \frac{\psi(v)}{v} \, dv$$

and for $r > 0$

$$G(r, x) = \int_0^\infty e^{ux - \phi(u)} u^{-1+r/\gamma} \, du.$$

As $\{e^{-rt} G(r, Y_t)\}_{t \geq 0}$ is a martingale for any $r > 0$, we can introduce the locally equivalent measure $\mathbb{Q}$ by

$$\left. \frac{d\mathbb{Q}_y}{d\mathbb{P}_y} \right|_{\mathcal{F}_t} = e^{-rt} \frac{G(r, Y_t)}{G(r, y)}.$$

We see that (4.15) can be written as

$$U(y) = G(r, y) \sup_{\tau \in \mathcal{T}} \mathbb{E}_y^{\mathbb{Q}} \left[\frac{g(Y_\tau)}{G(r, Y_\tau)} \mathbf{1}_{\{\tau < \infty\}} \right].$$

Theorem 4.1. *Suppose g is a function on $\mathbb{R}$ such that g/G attains its maximum at y^* and suppose that $\{Z_t\}_{t \geq 0}$ is a spectrally negative Lévy process satisfying (4.11) and*

$$\sigma > 0 \quad or \quad a - \int_{-1}^{0} z \, \Pi(dz) > \gamma y^*.$$

Then for any $Y_0 = y < y^*$ the optimal stopping time in (4.15) is given by

$$\sigma^* = \sigma_{y^*}^+ = \inf\{t \geq 0 : Y_t = y^*\}.$$

Furthermore,

$$U(y) = \frac{g(y^*)}{G(r, y^*)} G(r, y).$$

Proof. Let $y < y^*$. It suffices to prove that σ^* is almost surely finite under $\mathbb{P}_y$ and $\mathbb{Q}_y$. The first statement is contained in Theorem 2 in [83]. The proof of the second statement is similar to the end of the proof of Theorem 4.1. □

Remark 4.2. When the driving spectrally negative Lévy process is strictly stable, we find that

$$\begin{aligned} G(r, x) &= \int_0^\infty e^{ux - \phi(u)} u^{-1+r/\gamma} \, du \\ &= \int_0^\infty e^{ux - \gamma^{-1} \int_0^u s^{\alpha - 1} ds} u^{-1+r/\gamma} \, du \\ &= \int_0^\infty e^{ux - (\alpha\gamma)^{-1} u^\alpha} u^{-1+r/\gamma} \, du. \end{aligned}$$

We can, for example, apply Theorem 4.1 to functions g of the form

$$g(x) = \begin{cases} 0 & \text{for } x < k, \\ b(x) & \text{for } x \geq k, \end{cases}$$

where $k \in \mathbb{R}$ and where b is a continuous function such that $\lim_{x\to\infty} e^{ax}b(x) = 0$ for some $a \in \mathbb{R}$.

Remark 4.3. Denote by $Y^{(\alpha)}$ the generalised Ornstein–Uhlenbeck process which has a spectrally negative strictly stable process $X^{(\alpha)}$ with index $\alpha \in (1,2)$ as its driving Lévy process with $\gamma = 1/\alpha$ and $Y_0^{(\alpha)} = 0$. Then $e^{-t/\alpha}(X^{(\alpha)}(e^t-1))$ and $Y_t^{(\alpha)}$ are equal in distribution. Indeed, when $\lambda \geq 0$, we both have that

$$\begin{aligned}
\mathbb{E}[e^{\lambda Y_t^{(\alpha)}}] &= \exp\left(\int_0^t \psi(\lambda e^{-\gamma s})\, ds\right) \\
&= \exp\left(\lambda^\alpha \int_0^t e^{-\alpha\gamma s}\, ds\right) \\
&= \exp\left(\frac{1}{\alpha\gamma}\lambda^\alpha(1-e^{-\alpha\gamma t})\right) \\
&= \exp\left(\lambda^\alpha(1-e^{-t})\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\lambda e^{-t/\alpha}(X^{(\alpha)}(e^t-1))\right)\right] &= \exp\left(\lambda^\alpha(e^{-t/\alpha})^\alpha(e^t-1)\right) \\
&= \exp\left(\lambda^\alpha(1-e^{-t})\right).
\end{aligned}$$

We deduce that

$$\sup_\tau \mathbb{E}\left[\frac{X_\tau^{(\alpha)}}{\tau+1}\right] = \sup_\tau \mathbb{E}\left[e^{-\tau}X^{(\alpha)}(e^\tau-1)\right] = \sup_\tau \mathbb{E}\left[e^{-(1-\alpha^{-1})\tau}Y_\tau^{(\alpha)}\right].$$

Hence, for a spectrally negative strictly stable process we can also solve (4.1) by applying Theorem 4.1 to the case $g(x) = x$ and $r = (\alpha-1)/\alpha$.

Remark 4.4. In fact, we can state an alternative version of Theorem 4.1 when we drop condition (4.11). This condition was necessary to ensure finiteness of the function G (see p. 290 in [83]). Denote by $Z^{(n)}$ the (spectrally negative Lévy) process obtained from Z by ignoring all its jumps with size in $(-\infty,-n)$, i.e. the jump measure $\Pi^{(n)}$ of $Z^{(n)}$ is given by $\Pi^{(n)}(dx) = \mathbf{1}_{\{x\geq -n\}}\Pi(dx)$. Then $Z^{(n)}$ does satisfy (4.11). To see this, we first remark that for any $t \geq 0$

$$\mathbb{E}[\log(1+(-Z_t^{(n)})^+)] \leq 1 + \mathbb{E}[1 \vee |Z_t^{(n)}|].$$

Next, since $1 \vee |x|$ is a submultiplicative function and since

$$\int_{\{|x|\geq 1\}} (1 \vee |x|)\, \Pi^{(n)}(dx) \leq (1+n)\Pi(-n,-1) < \infty,$$

it follows from Theorem 25.3 in [104] that $Z^{(n)}$ satisfies (4.11). We can define a modification $\tilde{G}$ of G by

$$\tilde{G}(r,x) := \int_0^\infty e^{ux-\tilde{\phi}(u)}u^{-1+r/\gamma}\, du,$$

where

$$\begin{aligned}\gamma\tilde{\phi}(u) &= au + \frac{\sigma^2}{4}u^2 + \int_0^u \frac{1}{v}\int_{(-1,0)} (e^{vx} - 1 - vx)\,\Pi(dx)\,dv \\ &\quad - \int_{(-\infty,-1)} \left(\log(u) + \int_{-xu}^1 \frac{1}{y}e^{-y}\,dy\right)\Pi(dx). \qquad (4.17)\end{aligned}$$

The function $\tilde{G}$ is finite even when (4.11) fails (see again Lemma 1 in [83]) and $\tilde{G}(r,x) = kG(r,x)$ for some constant $k > 0$ whenever (4.11) does hold (compare (4.17) with the left hand side of (4.12)). Denote by $Y^{(n)}$ the generalised Ornstein–Uhlenbeck process driven by $Y^{(n)}$. We then deduce that there exist some constants $k_n > 0$ such that (with the obvious notation)

$$e^{-rt}\tilde{G}^{(n)}(r, Y_t^{(n)}) = e^{-rt}k_n G^{(n)}(r, Y_t^{(n)})$$

and thus $\{e^{-rt}\tilde{G}^{(n)}(r, Y_t^{(n)})\}_{t\geq 0}$ is a martingale for any $n \in \mathbb{N}$. Since $\Pi(-\infty,-1) < \infty$, we have that $\Pi^n \to \Pi$ as $n \to \infty$, which in turn implies that $\lim_{n\to\infty}\tilde{G}^{(n)} = \tilde{G}$. Since $\lim_{n\to\infty} Y_t^{(n)} = Y_t$ a.s., we deduce that condition (4.11) can be dropped if we replace G by $\tilde{G}$ in Theorem 4.1.

4.5 Smooth fit

We continue this chapter with a discussion on smooth fit for optimal stopping problems of the form

$$V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x[e^{-q\tau}f(Y_\tau)], \quad q > 0, \qquad (4.18)$$

where Y is a generalised Ornstein–Uhlenbeck process driven by a real valued Lévy process and where we assume that f is a C^1 function on $\mathbb{R}$ with a uniformly bounded derivative. The property of smooth fit means that the value function V of an optimal stopping problem with a (smooth) pay-off function f is differentiable on ∂D and satisfies $V'(x) = f'(x)$ for $x \in \partial D$, with $D = \{x \in \mathbb{R} : V(x) = f(x)\}$. It was conjectured in [1] that for 'nice enough' optimal stopping problems for a Lévy process Z, smooth fit holds at $x \in \partial D$ whenever x is regular for $\operatorname{int} D$ for Z. However, regularity is not a sufficient condition to imply smooth fit for diffusions, see [89] for a counterexample. In that paper it was also shown that a sufficient condition for smooth fit for a diffusion is that the diffusion leaves a symmetric interval upwards with probability 1/2 in the limit when the length of this interval goes to zero.

Since Y is a strong Markov process which is right-continuous and left-continuous over stopping times, general theory of optimal stopping (see Section 2.2 in [92]) implies that the optimal stopping region D for (4.18) is again given by

$$D = \{x \in \mathbb{R} : V(x) = f(x)\}.$$

In Theorem 4.5 below we give conditions under which, in this case, smooth fit holds. For convenience, we assume that D is of the form $D = (-\infty, y]$ for some

$y \in \mathbb{R}$. However, by inspection of the proof it can be checked that Theorem 4.5 still holds when D consists of several closed intervals.
Define for $\varepsilon > 0$

$$\tau_\varepsilon := \inf\{t \geq 0 : Y_t \notin [y-\varepsilon, y+\varepsilon]\}.$$

Furthermore,

$$\tau^+ := \inf\{t \geq 0 : Y_t > y+\varepsilon\}$$

and

$$\tau^- := \inf\{t \geq 0 : Y_t < y-\varepsilon\}.$$

In Theorem 4.5 below we prove that the properties

$$\lim_{\varepsilon\downarrow 0} \mathbb{P}_y(\tau^+ < \tau^-) = \frac{1}{2}, \tag{4.19}$$

$$\lim_{\varepsilon\downarrow 0} \frac{\mathbb{E}_y[e^{-q\tau_\varepsilon}] - 1}{\varepsilon} = 0, \tag{4.20}$$

$$\lim_{\varepsilon\downarrow 0} \frac{Y_{\tau^+} - y}{\varepsilon} = 1 \quad \mathbb{P}_y \text{ a.s.} \qquad \text{and} \tag{4.21}$$

$$\lim_{\varepsilon\downarrow 0} \frac{Y_{\tau^-} - y}{\varepsilon} = -1 \quad \mathbb{P}_y \text{ a.s.} \tag{4.22}$$

are sufficient to imply smooth fit for the optimal stopping problem (4.18). We refer to Remark 4.6 for a discussion on these conditions. We have the following result.

Theorem 4.5. *Suppose f is a C^1 function with bounded derivative and suppose that the generalised Ornstein–Uhlenbeck process Y (as described above) satisfies conditions (4.19)–(4.22). Then the function V defined in (4.18) is differentiable at y and $V'(y) = f'(y)$.*

Proof. Suppose the conditions mentioned in the Theorem are fulfilled. The functions f and V are equal on $(-\infty, y]$ and thus

$$\lim_{\varepsilon\downarrow 0} \frac{V(y) - V(y-\varepsilon)}{\varepsilon} = f'(y).$$

Since $V(y) = f(y)$ and $V(x) \geq f(x)$ for all x we have that

$$\frac{V(x) - V(y)}{x-y} \geq \frac{f(x) - f(y)}{x-y} \quad \text{for all } x > y,$$

which implies that

$$\liminf_{\varepsilon\downarrow 0} \frac{V(y+\varepsilon) - V(y)}{x-y} \geq f'(y). \tag{4.23}$$

It thus suffices to show that

$$\limsup_{\varepsilon\downarrow 0} \frac{V(y+\varepsilon) - V(y)}{\varepsilon} \leq f'(y). \tag{4.24}$$

The process $\{e^{-qt}V(Y_t)\}_{t\geq 0}$ is a supermartingale, hence

$$\begin{aligned}\mathbb{E}_y[e^{-q\tau_\varepsilon}V(Y_{\tau_\varepsilon})] &\leq V(y)\\ &= V(y)\mathbb{E}_y[e^{-q\tau^+}\mathbf{1}_{\{\tau^+<\tau^-\}}]\\ &\quad + f(y)(1-\mathbb{E}_y[e^{-q\tau^+}\mathbf{1}_{\{\tau^+<\tau^-\}}]). \qquad (4.25)\end{aligned}$$

Furthermore, since $f = V$ on $(-\infty, y]$, we have

$$\begin{aligned}\mathbb{E}_y[e^{-q\tau_\varepsilon}V(Y_{\tau_\varepsilon})] &= \mathbb{E}_y[e^{-q\tau^+}V(Y_{\tau^+})\mathbf{1}_{\{\tau^+<\tau^-\}}] + \mathbb{E}_y[e^{-q\tau^-}V(Y_{\tau^-})\mathbf{1}_{\{\tau^-<\tau^+\}}]\\ &= \mathbb{E}_y[e^{-q\tau^+}V(Y_{\tau^+})\mathbf{1}_{\{\tau^+<\tau^-\}}] + \mathbb{E}_y[e^{-q\tau^-}f(Y_{\tau^-})\mathbf{1}_{\{\tau^-<\tau^+\}}]\\ &= V(y+\varepsilon)\mathbb{E}_y[e^{-q\tau^+}\mathbf{1}_{\{\tau^+<\tau^-\}}] + f(y-\varepsilon)\mathbb{E}_y[e^{-q\tau^-}\mathbf{1}_{\{\tau^-<\tau^+\}}]\\ &\quad + \mathbb{E}_y[e^{-q\tau^+}(V(Y_{\tau^+}) - V(y+\varepsilon))\mathbf{1}_{\{\tau^+<\tau^-\}}]\\ &\quad + \mathbb{E}_y[e^{-q\tau^-}(f(Y_{\tau^-}) - f(y-\varepsilon))\mathbf{1}_{\{\tau^-<\tau^+\}}]. \qquad (4.26)\end{aligned}$$

From (4.25) and (4.26) we see that

$$\begin{aligned}&(V(y+\varepsilon) - V(y))\mathbb{E}_y[e^{-q\tau^+}\mathbf{1}_{\{\tau^+<\tau^-\}}]\\ &\quad \leq (f(y) - f(y-\varepsilon))(1-\mathbb{E}_y[e^{-q\tau^*}\mathbf{1}_{\{\tau^+<\tau^-\}}])\\ &\qquad - \mathbb{E}_y[e^{-q\tau^+}(V(Y_{\tau^+}) - V(y+\varepsilon))\mathbf{1}_{\{\tau^+<\tau^-\}}]\\ &\qquad\quad - \mathbb{E}_y[e^{-q\tau^-}(f(Y_{\tau^-}) - f(y-\varepsilon))\mathbf{1}_{\{\tau^-<\tau^+\}}]\\ &\qquad\qquad + f(y-\varepsilon)(1-\mathbb{E}_y[e^{-q\tau_\varepsilon}]). \qquad (4.27)\end{aligned}$$

Since the derivative of f is uniformly bounded, f is Lipschitz-continuous and we denote by K its Lipschitz constant. Of course, the optimal stopping time $\inf\{t \geq 0 : Y_t \leq y\}$ of (4.18) depends on the initial value Y_0 and we denote by τ_x^* the optimal stopping time when $Y_0 = x$ for $x \in \mathbb{R}$. Let $x, z \in \mathbb{R}$. Then, from the definition of Y in (4.8) and from the fact that τ_z^* is also a stopping time for the process Y under $\mathbb{P}_z$ we find

$$\begin{aligned}V(z) - V(x) &\leq \mathbb{E}_0[e^{-q\tau_z^*}(f(Y_{\tau_z^*} + ze^{-\gamma\tau_z^*}) - f(Y_{\tau_z^*} + xe^{-\gamma\tau_z^*}))]\\ &\leq \mathbb{E}_0\left[e^{-q\tau_z^*}|f(Y_{\tau_z^*} + ze^{-\gamma\tau_z^*}) - f(Y_{\tau_z^*} + xe^{-\gamma\tau_z^*})|\right]\\ &\leq K\mathbb{E}_0\left|ze^{-\gamma\tau_z^*} - xe^{-\gamma\tau_z^*}\right|\\ &\leq K|z-x|.\end{aligned}$$

Since we can interchange the roles of x and z above, we deduce that

$$|V(z) - V(x)| \leq K|x-z| \quad \text{ for all } x, z \in \mathbb{R}.$$

Because f and V are Lipschitz-continuous it follows from (4.21) and (4.22) that

$$\lim_{\varepsilon\downarrow 0}\frac{|V(Y_{\tau^+}) - V(y+\varepsilon)|}{\varepsilon} = 0 \quad \mathbb{P}_a \text{ a.s.}$$

and

$$\lim_{\varepsilon \downarrow 0} \frac{|f(Y_{\tau^-}) - f(y-\varepsilon)|}{\varepsilon} = 0 \quad \mathbb{P}_a \text{ a.s..}$$

As f and V are bounded functions, it follows from dominated convergence that

$$\mathbb{E}_y[e^{-q\tau^+}(V(Y_{\tau^+}) - V(y+\varepsilon))\mathbf{1}_{\{\tau^+<\tau^-\}}] + \mathbb{E}_y[e^{-q\tau^-}(f(Y_{\tau^-}) - f(y-\varepsilon))\mathbf{1}_{\{\tau^-<\tau^+\}}]$$

goes to zero as ε goes to zero. From (4.19), (4.20) and (4.27) we now deduce that

$$\begin{aligned}
\limsup_{\varepsilon \downarrow 0} \frac{V(y+\varepsilon) - V(y)}{\varepsilon} &\leq \limsup_{\varepsilon \downarrow 0} \frac{f(y) - f(y-\varepsilon)}{\varepsilon} \frac{1 - \mathbb{E}_y[e^{-q\tau^+}\mathbf{1}_{\{\tau^+<\tau^-\}}]}{\mathbb{E}_y[e^{-q\tau^+}\mathbf{1}_{\{\tau^+<\tau^-\}}]} \\
&\quad + \limsup_{\varepsilon \downarrow 0} \frac{f(y-\varepsilon)}{\mathbb{E}_y[e^{-q\tau^+}\mathbf{1}_{\{\tau^+<\tau^-\}}]} \frac{1 - \mathbb{E}_y[e^{-q\tau_\varepsilon}]}{\varepsilon} \\
&= f'(a),
\end{aligned}$$

which concludes the proof of Theorem 4.5. □

Remark 4.6. In this remark, we indicate why it is vital for the driving Lévy process to have a Gaussian component for the conditions (4.19)–(4.22) to hold. The stochastic integral in the definition (4.8) of the generalised Ornstein–Uhlenbeck process Z can be written as

$$\int_0^t e^{\gamma s}\, dZ_s = \int_0^t e^{\gamma s}\, dB_s + \int_0^t \int_{(-\infty,-1]} x e^{\gamma s}\, N(ds, dx) + \int_0^t \int_{(-1,0)} x e^{\gamma s}\, \tilde{N}(ds, dx),$$

where N and $\tilde{N}$ are the Poisson random measure of large (in absolute size) jumps and the compensated Poisson random measure of small jumps of Y, respectively. When t is small, $e^{\gamma s} \sim 1$ on $[0,t]$, and hence Y at small times behaves like the underlying Lévy process Z. In the following lemma we show for which Lévy processes conditions (4.19)–(4.22) hold.

Lemma 4.7. *Let L be a Lévy process and, with some abuse of notation, denote by τ_ε, τ^+ and τ^- the exit times for L (just as they were defined for Y). The following statements are equivalent:*

i) L has a Gaussian component.

ii) L satisfies conditions (4.19)–(4.22).

Proof. Let L be a Lévy process. We first prove that i) implies ii).
Suppose L has a Gaussian component. Theorem 36 (i) in [41] then implies that

$$\lim_{t \downarrow 0} \mathbb{P}(L_t \geq 0) = \frac{1}{2}$$

and thus

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}_y(\tau^+ < \tau^-) = \lim_{\varepsilon \downarrow 0} \mathbb{P}_y(L_{\tau_\varepsilon} \geq 0) = \frac{1}{2},$$

which is (4.19).
For (4.20), we recall a result from [99], see Lemma 8 in [41]. Denote by (a, σ, Π) the Lévy triple of L. Then there exists a constant $C > 0$ such that for all $\varepsilon > 0$

$$\mathbb{E}_y[\tau_\varepsilon] \leq \frac{C}{k(\varepsilon)},$$

where $k(x) = A(x)/x + U(x)/x^2$, $A(x) = a + D(1) - \int_x^1 D(y)\,dy$ and $U(x) = \sigma^2 + 2\int_0^x yL(y)\,dy$. Here $D(x) = N(x) - M(x)$ and $L(x) = N(x) + M(x)$, where $N(x) = \Pi((x,\infty))$ and $M(x) = \Pi((-\infty,-x))$. The important thing to remark is that

$$\lim_{x\downarrow 0} U(x) = \sigma^2 \quad \text{and} \quad \lim_{x\downarrow 0} xA(x) = 0.$$

Indeed, by Fubini's theorem

$$\begin{aligned}
\lim_{x\downarrow 0} U(x) &= \sigma^2 + 2\lim_{x\downarrow 0}\int_0^x y\,L(y)dy \\
&= \sigma^2 + 2\lim_{x\downarrow 0}\int_0^x\int_{(y,\infty)} y\,\Pi(dz)\,dy + 2\lim_{x\downarrow 0}\int_0^x\int_{(-\infty,-y)} y\,\Pi(dz)\,dy \\
&= \sigma^2 + 2\lim_{x\downarrow 0}\int_{(0,\infty)}\int_0^{z\wedge x} y\,dy\,\Pi(dz) + 2\lim_{x\downarrow 0}\int_{(-\infty,0)}\int_0^{-z\wedge x} y\,dy\,\Pi(dz) \\
&= \sigma^2 + \lim_{x\downarrow 0}\int_0^\infty (z^2\wedge x^2)\,\Pi(dz) + \lim_{x\downarrow 0}\int_{(-\infty,0)} (z^2\wedge x^2)\,\Pi(dz) \\
&= \sigma^2,
\end{aligned}$$

where the ultimate equality follows from (4.2) and the dominated convergence theorem. Similarly, by integration by parts and (4.2) we find

$$\begin{aligned}
\lim_{x\downarrow 0} xA(x) &= -\lim_{x\downarrow 0} x\int_x^1 D(y)\,dy \\
&= \lim_{x\downarrow 0} x^2\Pi((x,\infty)) - \lim_{x\downarrow 0} x^2\Pi((-\infty,-x)) \\
&\quad - \lim_{x\downarrow 0} x\int_{(x,1]} y\,\Pi(dy) + \lim_{x\downarrow 0} x\int_{[-1,-x)} y\,\Pi(dy) \\
&= 0.
\end{aligned}$$

Since $1 - e^{-x} \leq x$ for $x \geq 0$, it then follows that

$$\begin{aligned}
\limsup_{\varepsilon\downarrow 0}\frac{1-\mathbb{E}_y[e^{-q\tau_\varepsilon}]}{\varepsilon} &\leq q\limsup_{\varepsilon\downarrow 0}\frac{\mathbb{E}_y[\tau_\varepsilon]}{\varepsilon} \\
&\leq qC\lim_{\varepsilon\downarrow 0}\frac{1}{k(\varepsilon)\varepsilon} \\
&= qC\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\varepsilon A(\varepsilon) + U(\varepsilon)} \\
&= 0,
\end{aligned}$$

which implies (4.20).

From the remark on p. 175 in [18] it follows that the upward/downward ladder height process of L has strictly positive/negative drift. We invoke Theorem 42 in [41] to deduce (4.21) and (4.22).

Next, we show that ii) implies i). Suppose that (4.21) and (4.22) hold. Theorem 42 in [41] then implies that the upward/downward ladder height process of L has a strictly positive/negative drift, from which it follows that L must have a Gaussian component. □

We conclude this section by remarking that the (method of) proof of Theorem 4.5 can also be applied to more general strong Markov processes with jumps.

4.6 Example of an optimal stopping problem for which the first hitting time of a certain point is optimal

In this section, we use the method of measure transformation to show that for a particular optimal stopping problem for a spectrally negative Lévy process Z, an optimal stopping time is given by the first time Z hits a certain point. Assume that Z drifts to $+\infty$, i.e. $\lim_{t\to\infty} Z_t = \infty$. We assume that the jump measure Π of Z satisfies

$$\int_{\{x\leq -1\}} e^{\lambda x}\,\Pi(dx) < \infty \quad \text{for all } \lambda < 0. \tag{4.28}$$

From this assumption, it follows from Theorem 25.3 in [104] that

$$\psi(\lambda) := \mathbb{E}[e^{\lambda X_1}]$$

is well defined for all $\lambda \in \mathbb{R}$. It is not difficult to check that ψ is a convex function which is infinitely differentiable on $\mathbb{R}$. Furthermore, since a spectrally negative Lévy process does not have monotone paths, we have that $\mathbb{P}(Z_1 < 0) > 0$ and $\mathbb{P}(Z_1 > 0) > 0$, and thus

$$\lim_{x\to\pm\infty} \psi(\lambda) = \infty.$$

We conclude that ψ has a unique minimum (say at $\lambda = \lambda^*$). Since Z was assumed to drift to ∞, it holds that $\psi'(0) > 0$ and thus $\lambda^* < 0$. The function ψ is monotone decreasing on $(-\infty, \lambda^*]$ and monotone increasing on $[\lambda^*, \infty)$.

Consider the optimal stopping problem (of American put type)

$$V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x[e^{-\psi(\lambda^*)\tau}(K - e^{X_\tau})^+ \mathbf{1}_{\{\tau<\infty\}}], \tag{4.29}$$

where $K > 0$. Note that $\psi(\lambda^*) < 0$ and hence there is a trade-off between the process drifting towards the region with zero pay-off and the increase in value due to the negative discount factor. Since, for any $\lambda \in \mathbb{R}$, the process $\{e^{-\psi(\lambda)t+\lambda(X_t-x)}\}_{t\geq 0}$

is a martingale with mean 1 under $\mathbb{P}_x$, we can introduce the probability measure $\tilde{\mathbb{P}}_x$ by

$$\left.\frac{d\tilde{\mathbb{P}}_x}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = e^{-\psi(\lambda^*)t+\lambda^*(X_t-x)}.$$

We use the monotone convergence theorem to deduce that for any $\tau \in \mathcal{T}$

$$\begin{aligned}
&\mathbb{E}_x[e^{-\psi(\lambda^*)\tau}(K-e^{X_\tau})^+\mathbf{1}_{\{\tau<\infty\}}] \\
&\quad= \lim_{t\to\infty}\mathbb{E}_x[e^{-\psi(\lambda^*)\tau}(K-e^{X_\tau})^+\mathbf{1}_{\{\tau<t\}}] \\
&\quad= \lim_{t\to\infty}\tilde{\mathbb{E}}_x[e^{-\psi(\lambda^*)\tau+\psi(\lambda^*)t-\lambda^*(X_t-x)}(K-e^{X_\tau})^+\mathbf{1}_{\{\tau<t\}}] \\
&\quad= e^{\lambda^* x}\lim_{t\to\infty}\tilde{\mathbb{E}}_x[\tilde{\mathbb{E}}_x[e^{-\psi(\lambda^*)\tau+\psi(\lambda^*)t-\lambda^* X_t}(K-e^{X_\tau})^+\mathbf{1}_{\{\tau<t\}}|\mathcal{F}_\tau]] \\
&\quad= e^{\lambda^* x}\lim_{t\to\infty}\tilde{\mathbb{E}}_x[e^{-\psi(\lambda^*)\tau}(K-e^{X_\tau})^+\tilde{\mathbb{E}}_x[e^{\psi(\lambda^*)t-\lambda^* X_t}\mathbf{1}_{\{\tau<t\}}|\mathcal{F}_\tau]] \\
&\quad= e^{\lambda^* x}\lim_{t\to\infty}\tilde{\mathbb{E}}_x[e^{-\psi(\lambda^*)\tau}(K-e^{X_\tau})^+e^{\psi(\lambda^*)\tau-\lambda^* X_\tau}\mathbf{1}_{\{\tau<t\}}] \\
&\quad= e^{\lambda^* x}\tilde{\mathbb{E}}_x[e^{-\lambda^* X_\tau}(K-e^{X_\tau})^+\mathbf{1}_{\{\tau<\infty\}}].
\end{aligned}$$

We find that we can rewrite (4.29) as

$$V(x) = e^{\lambda^* x}\sup_{\tau\in\mathcal{T}}\tilde{\mathbb{E}}_x[e^{-\lambda^* X_\tau}(K-e^{X_\tau})^+\mathbf{1}_{\{\tau<\infty\}}].$$

The function $m(x) := e^{-\lambda^* x}(K-e^x)$ attains its unique maximum at

$$x^* := \log K + \log(-\lambda^*) - \log(-\lambda^*+1) < \log K$$

and thus

$$V(x) \le e^{\lambda^* x}e^{-\lambda^* x^*}(K-e^{x^*}). \tag{4.30}$$

Under $\tilde{\mathbb{P}}$, the process $\{Z_t\}_{t\ge0}$ still is a spectrally negative Lévy process and we denote its Laplace exponent by $\tilde{\psi}$. Since

$$\tilde{\mathbb{E}}[e^{\lambda Z_1}] = e^{-\psi(\lambda)}\mathbb{E}[e^{(\lambda+\lambda^*)Z_1}] = e^{\psi(\lambda+\lambda^*)-\psi(\lambda)},$$

it follows that

$$\tilde{\mathbb{E}}[Z_1] = \tilde{\psi}'(0) = \psi'(\lambda^*) = 0.$$

For $x \in \mathbb{R}$, we denote the first hitting time of x by T_x, i.e.

$$T_x = \inf\{t \ge 0 : Z_t = x\}.$$

Since an oscillating, spectrally one-sided process hits any point in an almost surely finite time, we find that $\tilde{\mathbb{P}}(T_x) < \infty$ for all $x \in \mathbb{R}$. We deduce that T_{x^*} is a stopping time at which the upper bound (4.30) is attained. We have shown the following result.

Theorem 4.8. *Suppose Z is a spectrally negative Lévy process drifting to $+\infty$ of which the jump measure satisfies condition (4.28). Then an optimal stopping time in (4.29) is given by*

$$T_{x^*} = \inf\{t \geq 0 : Z_t = x^*\},$$

where $x^* = \log K + \log(-\lambda^*) - \log(-\lambda^* + 1)$ and ψ attains its minimum at λ^*. Furthermore,

$$V(x) = \frac{K}{1-\lambda^*} e^{\lambda^*(x-x^*)}.$$

Remark 4.9. The assumption that Z has no positive jumps is not vital here. For a process with two-sided jumps, we can replace condition (4.28) by

$$\int_{\{|x|\geq 1\}} e^{\lambda x}\, \Pi(dx) < \infty \quad \text{for all } \lambda \in \mathbb{R}.$$

A second condition Z needs to satisfy is that, under $\tilde{\mathbb{P}}$, it hits any point in an almost surely finite time.

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Chapter 5

McKean stochastic game for spectrally negative Lévy processes*

We consider the stochastic game analogue of McKean's optimal stopping problem when the underlying source of randomness is a spectrally negative Lévy process. Compared with the solution for linear Brownian motion given in [61], one finds two new phenomena. Firstly, the breakdown of smooth fit and secondly, the stopping domain for one of the players 'thickens' from a singleton to an interval, at least in the case that there is no Gaussian component.

5.1 Introduction

Let $X = \{X_t : t \geq 0\}$ be a Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\{\mathcal{F}_t : t \geq 0\}$ is the filtration generated by X satisfying the usual conditions. For $x \in \mathbb{R}$ denote by $\mathbb{P}_x$ the law of X when it is started at x and write simply $\mathbb{P}_0 = \mathbb{P}$. Accordingly, we shall write $\mathbb{E}_x$ and $\mathbb{E}$ for the associated expectation operators. We shall assume throughout that X is *spectrally negative*, meaning here that it has no positive jumps and that it is not the negative of a subordinator. It is well known that the latter implies that the Laplace exponent $\psi(\theta) := \log \mathbb{E}(e^{\theta X_1})$ is finite for $\theta \geq 0$ and that ψ is a convex function which is infinitely differentiable on $(0, \infty)$. The Laplace exponent of X is of the form

$$\psi(\lambda) = a\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_{(-\infty,0)} (e^{\lambda x} - 1 - \lambda x \mathbf{1}_{\{x>-1\}})\, \Pi(dx), \quad \text{for } \lambda \geq 0 \qquad (5.1)$$

*Based on joint work [14] with A. E. Kyprianou.

where $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and where the jump measure Π of X has zero mass on $[0,\infty)$ as we have assumed that X is spectrally negative and satisfies

$$\int_{(-\infty,0)} (1 \wedge x^2)\, \Pi(dx) < \infty. \tag{5.2}$$

Whenever $\sigma^2 > 0$, we call it the Gaussian component of X.

Denote by $\mathcal{T}_{0,\infty}$ the set of all $[0,\infty]$-valued stopping times with respect to $\mathbb{F}$. We are interested in establishing a solution to a special class of stochastic games which are driven by spectrally negative Lévy processes. Specifically, for a given $K > 0$, $\delta > 0$ and $r \geq 0$, we study the stochastic game consisting of two players and expected pay-off

$$\mathbb{E}_x[e^{-r\tau}(K - e^{X_\tau})^+ \mathbf{1}_{\{\tau \leq \sigma\}} + e^{-r\sigma}((K - e^{X_\sigma})^+ + \delta)\mathbf{1}_{\{\sigma < \tau\}}], \tag{5.3}$$

which the max-player maximises over $\tau \in \mathcal{T}_{0,\infty}$ and the min-player minimises over $\sigma \in \mathcal{T}_{0,\infty}$. The order in which this optimisation takes place could be of importance, as it may occur that

$$\sup_{\tau \in \mathcal{T}_{0,\infty}} \inf_{\sigma \in \mathcal{T}_{0,\infty}} \mathbb{E}_x[e^{-r\tau}(K - e^{X_\tau})^+ \mathbf{1}_{\{\tau \leq \sigma\}} + e^{-r\sigma}((K - e^{X_\sigma})^+ + \delta)\mathbf{1}_{\{\sigma < \tau\}}] \tag{5.4}$$

is strictly smaller than

$$\inf_{\sigma \in \mathcal{T}_{0,\infty}} \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_x[e^{-r\tau}(K - e^{X_\tau})^+ \mathbf{1}_{\{\tau \leq \sigma\}} + e^{-r\sigma}((K - e^{X_\sigma})^+ + \delta)\mathbf{1}_{\{\sigma < \tau\}}]. \tag{5.5}$$

However, we prove (under an assumption on r) the existence of a saddle point. That is, we show that a pair of stopping times (τ^*, σ^*) exists such that

$$\begin{aligned}
&\mathbb{E}_x[e^{-r\tau}(K - e^{X_\tau})^+ \mathbf{1}_{\{\tau \leq \sigma^*\}} + e^{-r\sigma^*}((K - e^{X_{\sigma^*}})^+ + \delta)\mathbf{1}_{\{\sigma^* < \tau\}}] \\
&\quad \leq \mathbb{E}_x[e^{-r\tau^*}(K - e^{X_{\tau^*}})^+ \mathbf{1}_{\{\tau^* \leq \sigma^*\}} + e^{-r\sigma^*}((K - e^{X_{\sigma^*}})^+ + \delta)\mathbf{1}_{\{\sigma^* < \tau^*\}}] \\
&\quad \leq \mathbb{E}_x[e^{-r\tau^*}(K - e^{X_{\tau^*}})^+ \mathbf{1}_{\{\tau^* \leq \sigma\}} + e^{-r\sigma}((K - e^{X_\sigma})^+ + \delta)\mathbf{1}_{\{\sigma < \tau^*\}}]
\end{aligned}$$

for all $\tau, \sigma \in \mathcal{T}_{0,\infty}$ and for all $x \in \mathbb{R}$.

When such a pair of stopping times exist, we say that it solves the stochastic game (5.3) and that these stopping times are optimal. Existence of a saddle point implies that (5.4) equals (5.5) and we denote this common value by V, i.e.

$$V(x) = \mathbb{E}_x[e^{-r\tau^*}(K - e^{X_{\tau^*}})^+ \mathbf{1}_{\{\tau^* \leq \sigma^*\}} + e^{-r\sigma^*}((K - e^{X_{\sigma^*}})^+ + \delta)\mathbf{1}_{\{\sigma^* < \tau^*\}}]$$

for $x \in \mathbb{R}$.
We shall assume (unless otherwise stated) that the parameter r satisfies

$$0 \leq \psi(1) \leq r \text{ and } r > 0. \tag{5.6}$$

Remark 5.1. The assumption that $r > 0$ and the fact that the functions $(K - e^x)^+$ and $(K - e^x)^+ + \delta$ are bounded together imply that it does not matter which pay-off we assign to the event $\{\sigma = \tau = \infty\}$. However, in the case $r = 0$, this is an important issue, for which we refer to Section 5.10 at the end of this chapter.

When $\psi(1) = r > 0$, the stochastic game with pay-off given by (5.3) can be understood to characterise the so-called game option under the risk neutral measure in a simple market consisting of a risky asset whose value is given by $\{e^{X_t} : t \geq 0\}$ and a riskless asset which grows at rate r (cf. [59]). The latter game option is an American-type contract with infinite horizon which offers the holder the right but not the obligation to claim $(K - e^{X_\tau})^+$ at any stopping time $\tau \in \mathcal{T}_{0,\infty}$, but in addition it also gives the writer the right but not the obligation to force a payment of $(K - e^{X_\sigma})^+ + \delta$ at any stopping time $\sigma \in \mathcal{T}_{0,\infty}$. However, in this thesis we do not discuss the financial consequences of the mathematical object (5.3).

The stochastic game (5.3) is closely related to the McKean optimal stopping problem

$$U(x) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_x[e^{-r\tau}(K - e^{X_\tau})^+], \tag{5.7}$$

which, when $r = \psi(1)$, characterises the value of a perpetual American put option (cf. [77]). Indeed, should it be the case that the stochastic saddle point for (5.3) is achieved when $\sigma = \infty$, then $U = V$.

Thanks to a plethora of research papers on the latter topic it is known that an optimal stopping strategy for (5.7) is then given by

$$\tau^* = \inf\{t > 0 : X_t < \log\left(K\mathbb{E}[e^{\underline{X}_{\mathbf{e}_r}}]\right)\}$$

where $\underline{X}_t = \inf_{s \leq t} X_s$ and $\mathbf{e}_r$ is an exponentially distributed random variable with parameter r which is independent of X. We refer to [31] and [81] for the case that X is spectrally negative and the case that X is a general Lévy process respectively. The stochastic game (5.3) may therefore be thought of as a natural extension of the McKean optimal stopping problem and we henceforth refer to this as the *McKean stochastic game*. The McKean stochastic game for a Brownian motion was studied in [61].

The finite horizon version of (5.3) (i.e. the game with the same pay-off but for which both players choose stopping times valued in $[0, T]$ for some fixed $T > 0$) was solved in [64] for a Brownian motion by decomposing into two finite horizon optimal stopping problems.

5.2 The solution to the McKean stochastic game

Below in Theorems 5.4, 5.5 and 5.6, we give a qualitative and quantitative exposition of the solution to (5.3) under the assumption (5.6). Before doing so, we need to give a brief reminder of a class of special functions which appear commonly in connection with the study of spectrally negative Lévy processes and indeed in connection with the description below of the McKean stochastic game. For each $q \geq 0$, we introduce a function $W^{(q)} : \mathbb{R} \to [0, \infty)$ which satisfies for all $x \in \mathbb{R}$ and $a \geq 0$

$$\mathbb{E}_x[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+ < \tau_0^-\}}] = \frac{W^{(q)}(x \wedge a)}{W^{(q)}(a)}, \tag{5.8}$$

where

$$\tau_a^+ := \inf\{t > 0 : X_t > a\} \text{ and } \tau_0^- = \inf\{t > 0 : X_t < 0\},$$

cf. Chapter 8 of [63]. In particular, it is evident that $W^{(q)}(x) = 0$ for all $x < 0$. Note that a function satisfying equation (5.8) multiplied by a constant still satisfies that equation. Furthermore, it is known that for $q \geq 0$, $W^{(q)}(x)$ is almost everywhere differentiable on $(0, \infty)$, right-continuous at zero and, by specifying the appropriate multiplicative constant,

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)\, dx = \frac{1}{\psi(\beta) - q}$$

for all $\beta > \Phi(q)$, where $\Phi(q)$ is the largest root of the equation $\psi(\theta) = q$ (of which there are at most two). For convenience we shall write W instead of $W^{(0)}$. Associated to the functions $W^{(q)}$ are the functions $Z^{(q)} : \mathbb{R} \to [1, \infty)$ defined by

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y)\, dy$$

for $q \geq 0$. Together, the functions $W^{(q)}$ and $Z^{(q)}$ are collectively known as scale functions and predominantly appear in almost all fluctuation identities for spectrally negative Lévy processes. For example, it is also known that for all $x \in \mathbb{R}$ and $a, q \geq 0$,

$$\mathbb{E}_x[e^{-q\tau_0^-} \mathbf{1}_{\{\tau_a^+ > \tau_0^-\}}] = Z^{(q)}(x \wedge a) - \frac{Z^{(q)}(a)}{W^{(q)}(a)} W^{(q)}(x \wedge a) \tag{5.9}$$

and

$$\mathbb{E}_x[e^{-q\tau_0^-} \mathbf{1}_{\{\tau_0^- < \infty\}}] = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x), \tag{5.10}$$

where $q/\Phi(q)$ is to be understood in the limiting sense $\psi'(0) \vee 0$ when $q = 0$.

If we assume that

the jump measure X has no atoms when X has bounded variation

then it is known from existing literature ([32; 96]) that $W^{(q)} \in C^1(0, \infty)$ and hence $Z^{(q)} \in C^2(0, \infty)$. Further, if X has a Gaussian component, they both belong to $C^\infty(0, \infty)$. For computational convenience we shall proceed with the above assumption on X. It is also known that if X has bounded variation with drift[†] d, then $W^{(q)}(0) = 1/\mathrm{d}$ and otherwise $W^{(q)}(0) = 0$. Further,

$$W^{(q)\prime}(0+) = \begin{cases} \frac{2}{\sigma^2} & \text{if } \sigma > 0, \\ \frac{\Pi(-\infty,0)+q}{\mathrm{d}^2} & \text{if } X \text{ is of bounded variation with } \Pi(-\infty, 0) < \infty, \\ \infty & \text{otherwise.} \end{cases} \tag{5.11}$$

For completeness, we include the proof.

Lemma 5.2. *Let $q \geq 0$. The function $x \to W^{(q)}(x)$ is continuous on $\mathbb{R}$ if and only if X is of unbounded variation. Also, when X is of bounded variation, $W^{(q)}(0) = 1/\mathrm{d}$. Finally, $W^{(q)\prime}(0+)$ is given by (5.11).*

[†]Here and in the sequel we take the canonical representation of a bounded variation Lévy process $X_t = \mathrm{d}t - S_t$ for $t \geq 0$ where $\{S_t : t \geq 0\}$ is a driftless subordinator and d is a strictly positive constant which is referred to as the *drift*.

Proof. From a Tauberian theorem we know that

$$W^{(q)}(0) = \lim_{\lambda\to\infty} \frac{\lambda}{\psi(\lambda) - q}.$$

The process $\{X_t\}_{t\geq 0}$ is of unbounded variation when

$$\sigma > 0 \quad \text{or} \quad \int_{(-1,0)} |x|\,\Pi(dx) = \infty,$$

in which case $W^{(q)}(0) = 0$ follows directly from (5.1). When X is of bounded variation with drift d, the Laplace exponent of X can be written as

$$\psi(\lambda) = \lambda\left(\mathrm{d} - \int_0^\infty e^{-\lambda x}\Pi(-\infty,-x)\,dx\right)$$

and now the value of $W^{(q)}(0)$ follows. Similarly, it follows from a Tauberian theorem and integration by parts that

$$\begin{aligned}
W^{(q)\prime}(0+) &= \lim_{\lambda\to\infty} \lambda \int_{(0,\infty)} e^{-\lambda x} W^{(q)}\,(dx) \\
&= \lim_{\lambda\to\infty} \lambda \left(\int_{[0,\infty)} e^{-\lambda x} W^{(q)}\,(dx) - W^{(q)}(0)\right) \\
&= \lim_{\lambda\to\infty} \lambda \left(\int_0^\infty \lambda e^{-\lambda x} W^{(q)}(x)\,dx - W^{(q)}(0)\right) \\
&= \lim_{\lambda\to\infty} \left(\frac{\lambda^2}{\psi(\lambda)-q} - \lambda W^{(q)}(0)\right)
\end{aligned}$$

When X is of unbounded variation, we have just shown that $W^{(q)}(0) = 0$ and expression (5.11) now follows from Proposition 2 in Chapter 1 in [18]. When X is of bounded variation, we find that

$$\begin{aligned}
W^{(q)\prime}(0+) &= \lim_{\lambda\to\infty} \frac{\lambda^2 \int_{(0,\infty)} e^{-\lambda x}\Pi(-\infty,-x)\,dx + q\lambda}{\mathrm{d}\lambda(\mathrm{d} - \int_{(0,\infty)} e^{-\lambda x}\Pi(-\infty,-x)\,dx) - q\mathrm{d}} \\
&= \lim_{\lambda\to\infty} \frac{\lambda \int_{(0,\infty)} e^{-\lambda x}\Pi(-\infty,-x)\,dx + q}{\mathrm{d}^2 - \int_{(0,\infty)} e^{-\lambda x}\Pi(-\infty,-x)\,dx} \\
&= \frac{\Pi(-\infty,0) + q}{\mathrm{d}^2},
\end{aligned}$$

which implies (5.11). □

Consider the exponential change of measure

$$\left.\frac{d\mathbb{P}^1}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{X_t - \psi(1)t}. \tag{5.12}$$

Under $\mathbb{P}^1$, X is still a spectrally negative Lévy process and we mark its Laplace exponent and scale functions with the subscript 1. It holds that

$$\begin{aligned}\psi_1(\lambda) &= \log\left(\mathbb{E}^1[e^{\lambda X_1}]\right)\\ &= \log\left(e^{-\psi(1)}\mathbb{E}[e^{(\lambda+1)X_1}]\right)\\ &= \psi(1+\lambda)-\psi(1),\end{aligned} \tag{5.13}$$

for $\lambda \geq 0$ and, by taking Laplace transforms, we find for $q \geq 0$

$$W_1^{(q)}(x) = e^{-x}W^{(q+\psi(1))}(x) \quad \text{for all } x \in \mathbb{R}. \tag{5.14}$$

Indeed, for $\lambda \geq 0$ such that $\psi_1(\lambda > q)$,

$$\int_0^\infty e^{-\lambda x}e^{-x}W^{(q+\psi(1))}(x)\,dx = \frac{1}{\psi(1+\lambda)-q-\psi(1)} = \frac{1}{\psi_1(\lambda)-q}.$$

The reader is otherwise referred to Chapter VII of [18] or Chapter 8 of [63] for a general overview of scale functions of spectrally negative Lévy processes.

For comparison with the main results in Theorems 5.4, 5.5 and 5.6 below, we give the solution to the McKean optimal stopping problem as it appears in [31] and [81].

Theorem 5.3. *For the McKean optimal stopping problem under (5.6) we have*

$$U(x) = KZ^{(r)}(x-k^*) - e^x Z_1^{(r-\psi(1))}(x-k^*), \tag{5.15}$$

where

$$e^{k^*} = K\frac{r}{\Phi(r)}\frac{\Phi(r)-1}{r-\psi(1)}. \tag{5.16}$$

This is to be understood in the limiting sense when $r = \psi(1)$, i.e.

$$e^{k^*} = K\psi(1)/\psi'(1) \quad \textit{when } r = \psi(1).$$

We now return to the solution of the McKean stochastic game and present our main results in terms of scale functions.

Theorem 5.4. *Consider the McKean stochastic game under the assumption (5.6).*

(i) If $\delta \geq U(\log K)$, then $V = U$.

(ii) If $\delta < U(\log K)$ a stochastic saddle point to (5.3) is given by the pair

$$\tau^* = \inf\{t > 0 : X_t < x^*\} \textit{ and } \sigma^* = \inf\{t > 0 : X_t \in [\log K, y^*]\},$$

where x^* uniquely solves

$$Z^{(r)}(\log K - x) - Z_1^{(r-\psi(1))}(\log K - x) = \frac{\delta}{K},$$

$x^* > k^*$ (the optimal level of the corresponding McKean optimal stopping problem in Theorem 5.3) and $y^* \in [\log K, z^*]$, where z^* is the unique solution to

$$Z^{(r)}(z-\log K) - \frac{r}{\Phi(r)}W^{(r)}(z-\log K) = \frac{\delta}{K}. \tag{5.17}$$

The next theorem gives partial information on the value of y^*. Unfortunately, we are unable to give a complete characterisation of y^*.

Theorem 5.5. *Suppose in Theorem 5.4 that $\delta < U(\log K)$. If X has no Gaussian component, then $y^* > \log K$ and necessarily $\Pi(-\infty, \log K - y^*) > 0$, where Π is the jump measure of X.*

The question whether $y^* = \log K$ is more difficult to answer when the Gaussian component of X is positive and we refer to Section 5.8 for a discussion on this case.

For practical purposes, one would also like to be able to characterise y^* as the unique solution to some functional equation. Experience in the theory of optimal stopping shows that this often follows as a consequence of, for example, a smooth pasting condition. In this case, despite the fact that we are able to make decisive statements about pasting of the value function onto the upper and lower gain functions (see Theorem 5.6 below), the desired characterisation of y^* does not seem to be available.

Our last main result gives information concerning the analytical shape of the value function V. Define the function $j : \mathbb{R} :\to \mathbb{R}$ by

$$j(x) = KZ^{(r)}(x - x^*) - e^x Z_1^{(r-\psi(1))}(x - x^*) + \alpha e^{\Phi(r)(\log K - x^*)} W^{(r)}(x - \log K), \tag{5.18}$$

where

$$\alpha = e^{x^*} \frac{r - \psi(1)}{\Phi(r) - 1} - \frac{rK}{\Phi(r)},$$

which is to be understood in the limiting sense when $r = \psi(1)$, i.e.

$$\alpha = e^{x^*} \psi'(1) - K\psi(1) \quad \text{when } r = \psi(1).$$

Theorem 5.6. *For the McKean stochastic game under the assumption (5.6), when $\delta < U(\log K)$, V is continuous everywhere. In particular*

$$V(x) = KZ^{(r)}(x - x^*) - e^x Z_1^{(r-\psi(1))}(x - x^*) \tag{5.19}$$

for $x \in (-\infty, \log K]$ and $V(x) = \delta$ for $x \in [\log K, y^*]$. Further, if $y^* = \log K$, then for any $x \in \mathbb{R}$

$$V(x) = j(x).$$

Moreover,

(i) if X has unbounded variation, then there is smooth pasting at x^*. Further, there is smooth pasting at y^* if and only if $y^* > \log K$,

(ii) if X has bounded variation, then there is no smooth pasting at x^* and no smooth pasting at y^*.

Note that it is in fact possible to show that V is everywhere differentiable except possibly at x^*, y^* and $\log K$. This is clear from the expression for $V(x)$ on $x \in (-\infty, y^*)$. However, when $y^* > \log K$, for the region $V(x) \in (y^*, \infty)$ things are less clear without an expression for V. Nonetheless, in this case it is possible to use the compensation formula with the help of potential densities (which themselves can be written in terms of the scale functions) to write down a formula for

$$V(x) = \mathbb{E}_x \left[e^{-r\tau^-_{x^*}} (K - e^{X_{\tau^-_{x^*}}}) \mathbf{1}_{\{\tau^-_{x^*} < \tau_{[\log K, y^*]}\}} + \delta e^{-r\tau_{[\log K, y^*]}} \mathbf{1}_{\{\tau_{[\log K, y^*]} < \tau^-_{x^*}\}} \right], \tag{5.20}$$

where $\tau_{[\log K, y^*]} = \inf\{t > 0 : X_t \in [\log K, y^*]\}$. However, this formula is rather lengthy and, simply for the sake of brevity, we refrain from including it here. It may be possible to use this formula and the pasting conditions to find y^*. If we write $V(x, y^*)$ to show the dependence in (5.20) on the choice of y^*, then it holds that y^* satisfies

$$V(y^*, y^*+) = \delta \quad \text{when } X \text{ is of bounded variation}$$

and

$$\frac{\partial}{\partial x} V(y^*, y^*+) = 0 \quad \text{when } X \text{ is of unbounded variation.}$$

However, it seems difficult to show that a solution to these equations is unique.

There are a number of remarks which are worth making about the above three theorems.

Theorem 5.4 (i) follows as a consequence of the same reasoning that one sees for the case that X is a linear Brownian motion in [61]. That is to say, when $\delta \geq U(\log K)$ it follows that $U(x) \leq (K - e^x)^+ + \delta$ showing that the min-player would not be behaving optimally by stopping in a finite time. The proof of this fact is virtually identical to the proof given in [61] with the help of the Verification Lemma in the next section and so we leave this part of the proof of Theorem 5.4 (i) as an exercise.

We shall henceforth assume that $U(\log K) < \delta$.

For the McKean stochastic game when X is a linear Brownian motion and $r = \psi(1) > 0$, it was shown in [61] that when δ is small enough, the optimal stopping strategies for the max-player and min-player are respectively given by

$$\tau^* = \inf\{t > 0 : X_t < x^*\} \text{ and } \sigma^* = \inf\{t > 0 : X_t = \log K\},$$

for some $x^* < \log K$. Also it was shown there that the solution is convex and that there is smooth pasting at x^*. For spectrally negative Lévy processes in general, Theorems 5.4–5.6 show that considerably different behaviour occurs.

Firstly, as was already found in numerous papers concerning optimal stopping problems driven by spectrally one sided Lévy processes (cf. [1], [5] and [31]), smooth pasting breaks down when the Lévy process is of bounded variation. Secondly and more interestingly, the different form of the stopping region for the min-player can be understood intuitively by the following reasoning. In the linear Brownian motion case there is no possibility for the process started at $x > \log K$ to enter $(-\infty, \log K]$ without hitting $\{\log K\}$. The positive discount rate r and the constant pay-off on $[\log K, \infty)$ imply that in this case it does not make sense for the min-player to stop anywhere on $(\log K, \infty)$. However, when X has negative jumps there is a positive probability to jump below points. When X starts at a value which is slightly greater than $\log K$, there is the danger (for the min-player) that X jumps to a large negative value, which could in principle induce a big pay-off to the max-player. The trade-off between this fact and the positive discount rate r when there is no Gaussian component results in the interval hitting strategy for the min-player indicated by Theorem 5.5. Also, note in this instance that $\Pi(-\infty, \log K - y^*) > 0$ implies that when $X_0 > y^*$ the max-player may still jump over the stopping interval of the min-player and possibly stop the game by entering $(-\infty, x^*)$. This is also a new feature of the optimal strategies compared with the linear Brownian motion case as in the latter context, when $X_0 > y^*$, the max-player will never exercise before the min-player.

This chapter continues with the following structure. In the next section, we present a set of sufficient conditions to check for a solution to the McKean stochastic game. Following that, in Sections 5.4 and 5.5 we present a description of the candidate solution in the regions $x \leq \log K$ and $x > \log K$. To some extent, the solution may be decoupled into these two regions thanks to the spectral negativity of the underlying process. In Section 5.6 we show that the previously described candidate solution fulfils the sufficient conditions outlined in Section 5.3 thus proving Theorem 5.4. Finally, in Sections 5.7 and 5.9 we give the proofs of Theorems 5.5 and 5.6, respectively.

5.3 Verification technique

To keep calculations brief and in order to avoid repetition of ideas, it is worth stating upfront the fundamental technique which leads to establishing the existence and hence characterisation of a solution. This comes in the form of the following Verification Lemma.

Lemma 5.7 (Verification Lemma). *Consider the stochastic game (5.3) with $r > 0$. Suppose that $\tau^* \in \mathcal{T}_{0,\infty}$ and $\sigma^* \in \mathcal{T}_{0,\infty}$ are candidate optimal strategies for the stochastic game (5.3) and let*

$$V^*(x) = \mathbb{E}_x[e^{-r\tau^*}(K - e^{X_{\tau^*}})^+ \mathbf{1}_{\{\tau^* \leq \sigma^*\}} + e^{-r\sigma^*}((K - e^{X_{\sigma^*}})^+ + \delta)\mathbf{1}_{\{\sigma^* < \tau^*\}}].$$

Then the triple (V^*, τ^*, σ^*) is a solution to (5.3) if

(i) $V^*(x) \geq (K - e^x)^+$,

(ii) $V^*(x) \leq (K - e^x)^+ + \delta$,

(iii) $V^*(X_{\tau^*}) = (K - e^{X_{\tau^*}})^+$ *almost surely on* $\{\tau^* < \infty\}$,

(iv) $V^*(X_{\sigma^*}) = (K - e^{X_{\sigma^*}})^+ + \delta$ *almost surely on* $\{\sigma^* < \infty\}$,

(v) the process $\{e^{-r(t\wedge\tau^*)}V^*(X_{t\wedge\tau^*}) : t \geq 0\}$ *is a right-continuous submartingale and*

(vi) the process $\{e^{-r(t\wedge\sigma^*)}V^*(X_{t\wedge\sigma^*}) : t \geq 0\}$ *is a right-continuous supermartingale.*

Proof. Let $r > 0$. For convenience, write $G(x) = (K - e^x)^+$, $H(x) = (K - e^x)^+ + \delta$ and

$$\Theta^r_{\tau,\sigma} = e^{-r\tau} G(X_\tau)\mathbf{1}_{\{\tau \le \sigma\}} + e^{-r\sigma} H(X_\sigma)\mathbf{1}_{\{\sigma < \tau\}}.$$

We remark again that the assumption $r > 0$ implies that we do not have to worry about having to assign a value $G(X_\infty)$ to the event $\{\tau = \sigma = \infty\}$. From the supermartingale property (vi), Doob's optional stopping theorem, (iv) and (i) we know that for any $\tau \in \mathcal{T}_{0,\infty}$ and $t \ge 0$,

$$\begin{aligned} V^*(x) &\ge \mathbb{E}_x[e^{-r(t\wedge\tau\wedge\sigma^*)} V^*(X_{t\wedge\tau\wedge\sigma^*})] \\ &\ge \mathbb{E}_x[e^{-r(t\wedge\tau)} G(X_{t\wedge\tau})\mathbf{1}_{\{\sigma^* \ge t\wedge\tau\}} + e^{-r\sigma^*} H(X_{\sigma^*})\mathbf{1}_{\{\sigma^* < t\wedge\tau\}}]. \end{aligned}$$

It follows from Fatou's lemma by taking $t \uparrow \infty$ that

$$V^*(x) \ge \mathbb{E}_x[\Theta^r_{\tau,\sigma^*}].$$

Also, using (v), Doob's optional stopping theorem, (iii) and (ii), we have for any $\sigma \in \mathcal{T}_{0,\infty}$ and $t \ge 0$,

$$\begin{aligned} V^*(x) &\le \mathbb{E}_x[e^{-r(t\wedge\tau^*\wedge\sigma)} V^*(X_{t\wedge\tau^*\wedge\sigma})] \\ &= \mathbb{E}_x[e^{-r\tau^*} V^*(X_{\tau^*})\mathbf{1}_{\{\tau^* \le t\wedge\sigma\}} + e^{-r(t\wedge\sigma)} V^*(X_{t\wedge\sigma})\mathbf{1}_{\{\tau^* > t\wedge\sigma\}}] \\ &\le \mathbb{E}_x[e^{-r\tau^*} G(X_{\tau^*})\mathbf{1}_{\{\tau^* \le t\wedge\sigma\}} + e^{-r(t\wedge\sigma)} H(X_{t\wedge\sigma})\mathbf{1}_{\{\tau^* > t\wedge\sigma\}}]. \end{aligned}$$

Taking limits as $t \uparrow \infty$, applying the dominated convergence theorem, taking note of the non-negativity of G and of the fact that $\Theta^r_{\infty,\infty} = 0$, we have

$$V^*(x) \le \mathbb{E}_x[\Theta^r_{\tau^*,\sigma}],$$

from which it follows that (τ^*, σ^*) is a saddle point. □

5.4 Candidature on $x \le \log K$

Here, we describe analytically a proposed solution when $X_0 \in (-\infty, \log K]$.

Lemma 5.8. *For $x \in (-\infty, \log K]$ define*

$$w(x) = KZ^{(r)}(x - x^*) - e^x Z_1^{(r-\psi(1))}(x - x^*), \tag{5.21}$$

where $x^* > k^*$ uniquely solves

$$Z^{(r)}(\log K - x) - Z_1^{(r-\psi(1))}(\log K - x) = \frac{\delta}{K}. \tag{5.22}$$

Then w has the following properties on $(-\infty, \log K]$,

(i) $w(x) = \mathbb{E}_x[e^{-r\tau^+_{\log K}} \delta \mathbf{1}_{\{\tau^+_{\log K} < \tau^-_{x^*}\}} + e^{-r\tau^-_{x^*}}(K - e^{X_{\tau^-_{x^*}}})\mathbf{1}_{\{\tau^-_{x^*} < \tau^+_{\log K}\}}]$,

(ii) $w(x) \ge (K - e^x)^+$,

(iii) $w(x) \le (K - e^x)^+ + \delta$,

(iv) the derivative at x^* *is computed as follows:*

$$w'(x^*+) = -e^{x^*} \quad \text{if } X \text{ has unbounded variation}$$

and

$$w'(x^*) = -e^{x^*} + \frac{Kr - (r - \psi(1))e^{x^*}}{\mathrm{d}} \quad \text{if } X \text{ has bounded variation,}$$

where d *is the drift term,*

(v) w *is decreasing,*

(vi) $w(X_{\tau^+_{\log K}}) = \delta$ *on* $\{\tau^+_{\log K} < \infty, X_0 \le \log K\}$,

(vii) $w(X_{\tau^-_{x^*}}) = (K - e^{X_{\tau^-_{x^*}}})$ *on* $\{\tau^-_{x^*} < \infty\}$,

(viii) $\{e^{-r(t\wedge\tau^-_{x^*}\wedge\tau^+_{\log K})}w(X_{t\wedge\tau^-_{x^*}\wedge\tau^+_{\log K}}) : t \ge 0\}$ *is a* $\mathbb{P}_x$*-martingale for* $x \le \log K$ *and*

(ix) $\{e^{-r(t\wedge\tau^-_{x^*}\wedge\tau^+_{\log K})}w(X_{t\wedge\tau^+_{\log K}}) : t \ge 0\}$ *is a* $\mathbb{P}_x$*-supermartingale for* $x \le \log K$.

Proof. First, note that the left hand side of (5.22) is equal to

$$h(x) := \int_0^{\log K - x} (\psi(1)e^{-y} - r(e^{-y} - 1))W^{(r)}(y)\,dy$$

which is a decreasing continuous function in x. Further, $h(\log K) = 0$ and so we need to show that $h(-\infty) > \delta$ in order to deduce that x^* is uniquely defined. From Theorem 5.3 we have that $U(\log K) = Kh(k^*)$ (see (5.16) for the definition of k^*). Hence, by monotonicity and the assumption on the size of δ,

$$h(-\infty) \ge h(k^*) = U(\log K)/K > \delta/K.$$

It also follows immediately from this observation that $x^* > k^*$.

Next, denote by $w(x)$ the right hand side of (5.21). The remainder of the proof consists of verifying that w fulfils conditions (i) to (ix) of Lemma 5.8. We label the proof in parts accordingly.

(i) Using (5.8) and (5.9) and the exponential change of measure (5.12), we find that for $x \le \log K$

$$
\begin{aligned}
&\mathbb{E}_x[e^{-r\tau^+_{\log K}}\delta \mathbf{1}_{\{\tau^+_{\log K} < \tau^-_{x^*}\}} + e^{-r\tau^-_{x^*}}(K - e^{X_{\tau^-_{x^*}}})\mathbf{1}_{\{\tau^-_{x^*} < \tau^+_{\log K}\}}] \\
&\quad = \delta \frac{W^{(r)}(x-x^*)}{W^{(r)}(\log K - x^*)} + K\left(Z^{(r)}(x-x^*) - \frac{W^{(r)}(x-x^*)Z^{(r)}(\log K - x^*)}{W^{(r)}(\log K - x^*)}\right) \\
&\quad\quad - e^x \mathbb{E}^1_x[e^{-(r-\psi(1))\tau^-_{x^*}}\mathbf{1}_{\{\tau^-_{x^*} < \tau_{\log K, y^*}\}}] \\
&\quad = \delta \frac{W^{(r)}(x-x^*)}{W^{(r)}(\log K - x^*)} + K\left(Z^{(r)}(x-x^*) - \frac{W^{(r)}(x-x^*)Z^{(r)}(\log K - x^*)}{W^{(r)}(\log K - x^*)}\right) \\
&\quad\quad - e^x\left(Z_1^{(r-\psi(1))}(x-x^*) - \frac{W_1^{(r-\psi(1))}(x-x^*)Z_1^{(r-\psi(1))}(\log K - x^*)}{W_1^{(r-\psi(1))}(\log K - x^*)}\right) \\
&\quad = \frac{W^{(r)}(x-x^*)}{W^{(r)}(\log K - x^*)}\left(\delta - KZ^{(r)}(\log K - x^*) + KZ_1^{(r-\psi(1))}(\log K - x^*)\right) \\
&\quad\quad + KZ^{(r)}(x-x^*) - e^x Z_1^{(r-\psi(1))}(x-x^*) \\
&\quad = w(x),
\end{aligned}
$$

where the last equality follows from the definition of x^* in (5.22).

(ii) By definition

$$
w(x) = K - e^x + \int_0^{x-x^*} r(K - e^{x-y})W^{(r)}(y) + \psi(1)e^{x-y}W^{(r)}(y)\,dy.
$$

For any $x \le \log K$, the integrand on the right hand side above is positive and hence $w(x) \ge K - e^x$ for $x \le \log K$.

(iii) We also see that

$$
\begin{aligned}
w(x) - (K - e^x) &= \int_0^{x-x^*} (rK + e^{x-y}(\psi(1) - r))W^{(r)}(y)\,dy \\
&= \int_{x^*}^{x} (r(K - e^z) + \psi(1)e^z)W^{(r)}(x-z)\,dz
\end{aligned}
$$

is increasing in x on $[x^*, \log K]$, which implies that for any $x \le \log K$

$$
w(x) \le K - e^x + \int_0^{\log K - x^*} KrW^{(r)}(y) - cKW_1^{(r-\psi(1))}(y)\,dy = K - e^x + \delta,
$$

where $c = r - \psi(1) \ge 0$.

(iv) For any $x \in (-\infty, \log K)\backslash\{x^*\}$, the derivative of w is given by

$$
w'(x) = -e^x + KrW^{(r)}(x-x^*) - ce^{x^*}W^{(r)}(x-x^*) - ce^x\int_0^{x-x^*} e^{-y}W^{(r)}(y)\,dy.
$$

Taking limits as $x \downarrow x^*$, the stated result follows from Lemma 5.2.

(v) Taking the expression for the value function, U, of the McKean optimal stopping problem (5.7) recall that $x^* > k^*$ where k^* is the optimal level for (5.7). It is also known that U is convex and decreasing in x. Hence, for any $x > k^*$

$$U'(x) = KrW^{(r)}(x-k^*) - e^x Z_1^{(r-\psi(1))}(x-k^*) - ce^x W_1^{(r-\psi(1))}(x-k^*) < 0.$$

Since we have that $x^* > k^*$ we deduce that for $x > x^*$

$$\begin{aligned} w'(x) &= KrW^{(r)}(x-x^*) - e^x Z_1^{(r-\psi(1))}(x-x^*) - ce^x W_1^{(r-\psi(1))}(x-x^*) \\ &< KrW^{(r)}(x-x^*) - e^{x+k^*-x^*} Z_1^{(r-\psi(1))}(x-x^*) \\ &\quad - ce^{x+k^*-x^*} W_1^{(r-\psi(1))}(x-x^*) \\ &= U'(x+k^*-x^*) < 0. \end{aligned}$$

(vi) and (vii) These two conditions follow by inspection using (5.22) in the case of (vi) and the fact that $Z^{(q)}(x) = 1$ for all $x \leq 0$ in the case of (vii).

(viii) From (i), (vi) and (vii) we deduce from the strong Markov property that for $X_0 = x \leq \log K$ we have that

$$\begin{aligned} &\mathbb{E}_x[e^{-r\tau^+_{\log K}}\delta \mathbf{1}_{\{\tau^+_{\log K} < \tau^-_{x^*}\}} + e^{-r\tau^-_{x^*}}(K - e^{X_{\tau^-_{x^*}}})\mathbf{1}_{\{\tau^-_{x^*} < \tau^+_{\log K}\}} | \mathcal{F}_{t\wedge\tau^-_{x^*}\wedge\tau^+_{\log K}}] \\ &= e^{-r(t\wedge\tau^-_{x^*}\wedge\tau^+_{\log K})} w(X_{t\wedge\tau^-_{x^*}\wedge\tau^+_{\log K}}) \end{aligned}$$

and now by the tower property of conditional expectation we observe the required martingale property.

(ix) Noting that w is a $C^2(x^*, \log K)$ function, a standard computation involving Itô's formula shows that $(\Gamma - r)w = 0$ on $(x^*, \log K)$ thanks to the just established martingale property. For $x < x^*$ we have that

$$(\Gamma - r)w(x) = (\Gamma - r)(K - e^x) = (-r - \psi(1))e^x < 0,$$

where Γ is the infinitesimal generator of X. Despite the conclusion of part (iv) for the case of bounded variation, the function w is smooth enough to allow one to use the change of variable formula in the case of bounded variation, and the classical Itô formula in the case of unbounded variation (cf. [70] and [98]) to show that, in light of the above inequality, $\{e^{-r(t\wedge\tau^+_{\log K})} w(X_{t\wedge\tau^+_{\log K}}) : t \geq 0\}$ is a $\mathbb{P}_x$-supermartingale for $x \leq \log K$. □

5.5 Candidature on $x > \log K$

In this section we give an analytical and probabilistic description of a proposed solution when $X_0 > \log K$.

Lemma 5.9. *Define the function $v : \mathbb{R} \to [0, K]$ by*

$$v(x) = \inf_{\sigma \in \mathcal{T}_{0,\infty}} \mathbb{E}_x[e^{-r(\sigma\wedge\tau^-_{\log K})} w_\delta(X_{\tau^-_{\log K}\wedge\sigma})]$$

where $w_\delta(x) = w(x)$ *given in (5.21) for* $x \leq \log K$ *and* $w_\delta(x) = \delta$ *for* $x > \log K$. *Then* v *has the following properties,*

(i) $v(x) = w(x)$ *for* $x < \log K$,

(ii) $v(x) \geq (K - e^x)^+$ *for* $x \in \mathbb{R}$,

(iii) $v(x) \leq (K - e^x)^+ + \delta$ *for* $x \in \mathbb{R}$,

(iv) $v(x)$ *is non-increasing,*

(v) there exists a $y^* \geq \log K$ *such that*

$$v(x) = \mathbb{E}_x[e^{-r\tau^-_{y^*}} w_\delta(X_{\tau^-_{y^*}})],$$

(vi) if $y^* = \log K$, *then* X *has a Gaussian component and for* $x \in \mathbb{R}$

$$v(x) = j(x), \tag{5.23}$$

where j *was defined in (5.18),*

(vii) $y^* \leq z^*$, *where* z^* *was defined as the unique solution of (5.17),*

(viii) $v(X_{\tau^-_{x^*}}) = (K - e^{X_{\tau^-_{x^*}}})$ *on* $\{\tau^-_{x^*} = \tau^-_{y^*} < \infty, X_0 \leq \log K\}$,

(ix) $v(X_{\tau^-_{y^*}}) = \delta$ *on* $\{\tau_{[\log K, y^*]} = \tau^-_{y^*} < \infty\}$ *where*

$$\tau_{[\log K, y^*]} = \inf\{t > 0 : X_t \in [\log K, y^*]\},$$

(x) $\{e^{-r(t\wedge\tau^-_{y^*})} v(X_{t\wedge\tau^-_{y^*}}) : t \geq 0\}$ *is a* $\mathbb{P}_x$*-martingale for* $x > \log K$,

(xi) $\{e^{-r(t\wedge\tau^-_{\log K})} v(X_{t\wedge\tau^-_{\log K}}) : t \geq 0\}$ *is a* $\mathbb{P}_x$*-submartingale for* $x > \log K$.

Proof. (i) Note that when $x < \log K$ we have $\mathbb{P}_x(\tau^-_{\log K} = 0) = 1$ so that $v(x) = w(x)$.

(ii) and (iii) These are trivial to verify in light of (i).

(iv) Denote $X^*_t = X_{t\wedge\tau^-_{\log K}}$ for all $t \geq 0$. Since w_δ is a continuous function and since X^* is quasi-left continuous we can deduce that v is upper semi-continuous. Furthermore, w_δ is bounded and continuous so we can apply a variant of Corollary 1.2.7 in [92]† to conclude that there exists an optimal stopping time, say σ^*, which without loss of generality we assume to be not greater than $\tau^-_{\log K}$. By considering the stopping time $\sigma = \infty$, we see by its definition that $v(x) < K\mathbb{E}_x[e^{-r\tau^-_{\log K}}]$ and hence $\lim_{x\uparrow\infty} v(x) = 0$. From the latter we deduce that the set defined by

$$C := \{x > \log K : v(x) < \delta\}$$

†See in particular their remarks at the end of section 1.1.1 and at the end of sections 2.1.1 and 2.1.2.

is non-empty. The upper semi-continuity of v implies that this set is open. Corollary 1.2.7 of [92] also implies that σ^* is the first entry into the set $\mathbb{R}\backslash C$.

In what follows, if ζ is a stopping time for X we shall write $\zeta(x)$ to show the dependence of the stopping time on the value of $X_0 = x$. For $x > y > \log K$ we have that $\tau^-_{\log K}(x) \geq \tau^-_{\log K}(y)$ and thus, also appealing to the definition of v as an infimum,

$$\begin{aligned} v(x) - v(y) &\leq \mathbb{E}\Big[e^{-r(\tau^-_{\log K}(x)\wedge\sigma^*(y))}w_\delta(X_{\tau^-_{\log K}(x)\wedge\sigma^*(y)} + x) \\ &\qquad\qquad -e^{-r(\tau^-_{\log K}(y)\wedge\sigma^*(y))}w_\delta(X_{\tau^-_{\log K}(y)\wedge\sigma^*(y)} + y)\Big] \\ &\leq \mathbb{E}\left[e^{-r(\tau^-_{\log K}(y)\wedge\sigma^*(y))}(w_\delta(X_{\sigma^*(y)} + x) - w_\delta(X_{\sigma^*(y)} + y))\right] \quad (5.24) \\ &\leq 0, \end{aligned}$$

where in the second inequality we have used that $\sigma^*(y) \leq \tau^-_{\log K}(y) \leq \tau^-_{\log K}(x)$ and from Lemma 5.8 (v), w_δ is a decreasing function.

(v) The fact that v is non-increasing and that C, defined above, is open implies that there exists a $y^* \geq \log K$ such that $C = (y^*, \infty)$. In that case $\sigma^* = \tau^-_{y^*}$.

(vi) By the dynamic programming principle, taking into account the fact that $w_\delta = w$ for $x \leq \log K$, it follows that

$$v(x) = \mathbb{E}_x[e^{-r\tau^-_{x^*}}(K - e^{X_{\tau^-_{x^*}}})\mathbf{1}_{\{\tau^-_{x^*} < T_K\}} + e^{-rT_K}\delta\mathbf{1}_{\{T_K < \tau^-_{x^*}\}}].$$

It is shown in the Appendix that the right hand side above is equal to the right hand side of (5.23).

Now, assume that X has no Gaussian component and suppose for contradiction that $y^* = \log K$. If X has bounded variation with drift d, Lemma 5.2 entails that

$$\begin{aligned} v(\log K+) &= KZ^{(r)}(\log K - x^*) - KZ_1^{(r-\psi(1))}(\log K - x^*) + e^{\Phi(r)(\log K - x^*)}\frac{\alpha}{\mathrm{d}} \\ &= \delta + e^{\Phi(r)(\log K - x^*)}\frac{\alpha}{\mathrm{d}} > \delta. \end{aligned}$$

Note that we have used the fact that since $k^* < x^* < \log K$ where k^* is the optimal crossing boundary in the McKean optimal stopping problem (cf. Theorem 5.3), we have that $\alpha > 0$. Taking account of part (iii) of this Lemma we thus have a contradiction. When X has unbounded variation with no Gaussian component, we know from Lemma 5.2 that $W^{(r)\prime}(0+) = \infty$ and hence one can deduce that $v'(\log K+) = \infty$, which again leads to a violation of the upper bound in (iii).

(vii) First, we need to prove that z^* in (5.17) is well-defined and that $y^* \leq z^*$. Denote by $k(z)$ the left hand side of (5.17). We first show that $k(\log K+) > \delta/K$. As we remarked in the proof of (iv),

$$v(z) < K\mathbb{E}_z[e^{-r\tau^-_{\log K}}] = Kk(z),$$

where the equality follows from (5.10). We use (vi) to show that $v(\log K+) = \delta$. When X has no Gaussian component this follows from the fact that $y^* > \log K$

and when X has a Gaussian component this follows from continuity of the function j. It thus holds that $k(\log K+) > \delta/K$. Note that k is a continuous function on $(\log K, \infty)$. From (5.10) it follows that k decreases on $(\log K, \infty)$ and that $\lim_{z\to\infty} k(z) = 0$. Hence, there exists a unique $z^* \in (\log K, \infty)$ such that $k(z^*) = \delta/K$. For $z > z^*$

$$v(z) < Kk(z) < Kk(z^*) = \delta,$$

which implies $y^* \le z^*$.

(viii) and (ix) These are trivial statements.

(x) and (xi) These are standard results from the theory of optimal stopping. See for example Theorem 1.2.2. of [92]. □

5.6 Existence of a solution: proof of Theorem 5.4

Recall from earlier remarks that the first part of the theorem can be proved in the same way as the Brownian case in [61]. We therefore concentrate on the second part of the theorem.

We piece together the conclusions of Lemmas 5.8 and 5.9 in order to check the conditions of the Verification Lemma 5.7.

In particular, we consider the candidate triple (V^*, τ^*, σ^*) generated by the choices $\tau^* = \inf\{t > 0 : X_t < x^*\}$ and $\sigma^* = \inf\{t > 0 : X_t \in [\log K, y^*]\}$ where the constants x^* and y^* are given in Lemmas 5.8 and 5.9 respectively. Note also that, due to the fact that X is spectrally negative, we have

$$V^*(x) = v(x) \quad \text{for } x \in \mathbb{R}.$$

Conditions (i) – (iv) of Lemma 5.7 are now automatically satisfied and it remains to establish the supermartingale and submartingale conditions in (v) and (vi). For the former we note that if the initial value $x \in [x^*, \log K)$ then spectral negativity and Lemma 5.8 (ix) gives the required supermartingale property. If on the other hand $x > y^*$, then, since by Lemma 5.9 (ix) $e^{-rt}v(X_t)$ is a martingale up to the stopping time $\tau^-_{y^*}$ and since by Lemma 5.8 (ix), given $\mathcal{F}_{\tau^-_{y^*}} \cap \{X_{\tau^-_{y^*}} < \log K\}$, the process $\{e^{-r(t+\tau^-_{y^*})}v(X_{t+\tau^-_{y^*}})\}$ is a supermartingale, the required supermartingale property follows. For the submartingale property, it is more convenient to break the proof into the cases that $y^* = \log K$ and $y^* > \log K$.

For the case $y^* > \log K$, pick two arbitrary points $\log K < a < b < y^*$. Note from the proof of Lemma 5.8 (ix) that $(\Gamma - r)v(x) = 0$ on $x \in (x^*, \log K)$. Also, it is easy to verify that because of the monotonicity of v, it holds that $(\Gamma - r)v(x) > 0$ for $x \in (\log K, a)$. The submartingale property follows by piece-wise consideration of the path of X and the following two facts. Firstly, the above remarks on the value of $(\Gamma - r)v(x)$ together with an application of the Itô–Meyer–Tanaka formula (cf. [98]) imply that $\{e^{-rt}v(X_t) : t \ge 0\}$ is a submartingale when $X_0 \le a$ and $t < \sigma^+_b \wedge \tau^-_{x^*}$. Secondly, from Lemma 5.9 (xi) $\{e^{-rt}v(X_t) : t \ge 0\}$ is a submartingale when $X_0 \ge b$ and $t < \sigma^-_a \wedge \tau^-_{x^*}$.

To deal with the case that $y^* = \log K$, recall from Lemma 5.9 (vi) that X necessarily has a Gaussian component. As mentioned in Section 5.2, this is a sufficient condition to guarantee that both scale functions are infinitely differentiable on $(0,\infty)$. An application of Itô's formula together with the martingale properties mentioned in Lemmas 5.8 (viii) and 5.9 (x) show that $(\Gamma - r)v = 0$ on $(x^*, \log K) \cup (\log K, \infty)$. Using this fact together with the Itô–Meyer–Tanaka formula (cf. [98]) the submartingale property of $\{e^{-r(t\wedge\tau^-_{x^*})}v(X_{t\wedge\tau^-_{x^*}}) : t \geq 0\}$ follows from its semi-martingale decomposition which now takes the form

$$e^{-rt}v(X_t) = v(X_0) + M_t + \int_0^t e^{-rs}\left(v'(\log K+) - v'(\log K-)\right) dL_s^{\log K}$$

on $\{t < \tau^-_{x^*}\}$ where $L^{\log K}$ is the semi-martingale local time of X at $\log K$ and M is a martingale. Specifically, the integral is non-negative, as one may check from the expression given for v in (5.11), (5.18) and (5.23),

$$v'(\log K+) - v'(\log K-) = \frac{2}{\sigma^2}\alpha e^{\Phi(r)(\log K - x^*)} > 0. \tag{5.25}$$

Note that we have used the fact that $\alpha > 0$, which was established in the proof of Lemma 5.9 (vi). □

5.7 Proof of Theorem 5.5

It follows immediately from Lemma 5.9 that when $y^* = \log K$ we necessarily have that X has a Gaussian component.

Next, we show that $\Pi(-\infty, \log K - y^*) > 0$. Suppose that $X_0 \in (\log K, y^*)$. It follows that $\{e^{-rt}V(X_t) : t \leq \tau^-_{\log K}\}$ is a submartingale and that $V(x) = \delta$ on $[\log K, y^*]$. We deduce from Itô's formula (see for example Theorem 36 of [98]) that in the semi-martingale decomposition of the aforementioned submartingale, the drift term must be non-negative and hence for any $x \in (\log K, y^*)$

$$\begin{aligned} 0 &\leq (\mathcal{L} - r)V(x) \\ &= -r\delta + \int_{-\infty}^0 (V(x+y) - \delta)\,\Pi(dy) \\ &= -r\delta + \int_{-\infty}^{\log K - x} (V(x+y) - \delta)\,\Pi(dy). \end{aligned}$$

Since V is decreasing on $(-\infty, \log K)$, we find that $\Pi(-\infty, \log K - y^*) > 0$ as required. □

5.8 Remarks on y^* for the case that X has a Gaussian component

In the previous section we have shown that $y^* > \log K$ whenever X has no Gaussian component. In this section we show that when X has a Gaussian component, the distinction between $y^* = \log K$ and $y^* > \log K$ is a more subtle issue. This distinction is important since in the next section we will show that when X is of unbounded variation, the value function is differentiable at y^* if and only if $y^* > \log K$. Lemma 5.9 (vi) implies that $y^* = \log K$ exactly when the value function is equal to $j(x)$. Reviewing the calculations in the previous sections, one sees that it is the upper bound condition (ii) of Lemma 5.7 which may not hold for j and that all the other conditions are verifiable in the same way as before. A sufficient condition that Lemma 5.7 (ii) holds is that j is a decreasing function in which case of course $y^* = \log K$. Whenever X has no Gaussian component, the function j violates this upper bound condition as was shown in the proof of Lemma 5.9 (vi). This is caused by the behaviour of the scale function W at zero: when the Gaussian component of X is zero, either W is discontinuous or its right derivative at zero is infinite. Assume now that X has a Gaussian component. Then the behaviour of the scale function at zero implies that $j(\log K+) = \delta$ and that j has finite derivative on $(\log K, \infty)$. From these properties alone we are not able to deduce anything about the value of y^*. In fact, as we will show next, whether the upper bound condition is satisfied by j depends on the sign of $j'(\log K+)$. Whenever $j'(\log K+) > 0$, it must hold that $y^* > \log K$, since otherwise Lemma 5.9 (iii) and (vi) lead to a contradiction. We show that a sufficient condition for j to be decreasing, and hence for y^* to be equal to $\log K$, is given by $j'(\log K+) < 0$. Recall that $j(x) = w(x)$ on $(-\infty, \log K]$. From Lemma (5.8) (v) and from $j'(\log K+) < 0$ we deduce the existence of some $\gamma > 0$ such that j is decreasing on $(-\infty, \log K + \gamma]$. Next, let $\log K + \gamma \leq x < y \leq x + \gamma$. By the strong Markov property

$$j(y) - j(x) = \mathbb{E}[e^{-r\tau^-_{\log K - x}}(j(X_{\tau^-_{\log K-x}} + y) - j(X_{\tau^-_{\log K-x}} + x))].$$

From

$$X_{\tau^-_{\log K-x}} + x < X_{\tau^-_{\log K-x}} + y \leq \log K - x + y \leq \log K + \gamma$$

we deduce that $j(y) - j(x) < 0$, which implies that j is a decreasing function on $\mathbb{R}$.

Remark 5.10. When X is a Brownian motion and $r = \psi(1) = \sigma^2/2$, the discussion above agrees with Theorem 2 in [61]. Indeed, in this case the scale functions are given by

$$W^{(\psi(1))}(x) = \frac{2}{\sigma^2}\sinh(x) \quad \text{and } Z^{(\psi(1))}(x) = \cosh(x)$$

for $x \geq 0$. It follows that

$$\begin{aligned} j'(\log K+) &= \psi(1)KW^{(\psi(1))}(\log K - x^*) - K + \frac{2\alpha K}{\sigma^2}e^{-x^*} \\ &= K\sinh(\log K - x^*) - K + 2K - K^2 e^{-x^*} \\ &= -\frac{K^2}{2}e^{-x^*} - \frac{1}{2}e^{x^*} + K. \end{aligned}$$

Since x^* solves $KZ^{(\psi(1))}(\log K - x) - K = \delta$ we deduce that

$$Ke^{-x^*} + \frac{1}{K}e^{x^*} = 2(\delta + K)$$

and thus

$$j'(\log K+) = -\delta < 0.$$

We conclude that the optimal stopping strategies are indeed given by

$$\tau^* = \tau^-_{x^*} \quad \text{and } \sigma^* = T_K.$$

Also, for the other cases $r \neq \sigma^2/2$, similar calculations lead to the results found in [61].

Unfortunately, there are rather few spectrally negative Lévy processes for which the scale functions are known in terms of elementary or special functions. Hence, in general, numerical analysis is needed to check whether the condition $j'(\log K) < 0$ holds.

5.9 Pasting properties at y^*: proof of Theorem 5.6

Using notation as in the proof of Lemmas 5.7 and 5.9, it follows from monotonicity of V and the definition of V as a saddle point that for $-\infty < x \leq y < \infty$

$$\begin{aligned} 0 \leq V(x) - V(y) \quad \leq \quad & \mathbb{E}[e^{-r\tau^*(x)}(G(X_{\tau^*(x)} + x) - G(X_{\tau^*(x)} + y))\mathbf{1}_{\{\tau^*(x) \leq \sigma^*(y)\}}] \\ & + \mathbb{E}[e^{-r\sigma^*(y)}(G(X_{\sigma^*(y)} + x) - G(X_{\sigma^*(y)} + y))\mathbf{1}_{\{\sigma^*(y) < \tau^*(x)\}}] \end{aligned}$$

and continuity of V follows from continuity of G and dominated convergence.

It has already been shown in Section 5.4 whilst proving Theorem 5.4 that there is smooth pasting at x^* if and only if X has unbounded variation. It then remains to establish the smoothness of V at y^*.

(i) Recall from (5.25) that

$$V'(\log K+) - V'(\log K-) = \frac{2}{\sigma^2}\alpha e^{\Phi(r)(\log K - x^*)} > 0$$

showing that there can be no smooth fit at y^*.

Next suppose that $y^* > \log K$. Our aim is to show that $V'(y^*+) = 0$. In order to do this we shall need two auxiliary results.

Lemma 5.11. *Suppose X is of unbounded variation and let $c < 0$. Then*

$$\lim_{\varepsilon \downarrow 0} \frac{\mathbb{P}(\tau^-_c = \tau^-_\varepsilon, \tau^-_c < \tau^+_\varepsilon)}{\varepsilon} = 0. \tag{5.26}$$

Proof. Let $c < 0$. Define

$$A_\varepsilon := \{\tau^-_c = \tau^-_\varepsilon, \tau^-_c < \tau^+_\varepsilon\} = \{X_{\tau_{\varepsilon^-}} < c, \tau^-_c < \tau^+_\varepsilon\}.$$

Define $\overline{X}_t = \sup_{s \leq t} X_s$. Let $L = \{L_t : t \geq 0\}$ be the local time at zero of the reflected process $\{\overline{X}_t - X_t : t \geq 0\}$. Denote by $\{(t, \epsilon_t) : t \geq 0\}$ the process of excursions from zero of $\{\overline{X}_t - X_t : t \geq 0\}$ on the local time scale. Note that excursions are of the form $\epsilon_t = \{\epsilon_t(s) : s \le \zeta_t\}$, where ζ_t is the duration of the excursion ϵ_t. For the generic excursion ϵ and $x > 0$ let

$$\rho_x := \inf\{s > 0 : \epsilon(s) > x\}$$

be the first time ϵ exceeds level x. Note that A_ε only happens if and only if there exists a left endpoint g of an excursion such that

(i) $L_g < \varepsilon$ (at time g the process must not have exceeded ε),

(ii) $\overline{\epsilon}_{L_h} < \overline{X}_h + \varepsilon$ for all $h < g$ in the support of dL (during excursions before time g, the process must stay above $-\varepsilon$),

(iii) $\epsilon_{L_g}(\rho_{\overline{X}_g+\varepsilon}) > \overline{X}_g + c$ (the first exit time below $-\varepsilon$ must be the first exit time below c).

Hence, we can use the compensation formula (with g and h denoting left end points of excursion intervals) to deduce that

$$\begin{aligned} P(A_\varepsilon) &= \mathbb{E}\left[\sum_{g<L_\varepsilon^{-1}} \mathbf{1}_{\{\overline{\epsilon}_{L_h}<\overline{X}_h+\varepsilon\ \forall h<g\}} \mathbf{1}_{\{\epsilon_{L_g}(\rho_{\overline{X}_g+\varepsilon})>\overline{X}_g+c\}}\right] \\ &= \mathbb{E}\left[\int_0^{L_\varepsilon^{-1}-} \mathbf{1}_{\{\overline{\epsilon}_{L_u}<\overline{X}_u+\varepsilon\ \forall u<s\}} \varphi(\overline{X}_s)\, dL_s\right], \end{aligned}$$

where $\varphi(x) = n(\epsilon(\rho_{x+\varepsilon}) > x + c)$. Using the fact that $\overline{X}_{L_t^{-1}} = t$, we find for small enough ε

$$\begin{aligned} 0 &\le \frac{1}{\varepsilon}\mathbb{P}(A_\varepsilon) \\ &= \frac{1}{\varepsilon}\mathbb{E}\left[\int_0^{\varepsilon\wedge L_\infty} \mathbf{1}_{\{\overline{\epsilon}_\theta<\theta+\varepsilon\ \forall\theta<t\}} \varphi(t+c)\, dt\right] \\ &\le \frac{1}{\varepsilon}\int_0^{\varepsilon} n(\epsilon(\rho_{t+\varepsilon}) > c)\, dt \\ &\le \frac{1}{\varepsilon}\int_0^{2\varepsilon} n(\epsilon(\rho_t) > c)\, dt. \end{aligned} \tag{5.27}$$

However, it is known (cf. [80]) that, since X has unbounded variation and hence is regular upwards, $\lim_{t\downarrow 0}\epsilon(\rho_t) = 0$, which in turn implies that

$$\lim_{\varepsilon\downarrow 0}\frac{P(A_\varepsilon)}{\varepsilon} = 0$$

as required. □

Lemma 5.12. *For any spectrally negative Lévy process*

$$\limsup_{\varepsilon\downarrow 0}\frac{W(2\varepsilon)}{W(\varepsilon)} \le 2.$$

Proof. We may assume without loss of generality that X does not drift to $-\infty$, i.e. $\Phi(0) = 0$. Denote by h the renewal function of the downward ladder height process and by κ its Laplace exponent. It is known (cf. [18]) that there is some constant $c > 0$ such that $W(x) = ch(x)$. The proof is completed by applying the strong Markov property at time $\tau = \inf\{t \geq 0 : H_t \leq -x\}$, which allows us to deduce

$$\begin{aligned} h(2x) &= \int_0^\infty (\mathbb{P}(H_t \geq -x) + \mathbb{P}(H_t \in [-2x, -x]))\, dt \\ &= \int_0^\infty (\mathbb{P}(H_t \geq -x) + \mathbb{P}(\tilde{H}_t \in [-2x - H_\tau, -x - H_\tau]))\, dt \\ &\leq 2h(x). \end{aligned}$$

Here $\tilde{H}$ denotes an independent copy of H. □

(i) We are now ready to conclude the proof of part (i) of Theorem 5.6. To this end suppose $y > \log K$ and X is of unbounded variation. Since $V = \delta$ on $[\log K, y^*]$ it suffices to show that the right derivative of V exists at y^* and that $V'(y^*+) = 0$. Since $V(y^*) = \delta$ and since $V(x) \leq \delta$ for any $x > \log K$ we have for any $x > y^*$

$$\frac{V(x) - V(y^*)}{x - y^*} \leq 0,$$

which implies that

$$\limsup_{x \downarrow y^*} \frac{V(x) - V(y^*)}{x - y^*} \leq 0.$$

To show that $V'(y^*) = 0$ we must thus show that

$$\liminf_{x \downarrow y^*} \frac{V(x) - V(y^*)}{x - y^*} \geq 0.$$

In order to achieve this, define for $\varepsilon < \log K - y^*$

$$\tau_\varepsilon^* = \inf\{t \geq 0 : X_t \notin [y^* - \varepsilon, y^* + \varepsilon]\}.$$

Furthermore,

$$\tau^+ := \inf\{t \geq 0 : X_t > y^* + \varepsilon\}$$

and

$$\tau^- := \inf\{t \geq 0 : X_t < y^* - \varepsilon\}.$$

We have that for small enough ε, $\{e^{-r(t \wedge \tau_\varepsilon)} V(X_{t \wedge \tau_\varepsilon})\}_{t \geq 0}$ is a $\mathbb{P}_{y^*}$-submartingale, hence by the optional sampling theorem

$$\begin{aligned} &\mathbb{E}_y^*[e^{-r\tau_\varepsilon} V(X_{\tau_\varepsilon})] \\ &\qquad \geq V(y^*) \\ &\qquad = V(y^*)\mathbb{E}_y^*[e^{-r\tau^+} \mathbf{1}_{\{\tau^+ < \tau^-\}}] + \delta(1 - \mathbb{E}_y^*[e^{-r\tau^+} \mathbf{1}_{\{\tau^+ < \tau^-\}}]). \end{aligned} \tag{5.28}$$

Furthermore, we use Lemma 5.11 and the fact that V is bounded by K to deduce

$$
\begin{aligned}
&\mathbb{E}^*_y[e^{-r\tau_\varepsilon}V(X_{\tau_\varepsilon})] \\
&\quad= V(y^*+\varepsilon)\mathbb{E}^*_y[e^{-r\tau^+}\mathbf{1}_{\{\tau^+<\tau^-\}}] + \mathbb{E}^*_y[e^{-r\tau^-}V(X_{\tau^-})\mathbf{1}_{\{\tau^-<\tau^+\}}] \\
&\quad= V(y^*+\varepsilon)\mathbb{E}^*_y[e^{-r\tau^+}\mathbf{1}_{\{\tau^+<\tau^-\}}] + \delta\mathbb{E}^*_y[e^{-r\tau^-}\mathbf{1}_{\{\tau^-_{\log K}<\tau^-<\tau^+\}}] \\
&\qquad + \mathbb{E}^*_y[e^{-r\tau^-}V(X_{\tau^-})\mathbf{1}_{\{\tau^-_{\log K}=\tau^-<\tau^+\}}] \\
&\quad\leq V(y^*+\varepsilon)\mathbb{E}^*_y[e^{-r\tau^+}\mathbf{1}_{\{\tau^+<\tau^-\}}] + \delta\mathbb{E}^*_y[e^{-r\tau^-}\mathbf{1}_{\{\tau_{\log K^-}<\tau^-<\tau^+\}}] \\
&\qquad + K\mathbb{P}(\tau^-_{\log K}=\tau^-<\tau^+) \\
&\quad\leq V(y^*+\varepsilon)\mathbb{E}^*_y[e^{-r\tau^+}\mathbf{1}_{\{\tau^+<\tau^-\}}] + \delta\mathbb{E}^*_y[e^{-r\tau^-}\mathbf{1}_{\{\tau^-<\tau^+\}}] + o(\varepsilon) \qquad (5.29)
\end{aligned}
$$

as $\varepsilon \downarrow 0$. The two expectations on the right hand side of (5.29) can be evaluated in terms of scale functions with the help of (5.8) and (5.9). Also, because X is of unbounded variation, it is known that $W^{(q)}(0) = 0$. Combining (5.28), (5.29) and using Lemma 5.12 we find

$$
\begin{aligned}
\liminf_{\varepsilon\downarrow 0}\frac{V(y^*+\varepsilon)-V(y^*)}{\delta\varepsilon} &\geq \liminf_{\varepsilon\downarrow 0}\frac{1-\mathbb{E}^*_y[e^{-r\tau_\varepsilon}]}{\varepsilon\mathbb{E}^*_y[e^{-r\tau^+}\mathbf{1}_{\{\tau^+<\tau^-\}}]} \\
&= \liminf_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left(Z^{(r)}(2\varepsilon)-1-\frac{W^{(r)}(2\varepsilon)}{W^{(r)}(\varepsilon)}(1-Z^{(r)}(\varepsilon))\right) \\
&= 0.
\end{aligned}
$$

This concludes the proof of part (i) of Theorem 5.6.

(ii) Suppose now that X has bounded variation. We know that necessarily X has no Gaussian component and hence by Theorem 5.5 that $y^* > \log K$. We see from (5.24) and continuity of V that for $\varepsilon > 0$

$$
\frac{V(y^*+\varepsilon)-\delta}{\varepsilon} \leq \mathbb{E}\left[e^{-r\tau^-_{y^*}(y^*)}\frac{w_\delta(X_{\tau^-_{y^*}(y^*)}+y^*+\varepsilon)-w_\delta(X_{\tau^-_{y^*}(y^*)}+y^*)}{\varepsilon}\right]
$$

where as before we are working under the measure $\mathbb{P}$ and indicate the dependency of stopping times on an initial position of X. Now, recalling that w_δ is a non-increasing function and is equal to V on $(-\infty, \log K)$, we have with the help of Theorem 5.5, dominated convergence and the fact that V is decreasing on $(-\infty, \log K)$ that

$$
\limsup_{\varepsilon\downarrow 0}\frac{V(y^*+\varepsilon)-\delta}{\varepsilon} \leq \mathbb{E}[e^{-r\tau^-_{y^*}(y^*)}V'(X_{\tau^-_{y^*}(y^*)}+y^*)\mathbf{1}_{\{X_{\tau^-_{y^*}(y^*)}+y^*<\log K\}}] < 0.
$$

Hence, in this case, there is continuous fit but no smooth fit at y^*. □

5.10 The case $r = 0$

We conclude the study of the McKean stochastic game with the case $r = 0$. We include this case here, since it illustrates the subtleties which arise when there is no discount factor. Some of the results in this section can also be found in [63] (but see Remark 5.17 below). In the case when $r = 0$, the game (5.3) needs to be formulated somewhat more carefully. The reason for this is that when there is no discounting, the game depends on what value we assign to the event $\{\tau = \sigma = \infty\}$. Since any (non-trivial) Lévy process satisfies either (see Theorem 12 in [18])

$$\lim_{t\to\infty} X_t = -\infty, \quad \text{or} \tag{5.30}$$

$$\lim_{t\to\infty} X_t = \infty, \quad \text{or} \tag{5.31}$$

$$\limsup_{t\to\infty} X_t = -\liminf_{t\to\infty} X_t = \infty, \tag{5.32}$$

we can redefine (5.3), using the notation of the proof of Lemma 5.7, to have pay-off function

$$\Theta^0_{\tau,\sigma} = G(X_\tau)\mathbf{1}_{\{\tau\le\sigma,\tau<\infty\}} + H(X_\sigma)\mathbf{1}_{\{\sigma<\tau\}} + L(\limsup_{t\to\infty} X_t, \liminf_{t\to\infty} X_t)\mathbf{1}_{\{\tau=\sigma=\infty\}}, \tag{5.33}$$

where we put $L(\infty,\infty) = \delta$ and $L(-\infty,-\infty) = K$ and $L(\infty,-\infty) = K$. See Remarks 5.14, 5.15 and 5.16 below for a discussion on the choice of the function L. We still denote by V the value of the corresponding game. From the Wiener–Hopf factorisation it follows that for spectrally negative Lévy processes, (5.30), (5.31) and (5.32) are equivalent to $\psi'(0) < 0, \psi'(0) > 0$ and $\psi'(0) = 0$ respectively. We assume that $\delta < K$, since otherwise stopping in a finite time would not be optimal for any of the players. The latter follows from the observation that when $\delta \ge K$, it holds that

$$\inf_x H(x) \ge \sup_x G(x),$$

from which it follows that

$$V(x) = L(\limsup_{t\to\infty} X_t, \liminf_{t\to\infty} X_t).$$

We remark that value function $U(x)$ of the McKean optimal stopping problem is still given by (5.15) when $r = 0$ and $\psi'(0) > 0$ (see [81]). We have the following result.

Theorem 5.13. *Consider the stochastic game with pay-off given by (5.33) and let* $0 < \delta < K$.

(i) When $\psi(1) = 0$*, the stopping times*

$$\tau^* = \infty \quad \text{and} \quad \sigma^* = \inf\{t > 0 : X_t > \log K\} \tag{5.34}$$

form a saddle point for the stochastic game (5.33). Furthermore,

$$V(x) = \begin{cases} K - e^x + \frac{\delta}{K}e^x & \text{when } x < \log K, \\ \delta & \text{when } x \ge \log K. \end{cases}$$

(ii) When $\psi'(0) > 0$ and $\delta \geq U(\log K)$, a saddle point consists of

$$\tau^* = \inf\{t > 0 : X_t < k^*\} \quad \textit{and} \quad \sigma^* = \infty, \tag{5.35}$$

with

$$e^{k^*} = \frac{K\psi'(0)}{\psi(1)}.$$

In this case, $V(x) = U(x)$.

*(iii) When $\psi(1) > 0$ and $\psi'(0) \leq 0$, **or** when $\psi'(0) > 0$ and $\delta < U(\log K)$, a saddle point is given by*

$$\tau^* = \inf\{t > 0 : X_t < z^*\} \quad \textit{and} \quad \sigma^* = \inf\{t > 0 : X_t > \log K\},$$

where z^* is the unique solution to

$$K\psi(1) \int_0^{\log K - z} e^{-y} W(y)\, dy = \delta. \tag{5.36}$$

Also,

$$V(x) = \begin{cases} K - e^x + \psi(1)e^x \int_0^{x-z^*} e^{-y} W(y)\, dy & \textit{when } x < \log K, \\ \delta & \textit{when } x \geq \log K. \end{cases} \tag{5.37}$$

Proof. Let $0 < \delta < K$. We prove Theorem 5.13 by a tweaked version of Theorem (5.7). Firstly, suppose that $\psi(1) \leq 0$. It then holds that $\psi'(0) < 0$ and thus $\lim_{t\to\infty} X_t = -\infty$. From Theorem 3.12 in [63], it follows that, with τ^* and σ^* given in (5.34), we have for $x \leq \log K$

$$\begin{aligned} V^*(x) :&= \mathbb{E}_x[\Theta^0_{\tau^*,\sigma^*}] \\ &= \delta \mathbb{P}_x(\tau^+_{\log K} < \infty) + K\mathbb{P}_x(\tau^+_{\log K} = \infty) \\ &= K - e^x + \frac{\delta}{K} e^x. \end{aligned}$$

It is straightforward to check that (with the notation of (5.33))

$$G(x) \leq V^*(x) \leq K \wedge H(x).$$

Furthermore, $\{V^*(X_t)\}_{t\geq 0}$ is a submartingale and $\{V^*(X_{t\wedge\sigma^*})\}_{t\geq 0}$ is a martingale. We deduce that for any $\tau \in \mathcal{T}$ it holds that

$$\begin{aligned} V^*(x) &= \mathbb{E}_x[V^*(X_{t\wedge\sigma^*\wedge\tau})] \\ &\geq \mathbb{E}_x[G(X_{t\wedge\tau})\mathbf{1}_{\{t\wedge\tau\leq\sigma^*\}} + \delta\mathbf{1}_{\{\sigma^*<\tau\wedge t\}}]. \end{aligned}$$

Taking limits and using dominated convergence, we find that

$$\begin{aligned} V^*(x) &\geq \mathbb{E}_x[G(X_\tau)\mathbf{1}_{\{\tau\leq\sigma^*,\tau<\infty\}} + G(X_\infty)\mathbf{1}_{\{\sigma^*=\tau=\infty\}} + \delta\mathbf{1}_{\{\sigma^*<\tau\}}] \\ &= \mathbb{E}_x[G(X_\tau)\mathbf{1}_{\{\tau\leq\sigma^*,\tau<\infty\}} + K\mathbf{1}_{\{\sigma^*=\tau=\infty\}} + \delta\mathbf{1}_{\{\sigma^*<\tau\}}] \\ &= \mathbb{E}_x[\Theta^0_{\tau,\sigma^*}]. \end{aligned}$$

Also, for any $\sigma \in \mathcal{T}$

$$\begin{aligned} V^*(x) &\leq \mathbb{E}_x[V^*(X_{t\wedge\sigma})] \\ &\leq \mathbb{E}_x[H(X_{t\wedge\sigma})\mathbf{1}_{\{\sigma<\infty\}} + K\mathbf{1}_{\{\sigma=\infty\}}]. \end{aligned}$$

Again by taking limits, we deduce

$$\begin{aligned} V^*(x) &\leq \mathbb{E}_x[H(X_\sigma)\mathbf{1}_{\{\sigma<\infty\}} + K\mathbf{1}_{\{\sigma=\infty\}}] \\ &= \mathbb{E}_x[\Theta^0_{\tau^*,\sigma}]. \end{aligned}$$

This shows that the stopping times in (5.34) form a saddle point.
Secondly, suppose that $\psi'(0) > 0$ and that $\delta \geq U(\log K)$ (but still, $\delta < K$). It readily follows that the function $U(x)$ (as given in (5.15)) has the properties

$$\delta \vee G(x) \leq U(x) \leq H(x),$$

and $\{U(X_t)\}_{t\geq 0}$ and $\{U(X_{t\wedge\tau^*})\}_{t\geq 0}$ are a supermartingale and a martingale, respectively, where τ^* is defined in (5.35). Part (ii) now follows similarly to part (i). Finally, suppose that either of the conditions in part (iii) holds. We then refer to pp. 264–269 in [63] for the necessary martingale properties of and bounds on V given in (5.37). The proof is similar to the proof of case (i) above. □

Remark 5.14. When $\psi'(0) = 0$, we see from Theorem 5.13 that the optimal stopping times τ^* and σ^* are both almost surely finite. This implies that, when $0 < \delta < K$, the game (5.33) does not depend on $L(-\infty, \infty)$.

In the case when $\psi'(0) = 0$ and $\delta \geq K$, the game does depend on $L(-\infty, \infty)$. Whenever $L(-\infty, \infty) \in [K, \delta]$, it is suboptimal for either player to exercise in finite time, and hence $\sigma^* = \tau^* = \infty$.

When $L(-\infty, \infty) < K$, it is still optimal for the min-player to take $\sigma = \infty$, but the max-player does not want to take $\tau = \infty$ as this would lead to a pay-off of $L(-\infty, \infty)$ which is strictly smaller than the pay-off corresponding to the strategy

$$\sigma_n = \inf\{t \geq 0 : X_t < -n\}$$

for large enough n. Note that the latter stopping time is almost surely finite due to the assumption that $\psi'(0) = 0$. We deduce that we can unambiguously assign a value to the game, since

$$\sup_{\tau\in\mathcal{T}_{0,\infty}} \inf_{\sigma\in\mathcal{T}_{0,\infty}} \mathbb{E}_x[\Theta^0_{\tau,\sigma}] = \inf_{\sigma\in\mathcal{T}_{0,\infty}} \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbb{E}_x[\Theta^0_{\tau,\sigma}] = K.$$

However, there is no pair of stopping times at which this value is attained. Following the terminology in [47], we say that the Stackelberg equilibrium holds as opposed to the Nash equilibrium. The case $L(-\infty, \infty) > \delta$ is similar.

Finally, when $\psi'(0) = 0$, $\delta \geq K$ and when $L(-\infty, \infty)$ is not defined (in which case we implicitly require each player to choose only an almost surely finite stopping time), the game does not even have a value. Indeed, it then holds that

$$\begin{aligned} &\sup_{\tau\in\mathcal{T}_{[0,\infty)}} \inf_{\sigma\in\mathcal{T}_{[0,\infty)}} \mathbb{E}_x[(K - e^{X_\tau})^+\mathbf{1}_{\{\tau\leq\sigma\}} + ((K - e^{X_\sigma})^+ + \delta)\mathbf{1}_{\{\sigma<\tau\}}] \\ &< \inf_{\sigma\in\mathcal{T}_{[0,\infty)}} \sup_{\tau\in\mathcal{T}_{[0,\infty)}} \mathbb{E}_x[(K - e^{X_\tau})^+\mathbf{1}_{\{\tau\leq\sigma\}} + ((K - e^{X_\sigma})^+ + \delta)\mathbf{1}_{\{\sigma<\tau\}}] \end{aligned}$$

where $\mathcal{T}_{[0,\infty)}$ is the set of almost surely finite stopping times with respect to $\mathbb{F}$. The inequality above follows by considering stopping times of the form τ and $\sigma = \tau + 1$ and vice versa and by the observation that for any $x, y \in \mathbb{R}$ we have

$$(K - e^x)^+ < K \leq (K - e^y)^+ + \delta.$$

Remark 5.15. The choices of $L(\infty, \infty)$ and $L(-\infty, -\infty)$ are, in some sense, the natural definition for the game (5.33). We have seen in the $r > 0$ case that $(-\infty, \log K)$ never belongs to the stopping region of the min-player and that $[\log K, \infty)$ never belongs to the stopping region of the max-player and, as such, it is convenient to define $L(\infty, \infty) = H(\infty)$ and $L(-\infty, -\infty) = G(-\infty)$.

The case $r = 0$ is also included in section 9.6 in [63] for the case when $\psi'(0) > 0$. Note, however, that the definition of the game there should be redefined as we did above. With the notation of this chapter the function L in [63] is implicitly defined by $L(\infty, \infty) = 0$. If we take this as the definition of L in (5.33), the solution to the game should be different. Consider for example a Brownian motion with positive drift. For $x > \log K$, the min-player can choose the strategy $\inf\{t \geq 0 : X_t = \log K\}$, which leads to a strictly lower value than the one corresponding to the min-player taking $\tau = \inf\{t \geq 0 : X_t \geq \log K\}$, because of the fact that a Brownian motion with positive drift and starting at a value $x > \log K$ with positive probability never hits $\log K$ combined with the fact that on this event, the pay-off in [63] is taken to be zero instead of δ.

Remark 5.16. As mentioned on p. 146 in [92], there is an analogy between optimal stopping problems and so-called obstacle problems. An example of the latter is by hanging a rope on an obstacle. The rope will then be bounded from below by the obstacle and, at the same time, its energy is minimised. Hence, we see the close link with solutions of optimal stopping problems (when the objective is to maximise pay-off) being the smallest superharmonic functions bounded from below by the pay-off function. In the case of stochastic games, there is a similar analogy as was shown in [88]. Indeed, consider the following situation: a cable is pulled outwards from points $(-\infty, K)$ and (∞, δ) while it is sandwiched between the functions $(K - e^x)^+$ and $(K - e^x)^+ + \delta$. This analogy shows from a different perspective why the solution of the game for $r = 0$ depends on the value we give to the event $\{\sigma = \tau = \infty\}$, and hence on the choice of L. Indeed, pulling the rope from $(\infty, 0)$ instead of (∞, δ) leads to a different shape. Intuitively, when the driving Lévy process is of unbounded variation, we can think of the cable being made of a very flexible material. However, when the Lévy process is of bounded variation, the corresponding cable is less flexible which will lead to kinks at both $(x^*, K - e^{x^*})$ and at (y^*, δ).

Remark 5.17. We remark that the conditions in Lemma 9.13 in [63] are not sufficient to show that candidate optimal strategies for a general game are in fact optimal. Indeed, when $\sigma^* = \infty$, then the last line on p. 262 reads

$$\mathbb{E}_x[e^{-qt} H(X_t) \mathbf{1}_{\{\sigma=\infty\}} + e^{-q(t\wedge\sigma)} H(X_{t\wedge\sigma}) \mathbf{1}_{\{\sigma<\infty\}}].$$

The conditions mentioned in Lemma 9.13 are not sufficient to imply that the expectation above converges to $\mathbb{E}_x[\Theta^q_{\infty,\sigma}]$ as $t \to \infty$. One way to resolve this issue, is to assume that

$$\lim_{t\to\infty} e^{-qt}H(X_t) = \lim_{t\to\infty} e^{-qt}G(X_t).$$

However, the McKean game with no discounting does not satisfy this condition. This issue on $\{\sigma = \tau = \infty\}$ is avoided when the candidate optimal strategies are almost surely finite.

Appendix: value function when $y^* = \log K$

Here, our objective is to prove (5.23), i.e. to show that

$$\begin{aligned}
&\mathbb{E}_x[e^{-r\tau^-_{x^*}}(K - e^{X_{\tau^-_{x^*}}})\mathbf{1}_{\{\tau^-_{x^*}<T_K\}} + e^{-rT_K}\delta\mathbf{1}_{\{T_K<\tau^-_{x^*}\}}] \\
&= \quad KZ^{(r)}(x-x^*) - e^x Z_1^{(r-\psi(1))}(x-x^*) \\
&\quad + \alpha e^{\Phi(r)(\log K - x^*)}W^{(r)}(x - \log K). \qquad (5.38)
\end{aligned}$$

First, we need a preliminary lemma. Recall that $T_K = \inf\{t > 0 : X_t = \log K\}$.

Lemma 5.18. *For all $x \in \mathbb{R}$ the following two identities hold:*

$$\mathbb{E}_x[e^{-rT_K}\mathbf{1}_{\{T_K<\tau^-_{x^*}\}}] = \frac{W^{(r)}(x-x^*)}{W^{(r)}(\log K - x^*)} - e^{\Phi(r)(\log K - x^*)}\frac{W^{(r)}(x-\log K)}{W^{(r)}(\log K - x^*)}$$

and

$$\begin{aligned}
\mathbb{E}_x[e^{-r\tau^-_{x^*}}\mathbf{1}_{\{\tau^-_{x^*}<T_K\}}] \quad &= \quad \left(c_r - \frac{r}{\Phi(r)}\right)e^{-\Phi(r)(\log K - x^*)}W^{(r)}(x-\log K) \\
&\quad + Z^{(r)}(x-x^*) - c_r W^{(r)}(x-x^*),
\end{aligned}$$

where

$$c_r = \frac{Z^{(r)}(\log K - x^*)}{W^{(r)}(\log K - x^*)}.$$

Proof. Denote by u_q^+ the q-potential density of the process killed at exiting the positive half-line. We know (Corollary 8.8 in [63]) that for $x, a \geq 0$

$$u_q^+(x,a) = e^{-\Phi(q)a}W^{(q)}(x) - W^{(q)}(x-a).$$

Proposition 1 in [95] allows us to deduce with some algebra that

$$\begin{aligned}
\mathbb{E}_x[e^{-rT_K}\mathbf{1}_{\{T_K<\tau^-_{x^*}\}}] \quad &= \quad \frac{u_r^+(x-x^*, \log K - x^*)}{u_r^+((\log K - x^*)-, \log K - x^*)} \\
&= \quad \frac{W^{(r)}(x-x^*)}{W^{(r)}(\log K - x^*)} - e^{\Phi(r)(\log K - x^*)}\frac{W^{(r)}(x-\log K)}{W^{(r)}(\log K - x^*)}.
\end{aligned}$$

From the Markov property it follows that $\mathbb{E}_x[e^{-r\tau_{x^*}}\mathbf{1}_{\{\tau^-_{x^*}<T_K\}}]$ is equal to

$$
\begin{aligned}
&\mathbb{E}_x[e^{-r\tau^-_{x^*}}] - \mathbb{E}_x[e^{-r\tau^-_{x^*}}\mathbf{1}_{\{T_K<\tau^-_{x^*}\}}]\\
&= Z^{(r)}(x-x^*) - \frac{r}{\Phi(r)}W^{(r)}(x-x^*) - \mathbb{E}_{\log K}[e^{-r\tau^-_{x^*}}]\mathbb{E}_x[e^{-rT_K}\mathbf{1}_{\{T_K<\tau^-_{x^*}\}}]\\
&= \left(\frac{Z^{(r)}(\log K-x^*)}{W^{(r)}(\log K-x^*)} - \frac{r}{\Phi(r)}\right)e^{\Phi(r)(\log K-x^*)}W^{(r)}(x-\log K)\\
&\quad + Z^{(r)}(x-x^*) - \frac{Z^{(r)}(\log K-x^*)}{W^{(r)}(\log K-x^*)}W^{(r)}(x-x^*)
\end{aligned}
$$

thus concluding the proof. □

Proof of (5.38). From Lemma 5.18

$$
\begin{aligned}
&\mathbb{E}_x[e^{-r\tau^-_{x^*}}(K-e^{X_{\tau^-_{x^*}}})\mathbf{1}_{\{\tau^-_{x^*}<T_K\}} + e^{-rT_K}\delta\mathbf{1}_{\{T_K<\tau^-_{x^*}\}}]\\
&= K\mathbb{E}_x[e^{-r\tau^-_{x^*}}\mathbf{1}_{\{\tau^-_{x^*}<T_K\}}] - e^x\mathbb{E}^1_x[e^{-c\tau x^{*-}}\mathbf{1}_{\{\tau^-_{x^*}<T_K\}}]\\
&\quad + \delta\frac{W^{(r)}(x-x^*)}{W^{(r)}(\log K-x^*)} - \delta e^{\Phi(r)(\log K-x^*)}\frac{W^{(r)}(x-\log K)}{W^{(r)}(\log K-x^*)}\\
&= K\left(\frac{Z^{(r)}(\log K-x^*)}{W^{(r)}(\log K-x^*)} - \frac{r}{\Phi(r)}\right)e^{\Phi(r)(\log K-x^*)}W^{(r)}(x-\log K)\\
&\quad + KZ^{(r)}(x-x^*) - K\frac{Z^{(r)}(\log K-x^*)}{W^{(r)}(\log K-x^*)}W^{(r)}(x-x^*)\\
&\quad -e^x\left(\frac{Z_1^{(r-\psi(1))}(\log K-x^*)}{W_1^{(r-\psi(1))}(\log K-x^*)} - \frac{r-\psi(1)}{\Phi_1(r-\psi(1))}\right)\\
&\quad \times e^{\Phi_1(r-\psi(1))(\log K-x^*)}W_1^{(r-\psi(1))}(x-\log K)\\
&\quad -e^xZ_1^{(r-\psi(1))}(x-x^*) + e^x\frac{Z_1^{(r-\psi(1))}(\log K-x^*)}{W_1^{(r-\psi(1))}(\log K-x^*)}W_1^{(r-\psi(1))}(x-x^*)\\
&\quad +\delta\frac{W^{(r)}(x-x^*)}{W^{(r)}(\log K-x^*)} - \delta e^{\Phi(r)(\log K-x^*)}\frac{W^{(r)}(x-\log K)}{W^{(r)}(\log K-x^*)},
\end{aligned}
$$

where Φ_1 plays the role of Φ under $\mathbb{P}^1$. Using (5.13) we have

$$\psi_1(\Phi(r)-1) = \psi(\Phi(r)) - \psi(1) = r-\psi(1)$$

and thus $\Phi_1(r-\psi(1)) = \Phi(r)-1$. By definition of x^* we have

$$Z^{(r)}(\log K-x^*) - Z_1^{(r-\psi(1))}(\log K-x^*) = \delta/K$$

and we use (5.14) to conclude that

$$
\begin{aligned}
&\mathbb{E}_x[e^{-r\tau^-_{x^*}}(K - e^{X_{\tau^-_{x^*}}})\mathbf{1}_{\{\tau^-_{x^*}<T_K\}} + e^{-rT_K}\delta\mathbf{1}_{\{T_K<\tau^-_{x^*}\}}] \\
&\quad = KZ^{(r)}(x-x^*) - e^x Z_1^{(r-\psi(1))}(x-x^*) \\
&\qquad + e^{\Phi(r)(\log K - x^*)}\frac{W^{(r)}(x-\log K)}{W^{(r)}(\log K - x^*)} \\
&\qquad \times \left(KZ^{(r)}(\log K - x^*) - \delta - KZ_1^{(r-\psi(1))}(\log K - x^*)\right) \\
&\qquad\quad + \frac{W^{(r)}(x-x^*)}{W^{(r)}(\log K - x^*)}\left(\delta - KZ^{(r)}(\log K - x^*) + KZ_1^{(r-\psi(1))}(\log K - x^*)\right) \\
&\qquad\qquad + Ke^{\Phi(r)(\log K - x^*)}W^{(r)}(x-\log K)\left(-\frac{r}{\Phi(r)} + Ke^{x^*}\frac{r-\psi(1)}{\Phi(r)-1}\right) \\
&\quad = KZ^{(r)}(x-x^*) - e^x Z_1^{(r-\psi(1))}(x-x^*) + \alpha K e^{\Phi(r)(\log K - x^*)}W^{(r)}(x-\log K)
\end{aligned}
$$

as required. □

Chapter 6

Shepp–Shiryaev stochastic game for spectrally negative Lévy processes*

In [61] the stochastic-game-analogue of Shepp and Shiryaev's optimal stopping problem (cf. [107] and [108]) was considered when driven by an exponential Brownian motion. We consider the same stochastic game, which we call the *Shepp–Shiryaev stochastic game*, but driven by a spectrally negative Lévy process and for a wider parameter range. Unlike [61], we do not appeal predominantly to stochastic analytic methods. Principally, this is due to difficulties in writing down variational inequalities of candidate solutions on account of then having to work with non-local integro-differential operators. We appeal instead to a mixture of techniques including fluctuation theory, stochastic analytic methods associated with martingale characterisations and reduction of the stochastic game to an optimal stopping problem.

6.1 Introduction

Let $X = \{X_t : t \geq 0\}$ be a Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ is the filtration generated by X satisfying the usual conditions. For $x \in \mathbb{R}$, denote by $\mathbb{P}_x$ the law of X when it is started at x and write simply $\mathbb{P}_0 = \mathbb{P}$. Accordingly, we shall write $\mathbb{E}_x$ and $\mathbb{E}$ for the associated expectation operators. We shall assume throughout that X is *spectrally negative*, meaning here that it has no positive jumps and that it is not the negative of a subordinator. It is well known that the latter allows us to talk about the Laplace exponent $\psi(\theta) := \log \mathbb{E}[e^{\theta X_1}]$ for $\theta \geq 0$, which will be of frequent use in the sequel.

*Based on joint work [15] with A. E. Kyprianou.

The Laplace exponent necessarily takes the form

$$\psi(\theta) = a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} (e^{\theta x} - 1 - x\theta\mathbf{1}_{\{x>-1\}})\,\Pi_X(dx)$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ is the Gaussian coefficient and Π_X is a measure concentrated on $(-\infty, 0)$ satisfying $\int_{(-\infty,0)}(1 \wedge x^2)\,\Pi_X(dx) < \infty$.

Denote by $\mathcal{T}_{0,\infty}$ the family of all $[0,\infty]$-valued stopping times with respect to $\mathbb{F}$. We are interested in establishing a solution to a special class of stochastic games which are driven by spectrally negative Lévy processes. Specifically, for a given $q > 0$ and $\delta > 0$, we study the stochastic game consisting of two players and expected pay-off given by

$$\mathbb{E}_x\left[e^{-q\tau+(x\vee\overline{X}_\tau)}\mathbf{1}_{\{\tau\leq\sigma,\,\tau<\infty\}} + e^{-q\sigma}\left(e^{x\vee\overline{X}_\sigma} + \delta e^{X_\sigma}\right)\mathbf{1}_{\{\sigma<\tau\}}\right] \quad \text{for } x \geq 0, \tag{6.1}$$

where $\overline{X}_t = \sup_{s\leq t} X_s$ denotes the running supremum of X and where $a \vee b = \max(a, b)$. The min-player's objective is to choose some $\sigma \in \mathcal{T}_{0,\infty}$ which minimises (6.1), whereas the max-player chooses some $\tau \in \mathcal{T}_{0,\infty}$ which maximises this quantity. Our aim is to prove the existence of a saddle point (τ^*, σ^*) such that

$$\begin{aligned}
&\mathbb{E}_x[e^{-q\tau+(x\vee\overline{X}_\tau)}\mathbf{1}_{\{\tau\leq\sigma^*,\,\tau<\infty\}} + e^{-q\sigma^*}\left(e^{x\vee\overline{X}_{\sigma^*}} + \delta e^{X_{\sigma^*}}\right)\mathbf{1}_{\{\sigma^*<\tau\}}] \\
&\quad\leq \mathbb{E}_x[e^{-q\tau^*+(x\vee\overline{X}_{\tau^*})}\mathbf{1}_{\{\tau^*\leq\sigma^*,\,\tau^*<\infty\}} + e^{-q\sigma^*}\left(e^{x\vee\overline{X}_{\sigma^*}} + \delta e^{X_{\sigma^*}}\right)\mathbf{1}_{\{\sigma^*<\tau^*\}}] \\
&\quad\leq \mathbb{E}_x[e^{-q\tau^*+(x\vee\overline{X}_{\tau^*})}\mathbf{1}_{\{\tau^*\leq\sigma,\,\tau^*<\infty\}} + e^{-q\sigma}\left(e^{x\vee\overline{X}_{\sigma}} + \delta e^{X_{\sigma}}\right)\mathbf{1}_{\{\sigma<\tau^*\}}]
\end{aligned}$$

for all $\tau, \sigma \in \mathcal{T}_{0,\infty}$ and for all $x \in \mathbb{R}$. When such a pair of stopping times exists, we say that it is the solution to the stochastic game (6.1) and we denote the corresponding value by

$$V(x) = \mathbb{E}_x[e^{-r\tau^*}(K - e^{X_{\tau^*}})^+\mathbf{1}_{\{\tau^*\leq\sigma^*\}} + e^{-r\sigma^*}((K - e^{X_{\sigma^*}})^+ + \delta)\mathbf{1}_{\{\sigma^*<\tau^*\}}]$$

for $x \in \mathbb{R}$.
Note that we have included the indicator $\mathbf{1}_{\{\tau<\infty\}}$ in (6.1) since $e^{-qt+(x\vee\overline{X}_t)}$ may not be well defined for $t = \infty$.

When $q = 0$, this issue does not occur since $e^{x\vee\overline{X}_t}$ is monotone in t, and in this case we are interested in the stochastic game which, for a given $\delta > 0$, has pay-off given by

$$e^{x\vee\overline{X}_\tau}\mathbf{1}_{\{\tau\leq\sigma\}} + \left(e^{x\vee\overline{X}_\sigma} + \delta e^{X_\sigma}\right)\mathbf{1}_{\{\sigma<\tau\}}. \tag{6.2}$$

The game (6.1) was solved for $q > \psi(1) \geq 0$ (under an extra technical assumption on the parameters) for a Brownian motion in [61]. In some sense, that case is easier, since for a Brownian motion we can use standard Itô calculus and general theory of optimal stopping to show that a solution to a related free boundary problem (with a differential operator) also solves the game (6.1). The solution to this free boundary problem is readily found in terms of exponential functions. For a Lévy process with jumps, the corresponding free boundary problem seems more difficult to solve directly (or even to establish existence of a solution), as it involves an integro-differential operator. Instead, we use a mixture of fluctuation theory, martingale techniques and reduction of the stochastic game to an optimal stopping problem to solve (6.1). As a by-product, we find that a technical assumption in [61] is not needed, see Remark 6.19.

When $\psi(1) = q > 0$, the stochastic game (6.1) can be understood to characterise the risk neutral price of a so-called game option in a simple market consisting of a risky asset the value of which is given by $\{e^{X_t} : t \geq 0\}$ and a riskless asset which grows at rate q (cf. [59]). The latter game option is an American-type contract with infinite horizon which offers the holder the right but not the obligation to claim $e^{x\vee\overline{X}_\tau}$ at any stopping time $\tau \in \mathcal{T}_{0,\infty}$. In addition, the contract also gives the writer the right but not the obligation to force a payment of $e^{x\vee\overline{X}_\sigma} + \delta e^{X_\sigma}$ at any stopping time $\sigma \in \mathcal{T}_{0,\infty}$; that is to say, what the holder would claim at that moment plus a penalty proportional to the current value of the asset. However, in this thesis we do not discuss the relevance of the stochastic games (6.1) and (6.2) in the context of mathematical finance.

The stochastic games (6.1) and (6.2) are closely related to the *Shepp–Shiryaev optimal stopping problem*

$$U(x) = \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbb{E}[e^{-q\tau+(x\vee\overline{X}_\tau)}\mathbf{1}_{\{\tau<\infty\}}], \tag{6.3}$$

which characterises the value of a perpetual Russian option (cf. [107; 108] in the Brownian case and [5] for the Lévy case). See also [42], [45] and [87] for the finite expiry case and [54] for a linear programming approach. Indeed, if it is the case that the stochastic saddle point in (6.1) is achieved at $\sigma = \infty$, then it holds that $U = V$. In the article [108], an idea which is instrumental in helping provide the solution to (6.3), is to change measure from $\mathbb{P}$ to the measure $\mathbb{P}^1$, where

$$\left.\frac{d\mathbb{P}^\lambda}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\lambda X_t - \psi(\lambda)t} \tag{6.4}$$

defines an equivalent measure on $\{\mathcal{F}_t : t \geq 0\}$ for any $\lambda \geq 0$. Under $\mathbb{P}^\lambda$, the process X still belongs to the class of spectrally negative processes and its Laplace exponent is given by

$$\psi_\lambda(\theta) = \psi(\theta+\lambda) - \psi(\lambda) \text{ for } \theta \geq -\lambda. \tag{6.5}$$

The effect of the change of measure is to reduce the dimension of the underlying driving Markov process of (6.3) from three to two. That is to say, the driving source of randomness changes from $\{(t, X_t, \overline{X}_t) : t \geq 0\}$ to $\{(t, (x\vee\overline{X}_t) - X_t) : t \geq 0\}$. The Shepp–Shiryaev optimal stopping problem can be solved whenever it is possible to solve

$$U(x) = \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbb{E}^1[e^{-\alpha\tau+Y^x_\tau}\mathbf{1}_{\{\tau<\infty\}}],$$

where

$$\alpha = q - \psi(1)$$

and

$$Y^x_\tau = (x \vee \overline{X}_\tau) - X_\tau.$$

The same effect occurs when the change of measure is applied to (6.1) and thus the pay-off function of the Shepp–Shiryaev game can be rewritten as

$$e^{-\alpha\tau+Y_\tau^x}\mathbf{1}_{\{\tau\leq\sigma,\,\tau<\infty\}}+e^{-\alpha\sigma}\left(e^{Y_\sigma^x}+\delta\right)\mathbf{1}_{\{\sigma<\tau\}}. \tag{6.6}$$

We call (6.1) or equivalently (6.6) the *Shepp–Shiryaev stochastic game* and the purpose of this chapter is to give a complete study of its solution within the specified parameter regime $q \geq 0$ and $\delta > 0$.

In the Brownian motion case, the finite horizon version of (6.1) (i.e. when both players have to choose stopping times valued in $[0, T]$ for some $T > 0$) was solved in the preprint [65] preceding [64] by decomposing it into two finite horizon optimal stopping problems, just as was done for the McKean stochastic game.

6.2 The solution to the Shepp–Shiryaev stochastic game

Below, in Theorems 6.2, 6.3 and 6.4 we give a qualitative and quantitative exposition of the solution to (6.1). Before doing so, we need to give a brief reminder of a class of special functions which appear commonly in connection with the study of spectrally negative Lévy processes and indeed in connection with the description of the Shepp–Shiryaev stochastic game as given below. For each $q \geq 0$ we introduce the functions $W^{(q)} : \mathbb{R} \to [0, \infty)$ which are known to satisfy for all $x \in \mathbb{R}$ and $a \geq 0$

$$\mathbb{E}_x[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}] = \frac{W^{(q)}(x\wedge a)}{W^{(q)}(a)} \tag{6.7}$$

where

$$\tau_a^+ := \inf\{t > 0 : X_t > a\} \text{ and } \tau_0^- = \inf\{t > 0 : X_t < 0\}$$

(cf. Chapter 8 of [63]). In particular, it is evident that $W^{(q)}(x) = 0$ for all $x < 0$. Further, it is known that $W^{(q)}$ is almost everywhere differentiable on $(0, \infty)$, there is right-continuity at zero and

$$\int_0^\infty e^{-\beta x}W^{(q)}(x)\,dx = \frac{1}{\psi(\beta)-q} \tag{6.8}$$

for all $\beta > \Phi(q)$, where $\Phi(q)$ is the largest root of the equation $\psi(\theta) = q$ (of which there are at most two). For convenience, we write W instead of $W^{(0)}$.

Associated to the functions $W^{(q)}$ are the functions $Z^{(q)} : \mathbb{R} \to [1, \infty)$ defined by

$$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)\,dy$$

for $q \geq 0$. Together, the functions $W^{(q)}$ and $Z^{(q)}$ are collectively known as scale functions and predominantly appear in almost all fluctuation identities for spectrally negative Lévy processes. For example, it is also known that for all $x \in \mathbb{R}$ and

$a, q \geq 0$,

$$\mathbb{E}_x[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_a^+ > \tau_0^-\}}] = Z^{(q)}(x \wedge a) - \frac{Z^{(q)}(a)}{W^{(q)}(a)} W^{(q)}(x \wedge a). \tag{6.9}$$

We shall henceforth assume that

the jump measure of X, Π_X, has no atoms when X has bounded variation.

Then it is known from existing literature (cf. [32]) that $W^{(q)} \in C^1(0,\infty)$ and hence $Z^{(q)} \in C^2(0,\infty)$. For computational convenience we shall proceed with the above assumption on X. Recall that X has bounded variation if and only if it can be written in the form $X_t = \mathrm{d}t - S_t$ for $t \geq 0$ where $\{S_t : t \geq 0\}$ is a driftless subordinator with jump measure ν satisfying $\nu(x,\infty) = \Pi_X(-\infty, -x)$ (and then must necessarily satisfy $\int_{(0,\infty)}(1 \wedge x)\, \nu(dx) < \infty$) and d is a strictly positive constant which is referred to as the *drift*. In that case, it is also known that $W^{(q)}(0) = 1/\mathrm{d}$ and otherwise, when X has unbounded variation, $W^{(q)}(0) = 0$.

For comparison with the main contributions of this chapter (Theorems 6.2, 6.3 and 6.4), we give below the statement concerning existence of solutions to the Shepp–Shiryaev optimal stopping problem (6.3), the essential part of which can be found in [5]. For convenience, we shall first introduce a subclass of spectrally negative Lévy processes. Denote by $\mathcal{G}$ the general class of spectrally negative Lévy processes and the subclass

$$\begin{aligned}\mathcal{H}_q \quad = \quad & \Bigg\{ X \in \mathcal{G} : \int_{(-\infty,0)} (1 \wedge |x|)\, \Pi_X(dx) = \infty \text{ or } \sigma > 0 \\ & \qquad\qquad \textit{or } \sigma = 0, \int_{(-\infty,0)} (1 \wedge |x|)\, \Pi_X(dx) < \infty \text{ and } q < \mathrm{d} \Bigg\}\end{aligned}$$

where we recall the constant d is the drift in the case of bounded variation. Also needed is the following class of stopping times defined for all $y \geq 0$ by

$$T_y^+ = \inf\{t > 0 : Y_t^x \geq y\} \text{ and } T_y^- = \inf\{t > 0 : Y_t^x \leq y\}.$$

Finally, introduce the continuous function

$$f(x) = Z^{(q)}(x) - qW^{(q)}(x), \tag{6.10}$$

which will play an important role in characterising optimal thresholds. Owing to the fact that $W^{(q)}(x) = e^{\Phi(q)x} W_{\Phi(q)}(x)$, where $W_{\Phi(q)}(x)$ plays the role of $W(x)$ under $\mathbb{P}^{\Phi(q)}$, we can differentiate f and easily deduce that, when $q > \psi(1) \vee 0$, the function f is strictly decreasing to $-\infty$ and hence within this regime

$$k^* := \inf\{x \geq 0 : f(x) \leq 0\} \in [0, \infty).$$

In particular, when $q > \psi(1) \vee 0$, $k^* = 0$ if and only if $X \in \mathcal{G} \backslash \mathcal{H}_q$. This follows from the fact that $Z^{(q)}(0) = 1$ and $W^{(q)}(0) = 0$ unless X has bounded variation in which case $W^{(q)}(0) = 1/\mathrm{d}$.

In the sequel, when U is attained by a stopping time in $\mathcal{T}_{0,\infty}$, we shall denote it by τ^*. That is to say, when it exists, τ^* satisfies

$$U(x) = \mathbb{E}[e^{-q\tau^* + (x\vee\overline{X}_{\tau^*})}\mathbf{1}_{\{\tau^*<\infty\}}].$$

Theorem 6.1. *Let $q \geq 0$.*

(i) When $q \leq \psi(1)$ we have $U(x) = \infty$ which is not attained by any $\tau \in \mathcal{T}_{0,\infty}$,

(ii) when $\psi(1) < q = 0$

$$U(x) = e^x + \frac{1}{\Phi(0)-1}e^{x(1-\Phi(0))},$$

for $x \geq 0$, which is not attained by any $\tau \in \mathcal{T}_{0,\infty}$,

(iii) when $X \in \mathcal{G}\backslash\mathcal{H}_q$, then for $x \geq 0$

$$U(x) = e^x \text{ and } \tau^* = 0,$$

(iv) when $q > \psi(1) \vee 0$ and $X \in \mathcal{H}_q$, then

$$U(x) = e^x Z^{(q)}(k^* - x) \text{ and } \tau^* = T^+_{k^*}.$$

Proof. Cases (iii) and (iv) are contained in Theorem 2 in [5]. Suppose $q \leq \psi(1)$. Since $\sup_{t\geq 0} Y^x_t = \sup_{t\geq 0}(x\vee\overline{X}_t) - X_t$ is $\mathbb{P}^1$-almost surely unbounded, the sequence of stopping times $\{T^+_n\}_{n\in\mathbb{N}}$ are $\mathbb{P}^1$-almost surely finite. Hence when $\alpha \leq 0$,

$$U(x) \geq \mathbb{E}^1[e^{-\alpha T^+_n + Y^x_{T^+_n}}] \geq e^n,$$

which implies (i).

Suppose $\psi(1) < q = 0$. Then

$$U(x) = \sup_{\tau\in\mathcal{T}_{0,\infty}} \mathbb{E}[e^{x\vee\overline{X}_\tau}\mathbf{1}_{\{\tau<\infty\}}] \geq \sup_{t\geq 0}\mathbb{E}[e^{x\vee\overline{X}_t}] = \mathbb{E}[e^{x\vee\overline{X}_\infty}].$$

As $\psi(1) < 0$, it follows that $\psi'(0+) < 0$ and hence by a well-known result for spectrally negative processes, $\overline{X}_\infty$ is exponentially distributed with parameter

$$\Phi(0) = \sup\{\theta \geq 0 : \psi(\theta) = 0\} > 1.$$

Since for any $t \geq 0$ it holds that

$$\mathbb{E}[e^{x\vee\overline{X}_t}] < \mathbb{E}[e^{x\vee\overline{X}_\infty}]$$

and since

$$\lim_{t\to\infty}\mathbb{E}[e^{x\vee\overline{X}_t}] = \mathbb{E}[e^{x\vee\overline{X}_\infty}],$$

we deduce (ii). □

Note that when $X \in \mathcal{G}\backslash\mathcal{H}_q$ it follows that

$$q \geq \mathrm{d} > 0 \vee (\mathrm{d} + \int_{(-\infty,0)} (e^x - 1)\,\Pi_X(dx)) = 0 \vee \psi(1)$$

and hence the four cases in the above theorem constitute an exhaustive partition of the regime $q \geq 0$ for the optimal stopping problem (6.3).

Now, turning to the solution of the Shepp–Shiryaev stochastic game, it turns out that it is necessary to divide the regime $q \geq 0$ into many more cases. We present our main results accordingly.

Theorem 6.2 (The case $q = 0$). *When $q = 0$ the solution to (6.1) is given as follows:*

(i) when $\psi(1) \geq 0$ we have for any $\delta > 0$ that $\sigma^* = 0$ and hence $V(x) = e^x + \delta$,

(ii) when $\psi(1) < 0$ and $(\Phi(0) - 1)\delta > 1$ we have that $\tau^* = \sigma^* = \infty$ and

$$V(x) = e^x + \frac{1}{(\Phi(0) - 1)} e^{x(1-\Phi(0))},$$

for $x \geq 0$,

(iii) when $\psi(1) < 0$ and $(\Phi(0) - 1)\delta \leq 1$ we have $\tau^* = \infty$, $\sigma^* = T_0^-$ and

$$V(x) = e^x + \delta e^{x(1-\Phi(0))}.$$

Theorem 6.3 (The case $0 < q < \psi(1)$). *Suppose $0 < q < \psi(1)$. Let f be defined as in (6.10).*

The equation

$$f(y) = 1 \quad , y > 0 \tag{6.11}$$

has a unique solution (which we denote by y^*).

(i) If $\delta > Z^{(q)}(y^*) - 1$ then

$$V(x) = \begin{cases} e^x + \delta & \text{when } x < a^*, \\ e^x Z^{(q)}(b^* - x) & \text{when } x \geq a^*, \end{cases}$$

where $0 < a^* < b^* < \infty$ satisfy

$$Z^{(q)}(b^* - a^*) = 1 + \delta e^{-a^*}, \tag{6.12}$$

$$b^* = a^* + y^*, \tag{6.13}$$

with $\sigma^* = T_{a^*}^-$ and $\tau^* = T_{b^*}^+$. Further, the function $V(x)$ is monotone increasing and $V(x) - e^x$ is monotone decreasing.

(ii) If $\delta \leq Z^{(q)}(y^*) - 1$, then there exists a unique $z^* \in (0, y^*]$ which satisfies $Z^{(q)}(z^*) = 1 + \delta$ and then

$$V(x) = e^x Z^{(q)}(z^* - x)$$

and the optimal stopping times are given by $\sigma^* = T_0^-$ and $\tau^* = T_{z^*}^+$.

Theorem 6.4 (The case $q > 0$ and $q \geq \psi(1)$). *Let $q > 0$. Recall that z^* is the unique solution of $Z^{(q)}(z) = 1 + \delta$ which always exists uniquely as $Z^{(q)}$ is a strictly increasing function with $Z^{(q)}(0) = 1$ and $Z^{(q)}(\infty) = \infty$. Also, recall that for $q > 0$ and $X \in \mathcal{H}_q$, the equation $f(x) = 0$ has a unique solution, denoted by k^*.*

(i) When $q = \psi(1)$ and $\delta > 0$ we have $\sigma^* = T_0^-$, $\tau^* = T_{z^*}^+$ and

$$V(x) = e^x Z^{(q)}(z^* - x),$$

(ii) when $q > \psi(1)$, $X \in \mathcal{H}_q$ and $\delta > Z^{(q)}(k^*) - 1$ so that $k^* < z^*$ we have $\sigma^* = \infty$, $\tau^* = T_{k^*}^+$ and

$$V(x) = U(x) = e^x Z^{(q)}(k^* - x),$$

(iii) when $q > \psi(1)$, $X \in \mathcal{H}_q$ and $\delta \leq Z^{(q)}(k^*) - 1$ so that $k^* \geq z^*$ we have $\sigma^* = T_0^-$, $\tau^* = T_{z^*}^+$ and

$$V(x) = e^x Z^{(q)}(z^* - x),$$

(iv) when $q > \psi(1)$ and $X \in \mathcal{G} \backslash \mathcal{H}_q$, we have for any $\delta > 0$ that $\tau^* = 0$, hence

$$V(x) = e^x.$$

Remark 6.5 (Intuition). We briefly discuss some of the intuition behind the results of Theorems 6.2, 6.3 and 6.4.

When $q = 0$, one might expect it not to be optimal for the max-player to stop, since the gain in (6.2) is non-decreasing in time. One would also expect the min-player to never stop when the penalty δ is too large, which is indeed the conclusion of Theorem 6.2 (ii). When $\psi(1) \geq 0$ we have

$$\mathbb{E}[e^{\overline{X}_t \vee x}] \geq \mathbb{E}[e^{X_t}] = e^{\psi(1)t},$$

which indicates that the min-player cannot gain by waiting and hence should stop immediately. When $\psi(1) < 0$ and δ is below a critical value, it becomes worthwhile for the min-player to stop. Since $\psi(1) < 0$ implies that $\mathbb{E}[e^{X_t}]$ decreases in t, it might be lucrative for the min-player not to stop immediately and it turns out that it is optimal for the min-player to stop when the reflected process Y reaches its minimum 0. Note that this stopping time is infinite with positive probability.

When $q \geq \psi(1)$ and $q > 0$, we observe the same phenomenon that the min-player stops when Y reaches 0 providing δ is below a critical value. This time, since $q > 0$, the min-player should also stop in an almost surely finite stopping time and indeed this happens at the first time Y exceeds a certain positive value (possibly by a jump).

When $0 < q < \psi(1)$, the discount factor α in (6.6) is negative and therefore the min-player should stop at an almost surely finite time. It also seems plausible that the min-player should stop sooner than when $\psi(1) \leq q$, resulting in an optimal stopping set of the form $[0, a^*]$. However, this only happens when the penalty δ is large enough. It might seem counter-intuitive that the min-player is more eager to stop when the penalty is large, but this strategy could be explained by reasoning that the min-player is tolerant to the negative discount factor α in (6.6) as long as δ is small enough. When δ becomes too large, the min-player needs to take evasive action by stopping sooner.

Remark 6.6 (Pasting conditions). Theorems 6.3 and 6.4 both say that the value function of the Shepp–Shiryaev stochastic game necessarily takes the form

$$V(x) = \begin{cases} \delta + e^x & \text{when } x < a, \\ e^x Z^{(q)}(b-x) & \text{when } x \geq a, \end{cases}$$

for some $0 \leq a \leq b < \infty$. As a consequence of the behaviour at the origin of the scale functions $Z^{(q)}$ and $W^{(q)}$, if follows that when $a > 0$, there is smooth pasting at a (in accordance with the fact that 0 is regular for $(0, \infty)$ for X). Further, when $b < \infty$, there always is continuity at b and smooth pasting at b if and only if X has unbounded variation (corresponding to the case that 0 is regular for $(-\infty, 0)$ for X). See [1] for a discussion on the relevance of path regularity to pasting conditions.

The rest of this chapter is structured as follows. In the next section we make note of a lemma which provides sufficient conditions for stopping times to be a saddle point for the optimal stopping game (6.6) under the change of measure. This lemma essentially allows us to 'verify' directly that the solutions presented in Theorems 6.3 and 6.4 are indeed optimal. In addition, we present the candidate functions which will be used in conjunction with the Verification Lemma to establish the solution. In Section 6.4 we give the proof of Theorem 6.3. Having done this, one sees that the proof of Theorem 6.4 is a straightforward variant of a part of the proof of Theorem 6.3. We only comment briefly in Section 6.5 on the proof of Theorem 6.4, which is otherwise left as an exercise for the reader. In Section 6.6 we give the proof of Theorem 6.2. The proof differs from the proofs of Theorems 6.3 and 6.4 in the sense that one may no longer appeal to the change of measure (6.4).

6.3 Preliminary results

Following classical ideas in optimal stopping, we verify that a candidate solution solves the Shepp–Shiryaev game by checking certain associated bounds and martingale properties. Specifically, we use the following Verification Lemma, which is a variant of the similar one in Chapter 5.

Lemma 6.7 (Verification Lemma). *Suppose that $\tau^* \in \mathcal{T}_{0,\infty}$ and $\sigma^* \in \mathcal{T}_{0,\infty}$ are candidate optimal strategies for the stochastic game (6.1) such that*

$$e^{Y_\sigma^x} \mathbf{1}_{\{\sigma < \tau^*\}} \tag{6.14}$$

is uniformly bounded by a constant for all $\sigma \in \mathcal{T}_{0,\infty}$ and $x \geq 0$. Let

$$V^*(x) = \mathbb{E}^1[e^{-\alpha\tau^* + Y_{\tau^*}^x} \mathbf{1}_{\{\tau^* \leq \sigma^*,\, \tau^* < \infty\}} + e^{-\alpha\sigma^*} \left(e^{Y_{\sigma^*}^x} + \delta\right) \mathbf{1}_{\{\sigma^* < \tau^*\}}].$$

Then the triple (V^*, τ^*, σ^*) is a solution to (6.1) if *(i)* $V^*(x) \geq e^x$,

(ii) $V^*(x) \leq e^x + \delta$,

(iii) $V^*(Y_{\tau^*}) = e^{Y_{\tau^*}}$ *almost surely on* $\{\tau^* < \infty\}$,

(iv) $V^*(Y_{\sigma^*}) = e^{Y_{\sigma^*}} + \delta$ *almost surely on* $\{\sigma^* < \infty\}$,

(v) the process $\{e^{-\alpha(t\wedge\tau^*)}V^*(Y_{t\wedge\tau^*})\}_{t\geq 0}$ *is a right-continuous submartingale under* $\mathbb{P}^1$ *and*

(vi) the process $\{e^{-\alpha(t\wedge\sigma^*)}V^*(Y_{t\wedge\sigma^*})\}_{t\geq 0}$ *is a right-continuous supermartingale under* $\mathbb{P}^1$.

Proof. Define for each $\tau, \sigma \in \mathcal{T}_{0,\infty}$

$$\Theta^\alpha_{\tau,\sigma} = e^{-\alpha\tau + Y^x_\tau}\mathbf{1}_{\{\tau\leq\sigma,\,\tau<\infty\}} + e^{-\alpha\sigma}\left(e^{Y^x_\sigma} + \delta\right)\mathbf{1}_{\{\sigma<\tau\}}.$$

From the supermartingale property (vi), Doob's optional stopping theorem, (i) and (iv) we know that for any $\tau \in \mathcal{T}_{0,\infty}$ and $t \geq 0$,

$$\begin{aligned} V^*(x) &\geq \mathbb{E}^1[e^{-\alpha(t\wedge\tau\wedge\sigma^*)}V^*(X_{t\wedge\tau\wedge\sigma^*})] \\ &\geq \mathbb{E}^1[e^{-\alpha(t\wedge\tau)+Y^x_{t\wedge\tau}}\mathbf{1}_{\{\sigma^*\geq t\wedge\tau,\,\tau<\infty\}} + e^{-\alpha\sigma^*}(e^{Y^x_{\sigma^*}} + \delta)\mathbf{1}_{\{\sigma^*<t\wedge\tau\}}]. \end{aligned}$$

By taking limits $t \to \infty$, it follows from Fatou's lemma that

$$V^*(x) \geq \mathbb{E}^1[\Theta^\alpha_{\tau,\sigma^*}].$$

Next, we show that

$$V^*(x) \leq \mathbb{E}^1[\Theta^\alpha_{\tau^*,\sigma}]. \tag{6.15}$$

If σ is such that

$$\mathbb{E}^1[e^{-\alpha\sigma}\mathbf{1}_{\{\sigma<\tau^*\}}] = \infty,$$

then (6.15) holds trivially (it then reads $V^*(x) \leq \infty$). Hence, we assume $\sigma \in \mathcal{T}_{0,\infty}$ satisfies

$$\mathbb{E}^1[e^{-\alpha\sigma}\mathbf{1}_{\{\sigma<\tau^*\}}] < \infty.$$

Using (v), Doob's optional stopping theorem, (ii) and (iii) we find

$$\begin{aligned} V^*(x) &\leq \mathbb{E}^1[e^{-\alpha(t\wedge\tau^*\wedge\sigma)}V^*(X_{t\wedge\tau^*\wedge\sigma})] \\ &= \mathbb{E}^1[e^{-\alpha\tau^*}V^*(X_{\tau^*})\mathbf{1}_{\{\tau^*\leq t\wedge\sigma\}} + e^{-\alpha(t\wedge\sigma)}V^*(X_{t\wedge\sigma})\mathbf{1}_{\{\tau^*>t\wedge\sigma\}}] \\ &\leq \mathbb{E}^1[e^{-\alpha\tau^*+Y^x_{\tau^*}}\mathbf{1}_{\{\tau^*\leq t\wedge\sigma\}} + e^{-\alpha(t\wedge\sigma)}(e^{Y^x_{t\wedge\sigma}} + \delta)\mathbf{1}_{\{\tau^*>t\wedge\sigma\}}]. \end{aligned}$$

Taking limits as $t \uparrow \infty$ and applying the monotone convergence theorem to the first term on the right hand side and the dominated convergence theorem (see (6.14)) to the second term on the right hand side, we find that indeed

$$V^*(x) \leq \mathbb{E}^1[\Theta^\alpha_{\tau^*,\sigma}]$$

and hence the saddle point is achieved with the strategies (τ^*, σ^*). □

Note that Lemma 6.7 implies that when $\delta \geq \sup_{x\geq 0}(U(x) - e^x)$, a solution to the game is given by taking $V^* = U$ and τ^* as in Theorem 6.1 and $\sigma^* = \infty$. This agrees with the intuition that the min-player will force a greater payment by stopping than the max-player would otherwise induce by stopping and hence it is better for the min-player not to stop at all.

Often, we shall apply the above Verification Lemma to solutions of the form $V(x,a,b)$ for $0 \leq a < x < b < \infty$ corresponding to taking strategies $\sigma = T_a^-$ and $\tau = T_b^+$ in (6.1). That is,

$$V(x,a,b) = \mathbb{E}^1[e^{-\alpha T_b^+ + Y^x_{T_b^+}} \mathbf{1}_{\{T_b^+ \leq T_a^-\}} + e^{-\alpha T_a^-}\left(e^{Y^x_{T_a^-}} + \delta\right)\mathbf{1}_{\{T_a^- < T_b^+\}}]. \tag{6.16}$$

Using fluctuation theory we prove the following result.

Lemma 6.8. *Let* $0 \leq a < x < b < \infty$. *Then*

$$\begin{aligned} V(x,a,b) &= e^x\left(Z^{(q)}(b-x) - W^{(q)}(b-x)\frac{Z^{(q)}(b-a)}{W^{(q)}(b-a)}\right. \\ &\qquad\qquad \left. +(1+\delta e^{-a})\frac{W^{(q)}(b-x)}{W^{(q)}(b-a)}\right). \end{aligned} \tag{6.17}$$

Proof. Let $0 \leq a < x < b < \infty$. Note that $\tau^-_{x-b} = T_b^+$ on the event $\{T_b^+ < T_a^-\}$ and $\tau^+_{x-a} = T_a^-$ on the event $\{T_a^- < T_b^+\}$. We change measure using (6.4), then use (6.7) and (6.9) to derive

$$\begin{aligned} V(x,a,b) &= \mathbb{E}^1[e^{-\alpha T_b^+ + Y^x_{T_b^+}} \mathbf{1}_{\{T_b^+ \leq T_a^-\}} + e^{-\alpha T_a^-}\left(e^{Y^x_{T_a^-}} + \delta\right)\mathbf{1}_{\{T_a^- < T_b^+\}}] \\ &= \mathbb{E}[e^{-q\tau^-_{x-b} + (x \vee \overline{X}_{\tau^-_{x-b}})} \mathbf{1}_{\{\tau^-_{x-b} \leq \tau^+_{x-a}\}}] \\ &\quad + (\delta + e^a)\mathbb{E}[e^{-q\tau^+_{x-a} + X_{\tau^+_{x-a}}} \mathbf{1}_{\{\tau^+_{x-a} < \tau^-_{b-x}\}}] \\ &= e^x\left(Z^{(q)}(b-x) - W^{(q)}(b-x)\frac{Z^{(q)}(b-a)}{W^{(q)}(b-a)}\right. \\ &\qquad\qquad \left. +(1+\delta e^{-a})\frac{W^{(q)}(b-x)}{W^{(q)}(b-a)}\right), \end{aligned}$$

which was to be shown. $\square$

6.4 Proof of Theorem 6.3

We begin with a preliminary lemma (from which the opening part of Theorem 6.3 follows) concerning the function $f(x)$ defined in (6.10).

Lemma 6.9. *Suppose* $0 < q < \psi(1)$. *Then the function* f *satisfies* $\lim_{x\to\infty} f(x) = \infty$ *and* $f(\varepsilon) < 1$ *for all small enough* $\varepsilon > 0$. *Furthermore,* f *has a minimum valued in* $(0,1)$ *which is uniquely attained, say at* m. *The function* f *is strictly increasing on* (m,∞). *In particular, the equation* $f(y) = 1$ *has a unique solution (denoted by* y^**) on* $(0,\infty)$, $f(x) \leq 1$ *for* $x \leq y^*$ *and* $f'(y^*) > 0$.

Proof. In the case when X has unbounded variation, $f(0) = 1$ and

$$f'(0+) = -qW^{(q)\prime}(0+) = -q \lim_{\lambda\to\infty} \frac{\lambda^2}{\psi(\lambda) - q} = \begin{cases} -\frac{2q}{\sigma^2} & \text{when } \sigma > 0, \\ -\infty & \text{when } \sigma = 0. \end{cases}$$

In the case of bounded variation with drift d, we have $f(0) = 1 - q/\mathrm{d} < 1$. Also $\mathrm{d} > \psi(1) > q$, so $f(0) > 0$. Hence in either case of bounded or unbounded variation, it follows that $f(\epsilon) < 1$ for some $\epsilon > 0$.

Recalling that $W^{(q)}(x) = e^{\Phi(q)x} W_{\Phi(q)}(x)$, we have for $x > 0$

$$\begin{aligned} f'(x) &= q(W^{(q)}(x) - W^{(q)\prime}(x)) \\ &= qe^{\Phi(q)x}((1 - \Phi(q))W_{\Phi(q)}(x) - W'_{\Phi(q)}(x)). \end{aligned}$$

It is also known that

$$W'_{\Phi(q)}(x) = W_{\Phi(q)}(x) n_{\Phi(q)}(h > x),$$

where $n_{\Phi(q)}$ is the excursion measure of $\overline{X} - X$ under $\mathbb{P}^{\Phi(q)}$. Hence,

$$f'(x) = qW^{(q)}(x)(1 - \Phi(q) - n_{\Phi(q)}(h > x)) \tag{6.18}$$

and thus, in particular, $f(\infty) = \infty$ implying that the function f attains its minimum. From (6.18) it also follows that if $f'(x) \geq 0$ for some x, then $f'(y) > 0$ for all $y > x$. From the first paragraph of this proof we deduce that the minimum of f is valued in $(-\infty, 1)$ and that this minimum is uniquely attained (say at m). We deduce that the equation $f(y) = 1$ has a unique solution on $(0, \infty)$ (denoted by y^*). Clearly, $y^* > m$ and it readily follows that $f'(y^*) > 0$ and that $f(x) \leq 1$ for all $x \leq y^*$.

We now show positivity of f. It is known from the Wiener–Hopf factorisation (cf. Chapter 8 of [63]) that

$$\frac{1}{q} > \frac{1}{q}\mathbb{P}(-\underline{X}_{\mathbf{e}_q} \in [0, x]) = \frac{1}{\Phi(q)} W^{(q)}(x) - \int_0^x W^{(q)}(y)\, dy,$$

where $\mathbf{e}_q$ is an independent and exponentially distributed random variable with parameter q and $\underline{X}_t = \inf_{s\leq t} X_s$. Since $0 < q < \psi(1)$, it follows that $-\Phi(q)^{-1} < -1$ and hence

$$f(x) > Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x) = 1 - q\left(\frac{1}{\Phi(q)} W^{(q)}(x) - \int_0^x W^{(q)}(y)\, dy\right) > 0,$$

which completes the proof. □

We now divide the forthcoming analysis into the two cases $\delta > Z^{(q)}(y^*) - 1$ and $\delta \leq Z^{(q)}(y^*) - 1$ corresponding to parts (i) and (ii) respectively of Theorem 6.3.

6.4.1 The case $\delta > Z^{(q)}(y^*) - 1$.

Under this sub-regime of $0 < q < \psi(1)$, we have the existence of $0 < a^* < b^* < \infty$ satisfying

$$Z^{(q)}(y^*) = 1 + \delta e^{-a^*}$$

and

$$b^* = a^* + y^*,$$

where y^* was defined as the unique solution in $(0, \infty)$ of (6.11). Note that this choice of a^* and b^* has the convenient implication that for $x \geq a^*$

$$V(x, a^*, b^*) = e^x Z^{(q)}(b^* - x).$$

From the latter, we see that on $[a^*, b^*)$

$$V(x, a^*, b^*) > e^x.$$

Moreover, $V'(x, a^*, b^*) = e^x f(b^* - x)$ and, on account of the fact that $f(b^* - x) < 1$ for all $x > a^*$, it follows that

$$V(x, a^*, b^*) < e^x + \delta \quad \text{for all } x > a^*. \tag{6.19}$$

Since f is positive, it also follows that $V(x, a^*, b^*)$ is increasing in x, and thus, in particular,

$$e^{a^*} + \delta = V(a^*, a^*, b^*) < V(b^*, a^*, b^*) = e^{b^*}. \tag{6.20}$$

Next, define the function $\theta : \mathbb{R} \mapsto \mathbb{R}$ by

$$\theta(x) = Z^{(q)}(b^* - x) - 1 - \delta e^{-x}. \tag{6.21}$$

We will shortly make use of the following lemma.

Lemma 6.10. *The function θ satisfies $\theta(a^*) = 0$ and*

$$\theta(x) < 0 \quad \textit{for all } x < a^*.$$

Proof. The statement $\theta(a^*) = 0$ rephrases (6.12). Next, differentiating and using the fact that $b^* - a^* = y^*$ (and hence $f(b^* - a^*) = 1$), we find

$$\theta'(a^*) = -qW^{(q)}(b^* - a^*) + \delta e^{-a^*} = 1 - Z^{(q)}(b^* - a^*) + \delta e^{-a^*} = 0.$$

From Lemma 6.9 we have $f'(y^*) > 0$ and hence $W^{(q)\prime}(b^* - a^*) < W^{(q)}(b^* - a^*)$, which in turn implies that

$$\theta''(a^*) = qW^{(q)\prime}(b^* - a^*) - \delta e^{-a^*} < qW^{(q)}(b^* - a^*) - \delta e^{-a^*} = 0. \tag{6.22}$$

In particular, $\theta(x) < 0$ for all $x \in (a^* - \varepsilon, a^*)$ and some sufficiently small $\varepsilon > 0$.

Suppose now, for contradiction, that $c = \sup\{x < a^* - \varepsilon : \theta(x) = 0\} > -\infty$. Then by Rolle's theorem, there exists some $d \in (c, a^*)$ such that $\theta(d) < 0$ and $\theta'(d) = 0$. On the other hand, for $x < a^*$ we also have

$$\begin{aligned}\theta(x) + \theta'(x) &= f(b^* - x) - 1 \\ &> f(b^* - a^*) - 1 \\ &= 0,\end{aligned}$$

where we have used Lemma 6.9. In particular with $x = d$ we find $\theta(d) > 0$ which is in contradiction with the definition of d. In conclusion, $c = a^*$ and $\theta(x) < 0$ for all $x < a^*$ as required. □

Our strategy for proving part (i) of Theorem 6.3 will be to look at an auxiliary optimal stopping problem and then use the information above to associate the solution of the aforementioned optimal stopping problem with the solution of the Shepp–Shiryaev stochastic game. To this end, let

$$I(x) = \inf_{\sigma \in \mathcal{T}_{0,\infty}} \mathbb{E}^1[e^{-\alpha\sigma} g(\tilde{Y}^x_\sigma)], \tag{6.23}$$

where $\tilde{Y}^x_\sigma := Y^x_{\sigma \wedge T^+_{b^*}}$, g is any continuous function such that

$$g(x) = \begin{cases} e^x + \delta & \text{when } x < a^*, \\ e^x & \text{when } x \geq b^* \end{cases}$$

and

$$e^x + \delta > g(x) > e^x Z^{(q)}(b^* - x)$$

for any $x \in (a^*, b^*)$.

Theorem 6.11. *There exists a solution to the optimal stopping problem (6.23) with the following properties.*

(i) For $x > a^*$, $I(x) = V(x, a^*, b^*)$ and hence $\sigma^* = T^-_{a^*}$.

(ii) For all $x \in (0, b^*)$, $I(x) > e^x$.

Proof. By taking $\sigma = 0$ in the expectation on the right hand side of (6.23), we see that $I(x) \leq (e^x + \delta)$. Hence, it follows that

$$I(x) = I(x) \wedge (e^x + \delta) = \inf_{\sigma \in \mathcal{T}_{0,\infty}} \mathbb{E}^1[(e^{-\alpha\sigma} g(\tilde{Y}^x_\sigma)) \wedge (e^x + \delta)]$$

and (6.23) is an optimal stopping problem for a strong Markov process where, for each fixed $x \geq 0$, the pay-off function is continuous and bounded but as a function of x the stochastic gain is locally bounded in x.

Taking note of (2.2.80) in [92], we may now invoke Corollary 2.9 in the same reference to deduce the existence of an optimal stopping time σ^* in (6.23) which is of the form

$$\sigma^* = \inf\{t > 0 : \tilde{Y}^x_t \in D\},$$

where $D = \{x : I(x) = g(x)\}$.

Since $\alpha = q - \psi(1) < 0$ and $g(x) \geq 1 + \delta$, we have that $0 \in D$. Now, define

$$s := \sup\{0 \leq x < b^* : I(x) = g(x)\}.$$

Taking $\sigma = T^-_{a^*}$ in the expectation on the right hand side of (6.23) leads to a value of $V(x, a^*, b^*)$ and thus for any $x > a^*$ it holds that $I(x) \leq V(x, a^*, b^*) < e^x + \delta$ where the last inequality follows by virtue of (6.19). As a consequence, we now see that $s \leq a^*$.

(i) We want to rule out the case that $s < a^*$ and then part (i) will follow. Suppose for contradiction that $s < a^*$. Then, on $[s, b^*]$ we have $I = V(\cdot, s, b^*)$. In particular, it holds that

$$\begin{aligned} I'(s+) &= V'(s, s, b^*) \\ &= e^s + \delta \\ &\quad + e^s \left(-qW^{(q)}(b-s) + \frac{W^{(q)\prime}(b-s)}{W^{(q)}(b-s)} \left(Z^{(q)}(b-s) - 1 - \delta e^{-s} \right) \right). \end{aligned}$$

The fact that $0 < f'(b^* - a^*)$ implies that $W^{(q)\prime}(b^* - s)/W^{(q)}(b^* - s) < 1$ and thus, using Lemma 6.10 we find

$$\begin{aligned} I'(s) &> e^s + \delta + e^s \left(-qW^{(q)}(b^* - s) + Z^{(q)}(b^* - s) - 1 - \delta e^{-s} \right) \\ &> e^s, \end{aligned}$$

where the last inequality is a consequence of the fact that $f(b^* - s) > f(b^* - a^*) = 1$. Since $I(s) = e^s + \delta$, the previous calculations indicate that I violates its upper bound $e^x + \delta$. We conclude that $s = a^*$ and thus $(a^*, b^*) \subseteq D^c$.

(ii) The next step in the proof is to show that for all $x \geq 0$

$$I(x) > e^x. \tag{6.24}$$

We prove (6.24) by contradiction. First, we show that there are only a finite number of intervals (l, r) satisfying $(l, r) \subset D^c$, $0 \leq l < r \leq a^*$, $I(l) = e^l + \delta$, $I(r) = e^r + \delta$ and such that there is some $x \in (l, r)$ for which $I(x) \leq e^x$. Indeed, since $\alpha < 0$, taking into account the fact that optimal stopping occurs whenever $\tilde{Y}^x$ hits the domain D and that X is spectrally negative, we deduce that for any $x \in (l, r)$, with (l, r) an interval satisfying the properties above,

$$I(x) \geq e^l + \delta.$$

For this inequality, we also use that $(a^*, b^*) \subset D^c$ implies

$$\inf_{x \in D \cap [r, \infty)} g(x) = e^r + \delta > e^l + \delta,$$

as for $x \geq a^*$ it holds that (see (6.20))

$$g(x) \geq V(x, a^*, b^*) \geq V(a^*, a^*, b^*) = e^{a^*} + \delta > e^l + \delta.$$

Hence, whenever $x \in (l, r)$ satisfies $I(x) < e^x$, then it must hold that $x \geq \log(e^l + \delta)$. In particular, (l, r) is necessarily of minimal length $\log(e^l + \delta) - l$ and therefore there can only be a finite number of intervals of this form. Now let (l^*, r^*) be the rightmost of such intervals. Choose $x \in (l^*, r^*)$ and define

$$T_{(l^*,r^*)} := \inf\{t > 0 : Y_t^x \notin (l^*, r^*)\}.$$

Note that $T_{(l^*,r^*)} \leq T_D$, where

$$T_D = \inf\{t > 0 : Y_t^x \in D\}.$$

Since

$$\{e^{-\alpha(t \wedge T_D)} I(Y^x_{t \wedge T_D})\}$$

is a $\mathbb{P}^1$-martingale (cf. Theorem 2.4 in [92]) we have

$$\begin{aligned}
I(x) &= \mathbb{E}^1[e^{-\alpha T_{(l^*,r^*)}} I(Y^x_{T_{(l^*,r^*)}})] \\
&= \mathbb{E}^1[e^{-\alpha T_{(l^*,r^*)}} \left((e^{Y^x_{T^-_{l^*}}} + \delta) \mathbf{1}_{\{T^-_{l^*} < T^+_{r^*}\}} + I(Y^x_{T^+_{r^*}}) \mathbf{1}_{\{T^+_{r^*} < T^-_{l^*}\}} \right)] \\
&\geq \mathbb{E}^1[e^{-\alpha T_{(l^*,r^*)}} \left((e^{Y^x_{T^-_{l^*}}} + \delta) \mathbf{1}_{\{T^-_{l^*} < T^+_{r^*}\}} + e^{Y^x_{T^+_{r^*}}} \mathbf{1}_{\{T^+_{r^*} < T^-_{l^*}\}} \right)] \\
&= V(x, l^*, r^*),
\end{aligned}$$

where for the inequality we used the fact that we have chosen (l^*, r^*) as the rightmost interval on which $I(x) > e^x$ fails. Since $r^* \leq b^*$, we have for $x \in (l^*, r^*)$

$$\begin{aligned}
&V(x, l^*, r^*) \\
&= e^x \left(Z^{(q)}(r^* - x) - W^{(q)}(r^* - x) \frac{Z^{(q)}(r^* - l^*)}{W^{(q)}(r^* - l^*)} + (1 + \delta e^{-l^*}) \frac{W^{(q)}(r^* - x)}{W^{(q)}(r^* - l^*)} \right) \\
&\geq e^x \left(Z^{(q)}(r^* - x) - \frac{W^{(q)}(r^* - x)}{W^{(q)}(r^* - l^*)} \left(Z^{(q)}(b^* - l^*) - 1 - \delta e^{-l^*} \right) \right) \\
&\geq e^x Z^{(q)}(r^* - x) \\
&> e^x,
\end{aligned}$$

where we have used Lemma 6.10. This contradiction has the desired implication that $I(x) > e^x$ for all $x < a^*$. □

In the next result we establish that there exists a saddle point for the Shepp–Shiryaev stochastic game.

Proposition 6.12. *The stochastic game (6.1) has a solution and its value satisfies $V(x) = I(x)$ for all $x \geq 0$.*

Proof. The proof uses the Verification Lemma 6.7 and the candidate triple $V^* = I$, σ^* equal to the optimal strategy in the solution of (6.23) and $\tau^* = T^+_{b^*}$.

From Theorem 6.11 we have that $I(x)$ fulfils conditions (i)–(iv) of Lemma 6.7. By standard optimal stopping theory and the strong Markov property, the submartingale property (v) automatically holds, see for example Theorem 2.4 of [92]. To justify the remaining condition (vi), one notes that on (a^*, ∞) the function $I(x) = e^x Z^{(q)}(b^* - x)$ is sufficiently smooth on (a^*, b^*) to apply the Itô formula (cf. [98]). It is standard to deduce from the strong Markov property that $\{e^{-\alpha t} e^{Y^x_t} Z^{(q)}(b^* - Y^x_t) : t < T^+_{b^*} \wedge T^-_{a^*}\}$ is a $\mathbb{P}^1$-martingale from which it follows that $(\Gamma - \alpha) I(x) = 0$ on (a^*, b^*), where Γ is the infinitesimal generator of $-X$ under $\mathbb{P}^1$. Note also that since $\{e^{-X_t + \psi(1)t} : t \geq 0\}$ is a martingale under $\mathbb{P}^1$, we have that $(\Gamma - \alpha)e^x = -qe^x$ for all $x \in \mathbb{R}$. It now follows that $(\Gamma - \alpha)I(x) \leq 0$ for $x \geq a^*$. Although the function I fails to have a continuous second derivative only at b^* it is still smooth enough to use I in conjunction with Itô's formula (cf. [98]). The required supermartingale condition can now be deduced from the semi-martingale decomposition of $\{e^{-\alpha t} e^{Y^x_t} Z^{(q)}(b^* - Y^x_t) : t < T^-_{a^*}\}$. Note that right-continuity of the paths of all the above semi-martingales is clear. □

Were it not for the fact that we have not yet proved that $I(x) = e^x + \delta$ for all $x \leq a^*$, we would be able to claim that the proof of Theorem 6.3 (i) is now complete. However, we must still rule out the possibility that $I(x) < e^x + \delta$ for some interval $\emptyset \neq (l, r) \subseteq [0, a^*]$. We finish this subsection by excluding this possibility and hence concluding the proof of Theorem 6.3 (i).

Theorem 6.13. *The value function $I(x) - e^x$ is decreasing and hence part (i) of Theorem 6.3 holds.*

Proof. Let $x > y \geq 0$. We use the notation $\sigma(x)$ to make explicit the dependency of the stopping time $\sigma \in \mathcal{T}_{0,\infty}$ on the initial position of the process Y^x. We then find that for any $x \geq 0$

$$\begin{aligned} V(x) \quad &\leq \quad \mathbb{E}[e^{-q\tau^*(x) + (x \vee \overline{X}_{\tau^*(x)})} \mathbf{1}_{\{\tau^*(x) \leq \sigma^*(y)\}}] \\ &\quad + \mathbb{E}[e^{-q\sigma^*(y)} \left(e^{x \vee \overline{X}_{\sigma^*(y)}} + \delta e^{X_{\sigma^*(y)}} \right) \mathbf{1}_{\{\sigma^*(y) < \tau^*(x)\}}] \end{aligned}$$

and similarly, for any $y \geq 0$

$$\begin{aligned} V(y) \quad &\geq \quad \mathbb{E}[e^{-q\tau^*(x) + (y \vee \overline{X}_{\tau^*(x)})} \mathbf{1}_{\{\tau^*(x) \leq \sigma^*(y)\}}] \\ &\quad + \mathbb{E}[e^{-q\sigma^*(y)} \left(e^{y \vee \overline{X}_{\sigma^*(y)}} + \delta e^{X_{\sigma^*(y)}} \right) \mathbf{1}_{\{\sigma^*(y) < \tau^*(x)\}}]. \end{aligned}$$

Now, let $x > y \geq 0$. Then

$$\begin{aligned} V(x) - V(y) \quad &\leq \quad \mathbb{E}[e^{-q\tau^*(x)} \left(e^{x \vee \overline{X}_{\tau^*(x)}} - e^{y \vee \overline{X}_{\tau^*(x)}} \right) \mathbf{1}_{\{\tau^*(x) \leq \sigma^*(y)\}}] \\ &\quad + \mathbb{E}[e^{-q\sigma^*(y)} \left(e^{x \vee \overline{X}_{\sigma^*(y)}} - e^{y \vee \overline{X}_{\tau^*(x)}} \right) \mathbf{1}_{\{\sigma^*(y) < \tau^*(x)\}}]. \end{aligned}$$

Since for any a it holds that

$$e^{x \vee a} - e^{y \vee a} \leq e^x - e^y,$$

we deduce

$$V(x) - V(y) \leq e^x - e^y. \tag{6.25}$$

Since $V(a^*) = e^{a^*} + \delta$ and since $V(x) \leq e^x + \delta$ for all x, it follows that $V(x) = e^x + \delta$ for all $x \in [0, a^*]$. The result follows. □

6.4.2 The case $\delta \leq Z^{(q)}(y^*) - 1$.

Let us now conclude this section and the proof of Theorem 6.3 by establishing the following result. Recall that we are still under the regime $0 < q < \psi(1)$.

Theorem 6.14. *If $\delta \leq Z^{(q)}(y^*) - 1$, then there exists a unique $z^* \in (0, y^*]$ which satisfies $Z^{(q)}(z^*) = 1 + \delta$. The value function is given by*

$$V(x) = e^x Z^{(q)}(z^* - x)$$

and optimal stopping times are $\sigma^* = T_0^-$ and $\tau^* = T_{z^*}^+$. In particular, part (ii) of Theorem 6.3 holds.

Proof. Since $1 + \delta \leq Z^{(q)}(y^*)$ and $Z^{(q)}(0) = 1$, it follows that there exists a $z^* \in (0, y^*]$ such that $1 + \delta = Z^{(q)}(z^*)$.

Next, note that from Lemma 6.8

$$V(x) = e^x Z^{(q)}(z^* - x) = V(x, 0, z^*)$$

and hence we can complete the proof by showing that the triple $(V, T_{z^*}^+, T_0^-)$ fulfils the conditions (i) – (vi) of Lemma 6.7. It is immediately clear from the definition of V that condition (i) holds. Next, note that $V'(x) = e^x f(z^* - x)$. Since by Lemma 6.9 the function f is strictly positive and since $V(0) = \delta$, it follows that $V(x) \leq \delta + e^x$ and hence condition (ii) of Lemma 6.7 holds. Conditions (iii) and (iv) are automatic.

To establish conditions (v) and (vi) of Lemma 6.7 one needs to appeal to the semi-martingale decomposition of $e^{-\alpha t} V(Y_t^x)$. In particular, again note that V is smooth enough to use in conjunction with the Itô formula and hence

$$\begin{aligned} e^{-\alpha t} V(Y_t^x) &= V(x) + \int_0^t (\Gamma - \alpha) V(Y_s^x)\, ds + \int_0^t V'(Y_s^x)\, d(x \vee \overline{X}_s) + M_t \\ &= V(x) + \int_0^t (\Gamma - \alpha) V(Y_s^x)\, ds + V'(0+)(x \vee \overline{X}_t) + M_t, \end{aligned} \tag{6.26}$$

where Γ is the generator of $-X$ under $\mathbb{P}^1$, $(\Gamma - \alpha)V(x) = 0$ for $x \in (0, z^*)$, $(\Gamma - \alpha)V(x) \leq 0$ for $x \geq 0$ and M is a martingale. Note also that the term $V'(Y_s^x)$ may be replaced by $V'(0+)$ as $x \vee \overline{X}_s$ increases only when $Y_s^x = 0$. From this, one sees in the above semi-martingale decomposition that the process $\{e^{-\alpha t} V(Y_t^x) : t \leq T_0^- \wedge T_{z^*}^+\}$ is a martingale and that $\{e^{-\alpha t} V(Y_t^x) : t \leq T_0^-\}$ is a supermartingale. Again, right-continuity of paths is obvious. As the second integral in (6.26) is equal to $V'(0+)(x \vee \overline{X}_t)$ (in particular, it is an increasing, continuous, adapted process), it follows that $\{e^{-\alpha t} V(Y_t^x) : t \leq T_{z^*}^+\}$ is also a right-continuous submartingale. This completes the proof of Theorem 6.3. □

6.5 Proof of Theorem 6.4

The proof goes along the lines of the proof of Theorem 6.14, principally appealing to the semi-martingale decomposition (6.26) for the specified triple (V, τ^*, σ^*). For Theorem 6.4 (iv) it is possible to compute exactly the quantities $(\Gamma - \alpha)V(x)$ and $V'(0+)$. In the remaining cases one may deduce the necessary properties of $(\Gamma - \alpha)V(x)$ as in the proof of Theorem 6.14 and that $V'(0+) = f(z^* \wedge k^*) \geq 0$ from the properties of the function f mentioned in Section 6.2.

6.6 Proof of Theorem 6.2

Recall that for $q = 0$, the pay-off of the game is given by

$$G^x_{t,s} = e^{\overline{X}_t \vee x}\mathbf{1}_{\{t \leq s\}} + (e^{\overline{X}_s \vee x} + \delta e^{X_s})\mathbf{1}_{\{s<t\}}. \tag{6.27}$$

Lemma 6.15. *A saddle point for the game (6.27) exists and it is optimal for the min-player to never stop, i.e.*

$$V(x) = \inf_{\sigma \in \mathcal{T}_{0,\infty}} \mathbb{E}[G^x_{\infty,\sigma}]. \tag{6.28}$$

Proof. Let $\sigma, \tau \in \mathcal{T}_{0,\infty}$ and let $t > 0$. Then on $\{\tau < \infty\}$

$$\begin{aligned}
G^x_{\tau+t,\sigma} - G^x_{\tau,\sigma} &= e^{\overline{X}_{\tau+t} \vee x}\mathbf{1}_{\{\tau+t \leq \sigma\}} - e^{\overline{X}_\tau \vee x}\mathbf{1}_{\{\tau \leq \sigma\}} \\
&\quad + (e^{\overline{X}_\sigma \vee x} + \delta e^{X_\sigma})\mathbf{1}_{\{\tau \leq \sigma < \tau+t\}} \\
&= -e^{\overline{X}_{\tau+t} \vee x}\mathbf{1}_{\{\tau \leq \sigma < \tau+t\}} + (e^{\overline{X}_{\tau+t} \vee x} - e^{\overline{X}_\tau \vee x})\mathbf{1}_{\{\tau \leq \sigma\}} \\
&\quad + (e^{\overline{X}_\sigma \vee x} + \delta e^{X_\sigma})\mathbf{1}_{\{\tau \leq \sigma < \tau+t\}} \\
&\geq -e^{\overline{X}_{\tau+t} \vee x}\mathbf{1}_{\{\tau \leq \sigma < \tau+t\}} + (e^{\overline{X}_{\tau+t} \vee x} - e^{\overline{X}_\tau \vee x})\mathbf{1}_{\{\tau \leq \sigma < \tau+t\}} \\
&\quad + (e^{\overline{X}_\sigma \vee x} + \delta e^{X_\sigma})\mathbf{1}_{\{\tau \leq \sigma < \tau+t\}} \\
&= (e^{\overline{X}_\sigma \vee x} - e^{\overline{X}_\tau \vee x} + \delta e^{X_\sigma})\mathbf{1}_{\{\tau \leq \sigma < \tau+t\}} \geq 0.
\end{aligned}$$

We have that

$$\inf_{\sigma \in \mathcal{T}_{0,\infty}} \mathbb{E}[G^x_{\infty,\sigma}] \leq \sup_{\tau \in \mathcal{T}_{0,\infty}} \inf_{\sigma \in \mathcal{T}_{0,\infty}} \mathbb{E}[G^x_{\tau,\sigma}] \leq \inf_{\sigma \in \mathcal{T}_{0,\infty}} \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}[G^x_{\tau,\sigma}] \leq \inf_{\sigma \in \mathcal{T}_{0,\infty}} \mathbb{E}[G^x_{\infty,\sigma}],$$

where the first inequality follows from the supremum and the last inequality is due to the monotonicity of $G^x_{\cdot,\sigma}$. This completes the proof. □

Remark 6.16 (Problems with change of measure). It is tempting to solve (6.28) by the change of measure we have used throughout this chapter, but the following example shows that when $q = 0$, the corresponding optimal stopping problem under $\mathbb{P}^1$ is a different one.

Let $q = 0$ and $\psi(1) < 0$ such that $\psi'(1) \geq 0$. Since $G^x_{s,t} \geq e^x$ for all s, t we immediately see that

$$\inf_{\sigma \in \mathcal{T}_{0,\infty}} \mathbb{E}[G^x_{\infty,\sigma}] \geq e^x.$$

However, the optimal stopping problem under the changed measure is given by

$$\inf_{\sigma\in\mathcal{T}_{0,\infty}} \mathbb{E}[e^{\psi(1)\sigma}(e^{Y^x_\sigma}+\delta)].$$

The latter optimal stopping problem has value zero, which can be seen by considering the sequence of stopping times $(\sigma_n)_{n\in\mathbb{N}}$ defined by

$$\sigma_n := \inf\{t\geq n : Y^x_t = 0\}$$

which is almost surely finite under $\mathbb{P}^1$. The reason for this phenomenon is that the equality

$$\mathbb{E}[e^{x\vee\overline{X}_\sigma}+\delta e^{X_\sigma}] = \mathbb{E}^1[e^{Y^x_\sigma}+\delta]$$

holds whenever $\mathbb{P}^1(\sigma<\infty) = \mathbb{P}(\sigma<\infty) = 1$. Since X drifts to $-\infty$ under $\mathbb{P}$ we have that $\mathbb{P}(\sigma_n<\infty)<1$ for any $n\in\mathbb{N}$.

On account of the above remark, we consider (6.28) as an optimal stopping problem for $(\overline{X}, X)$, just as was done in the first paper on the Russian option in [107]. We modify our notation and write for $y\geq x$

$$V(x,y) := \inf_{\sigma\in\mathcal{T}_{0,\infty}} \mathbb{E}[e^{x\vee(y+\overline{X}_\sigma)}+\delta e^{y+X_\sigma}]. \tag{6.29}$$

Again by standard theory on optimal stopping we know there exists a (possibly infinite) stopping time $\sigma^* = \sigma^*(x,y)$ at which the infimum in (6.29) is attained. We have the following verification lemma for (6.28), the proof of which is omitted as it is similar to the proof of Lemma 6.7.

Lemma 6.17. *Let $\sigma^*\in\mathcal{T}_{0,\infty}$ and let*

$$V^*(x,y) = \mathbb{E}[e^{x\vee(y+\overline{X}_{\sigma^*})}+\delta e^{y+X_{\sigma^*}}].$$

Then $(V^*(x,0),\sigma^*)$ is a solution to (6.28) if

(i) $V^*(x,y)\leq e^x+\delta e^y$,

(ii) the process $\{V^*(\overline{X}_t, X_t) : t\geq 0\}$ is a right-continuous submartingale.

Proof of Theorem 6.2. First, suppose $\psi(1)\geq 0$. Then $\{M_t\}_{t\geq 0}$ defined by

$$M_t = e^{x\vee(y+\overline{X}_t)}+\delta e^{y+X_t}$$

is a $\mathbb{P}$-submartingale. Indeed, for $0\leq s\leq t$

$$\begin{aligned}
\mathbb{E}[M_t|\mathcal{F}_s] &\geq e^{x\vee(\overline{X}_s+y)}+\delta e^{y+X_s}\mathbb{E}[e^{\tilde{X}_{t-s}}] \\
&= e^{x\vee(\overline{X}_s+y)}+\delta e^{y+X_s}e^{\psi(1)(t-s)} \\
&\geq M_s,
\end{aligned}$$

where $\tilde{X}$ denotes an independent copy of X. Hence, in this case Lemma 6.17 shows that $V(x) = e^x+\delta$ and $\sigma^* = 0$ form a solution to (6.28), which agrees with part (i) of Theorem 6.2.

Next, let $\psi(1) < 0$ and $\delta(\Phi(0) - 1) > 1$ and consider $\sigma^* = \infty$. Since $\overline{X}_\infty$ is exponentially distributed with parameter $\Phi(0)$, we find for $x \geq y$

$$\begin{aligned}
V^*(x,y) : &= \mathbb{E}[e^{x \vee (y + \overline{X}_\infty)} + \delta e^{y + X_\infty}] \\
&= \mathbb{E}[e^{x \vee (\overline{X}_\infty + y)}] \\
&= e^x \int_0^{x-y} \Phi(0) e^{-\Phi(0)z}\, dz + \int_{x-y}^\infty \Phi(0) e^{-\Phi(0)z + z + y}\, dz \\
&= e^x (1 - e^{-\Phi(0)(x-y)}) + \frac{\Phi(0)}{\Phi(0) - 1} e^x e^{-\Phi(0)(x-y)} \\
&= e^x + \frac{1}{\Phi(0) - 1} e^{-(\Phi(0)-1)x + \Phi(0)y}
\end{aligned}$$

and in particular, by the condition on δ

$$V^*(x,y) \leq e^x + \delta e^y.$$

As X is a strong Markov process, we have that

$$\overline{X}_\infty = \overline{X}_t \vee (X_t + \overline{X}'_\infty)$$

where $\overline{X}'_\infty$ is a copy of $\overline{X}_\infty$ which is independent of $\mathcal{F}_t$. Thus

$$\begin{aligned}
\mathbb{E}[e^{x \vee (\overline{X}_\infty + y)} | \mathcal{F}_t] &= \mathbb{E}[e^{x \vee (\overline{X}_t + y) \vee (X_t + y + \overline{X}'_\infty)} | \mathcal{F}_t] \\
&= V^*(x \vee (\overline{X}_t + y), X_t + y).
\end{aligned}$$

It now follows that $\{V^*(\overline{X}_t, X_t)\}_{t \geq 0}$ is a $\mathbb{P}$-martingale (and hence in particular a submartingale). Again using Lemma 6.17, we deduce part (ii) of Theorem 6.2.

Finally, let $\psi(1) < 0$ and $\delta(\Phi(0) - 1) \leq 1$ and take $x \geq y$. If we take

$$\sigma^* = \tau_x^+ = \inf\{t > 0 : X_t \geq x\} = T_0^-,$$

we have

$$\begin{aligned}
V^*(x,y) : &= \mathbb{E}[e^{\overline{X}_{\sigma^*}} + \delta e^{X_{\sigma^*}}] \\
&= e^x + \delta e^x \mathbb{P}(\tau_{x-y}^+ < \infty) \\
&= e^x + \delta e^{-(\Phi(0)-1)x + \Phi(0)y}
\end{aligned}$$

and again we have that $V^*(x,y) \leq e^x + \delta e^y$. Since $\{e^{\Phi(0)X_t}\}$ is a martingale, the submartingale property follows from Itô's formula and the fact that

$$\begin{aligned}
\int_0^t \frac{\partial}{\partial x} V^*(\overline{X_t}, X_t)\, d\overline{X}_t &= \int_0^t (e^{\overline{X}_t} + \delta(1 - \Phi(0)) e^{-(\Phi(0)-1)\overline{X}_t + \Phi(0)X_t})\, d\overline{X}_t \\
&= \int_0^t e^{\overline{X}_t} (1 + \delta - \Phi(0)\delta)\, d\overline{X}_t \geq 0,
\end{aligned}$$

where the second equality is due to the fact that $\overline{X}_t$ only increases when $X_t = \overline{X}_t$. This completes the proof of Theorem 6.2. □

6.7 Concluding remarks

Remark 6.18. In the proof of Theorem 6.13, the spectral negativity of the process is not used. This indicates that if a solution to the game exists, then the (possibly empty) sets I_1, I_2 and I_3 defined by

$$\begin{aligned} I_1 : &= \{x \in [0,\infty) : V(x) = e^x + \delta\}, \\ I_2 : &= \{x \in [0,\infty) : e^x < V(x) < e^x + \delta\}, \\ I_3 : &= \{x \in [0,\infty) : V(x) = e^x\} \end{aligned}$$

satisfy

$$x_1 < x_2 < x_3 \quad \text{for all } x_i \in I_i, \quad i = 1,2,3$$

and thus the solution to the game (if it exists) must be of the same nature for a more general Lévy process. An existence result for optimal stopping games in a general Markovian setting (including Lévy processes) can be found in [47]. In that paper, optimal stopping games are considered with a pay-off function of the general form

$$G_1(X_\tau)\mathbf{1}_{\{\tau<\sigma\}} + G_2(X_\sigma)\mathbf{1}_{\{\sigma<\tau\}} + G_3(X_\tau)\mathbf{1}_{\{\sigma=\tau\}},$$

where τ and σ are the strategies of the max-player and min-player, respectively. For the Shepp–Shiryaev game the strong Markov process is $(t, X_t, \overline{X}_t)$ and the functions G_1, G_2 and G_3 are given by

$$G_1(t,x,s) = G_3(t,x,s) = e^{-qt+s} \quad \text{and } G_2(t,x,s) = e^{-qt}(e^s + \delta e^x).$$

The assumptions

$$\mathbb{E}_x \sup_{t\geq 0} |G_i(X_t)| < \infty \qquad \text{for } x \in \mathbb{R} \quad (i = 1,2,3) \tag{6.30}$$

in [47] on the pay-off functions (to imply existence of a solution to the corresponding optimal stopping game) are consistent with the traditional assumption

$$\mathbb{E}_x \sup_{t\geq 0} |G(X_t)| < \infty$$

for existence results for optimal stopping problems with pay-off G. We find that we cannot always fit the Shepp–Shiryaev game in this framework because of the same reason the Russian optimal stopping problem does not always satisfy the assumptions of traditional optimal stopping theorems.

For example, when X is a spectrally negative Lévy process with $\psi(1) > q$, the conditions (6.30) are not satisfied since $\mathbb{E}\sup_{t\geq 0} |G_2(t, X_t, \overline{X}_t)|$ is bounded from below by $\sup_{t\geq 0} \delta\mathbb{E}[e^{-qt+X_t}] \geq \sup_{t\geq 0} \delta e^{(\psi(1)-q)t} = \infty$.

Remark 6.19. When X is a Brownian motion with parameter $\sigma > 0$ and drift μ, it can be directly checked by taking Laplace transforms that the scale functions for X are given by

$$W^{(q)}(x) = \frac{2}{\sigma^2\epsilon} e^{\beta x}\sinh(\epsilon x), \quad Z^{(q)}(x) = e^{\beta x}\cosh(\epsilon x) - \frac{\beta}{\epsilon} e^{\gamma x}\sinh(\epsilon x),$$

where $\epsilon = \frac{1}{\sigma}\sqrt{\mu^2/\sigma^2 + 2q}$ and $\beta = -\mu/\sigma^2$. In [61] the game is solved whenever $q > \psi(1) \geq 0$. Since, when δ is large enough, z^* satisfies $Z^{(q)}(z^*) = 1 + \delta$ we find that $k_* := e^{z^*}$ solves

$$k_*^{\beta}(k_*^{\epsilon} + k_*^{-\epsilon}) - \frac{\beta}{\epsilon} k_*^{\beta}(k_*^{\epsilon} - k_*^{-\epsilon}) = 2(1 + \delta),$$

which agrees with (7) in [61]. In [61] there is an additional technical condition (4) on the optimal stopping boundary k_*. The aforementioned condition pertains to the requirement that $V'(0+) \geq 0$ (which ultimately is required for the appropriate submartingale property to hold). Working here with general spectrally negative Lévy processes, and in particular with scale functions, has seemingly produced arguments which have circumvented the need for such a condition. Hence, from the results in this chapter, it follows that the claim in [61] (that this condition is necessary) is in fact false.

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Samenvatting

Wanneer is stoppen optimaal? Het is essentieel bij optimale stop problemen om op deze vraag een antwoord te geven. Een belangrijk voorbeeld van een optimaal stop probleem is een Amerikaanse optie. Stel dat je bepaalde aandelen bezit, maar vreest dat de koers ervan zal dalen. Om het risico hierop te vermijden, kan je een *Amerikaanse put-optie* (op dat aandeel) kopen, die je het recht geeft het aandeel voor een vooraf vastgestelde prijs te verkopen op een tijdstip naar keuze. In deze context wordt de bovenstaande vraag: wat is het optimale tijdstip om het aandeel te verkopen?

Om zo'n vraagstuk vanuit een wiskundig perspectief te kunnen bestuderen, dienen we eerst het waarneembare proces (de koers van het aandeel staat immers in de krant) te modelleren door een kansproces. Een veelgebruikt model voor de aandelenkoers is gebaseerd op een Brownse beweging (vernoemd naar de Schotse botanicus Robert Brown die in 1827 de chaotische beweging van stuifmeelkorrels in een vloeistof bestudeerde). Aan het begin van de twintigste eeuw was Louis Bachelier de eerste die de toepassing van een Brownse beweging in de financiering voorstelde. Ongeveer 70 jaar later werd het op een Brownse beweging gebaseerde Black–Scholes model ontwikkeld, waarvoor Robert Merton en Myron Scholes in 1997 de Nobelprijs voor de Economie ontvingen (Fischer Black kwam niet in aanmerking voor deze prijs omdat hij in 1995 overleden was). Uit recent empirisch onderzoek blijkt echter dat het Black–Scholes model een aantal tekortkomingen vertoont. Zo zijn grote, plotse koersbewegingen moeilijk te beschrijven door middel van een Brownse beweging en observeert men vaker extreme koersen dan het model doet vermoeden. Modellen gebaseerd op zogeheten *Lévy processen* kunnen deze eigenschappen beter beschrijven. In tegenstelling tot de Brownse beweging (die zelf ook een voorbeeld van een Lévy proces is), kunnen Lévy processen ook sprongen hebben. Lévy processen vindt men ook terug in de wachtrijtheorie, verzekeringswiskunde, fragmentaties en bij vertakkingsprocessen. Dit proefschrift gaat over aspecten en toepassingen van Lévy processen.

In Hoofdstuk 2 beschouwen we een verzekeringsmaatschappij. Eveneens aan het begin van de twintigste eeuw kwam Filip Lundberg met het idee om de waarde van zo'n firma te modelleren door een constante drift plus een samengesteld Poisson proces met negatieve sprongen (zo'n proces is constant tussen de negatieve sprongen in en is zelf ook een Lévy proces). De reden hiervoor is dat de sprongen in de waarde van de firma veroorzaakt worden door de claims en dus negatief zijn. De constante drift stelt de premies voor die de klanten betalen. We vervangen het samengesteld

Poisson proces door een algemeen Lévy proces zonder positieve sprongen; zo'n proces noemt men een *spectraal negatief Lévy proces.* We veronderstellen dat de firma een tijdelijke buffer heeft om faillissement te voorkomen op het moment dat de waarde van de firma negatief wordt. Daar deze buffer niet oneindig is, bestuderen we het *laatste moment* vóór een van te voren vastgesteld tijdstip T waarop de waarde van de firma negatief is. Om de berekeningen te vereenvoudigen, vervangen we T door een onafhankelijke, exponentieel verdeelde stochast τ_q. We vinden een formule voor het laatste moment vóór τ_q waarop de waarde negatief is evenals het laatste moment vóór τ_q waarop de waarde positief of gelijk aan nul is.

In Hoofdstuk 3 bestuderen we een algemeen *Lévy proces X gespiegeld in zijn infimum.* Dit is een proces dat gelijk is aan nul op de tijdstippen waarop X een nieuw minimum bereikt en dat zich op andere tijdstippen gedraagt als X. Toepassingen hiervan vindt men vooral als het te modelleren verschijnsel niet negatief kan worden: bijvoorbeeld het waterniveau van een dam of de lengte van een wachtrij in het postkantoor. Een andere toepassing vindt men in de financiering. Het nadeel van het kopen van aandelen is dat men het risico loopt het aandeel pas te verkopen als de prijs ervan alweer aan het dalen is. Een *Russische optie* voorkomt dit mogelijke probleem, doordat deze niet de waarde van het aandeel op een bepaald tijdstip oplevert, maar juist de maximale waarde die tot dan toe is bereikt. Larry Shepp en Albert Shiryaev toonden aan dat, door een handige verandering van maat, deze optie beschouwd kan worden als een optimaal stop probleem voor het gespiegeld proces. In dit hoofdstuk gebruiken we excursietheorie om een uitdrukking te vinden voor de maat die aangeeft hoeveel tijd het gespiegelde proces in een gegeven verzameling doorbrengt.

Hoofdstuk 4 gaat over optimale stop problemen voor spectraal negatieve Lévy processen en voor zogenaamde gegeneraliseerde Ornstein–Uhlenbeck processen gedreven door een spectraal negatief Lévy proces. We tonen aan dat voor een grote klasse uitbetalingsfuncties een door Beibel en Lerche geïntroduceerde methode gebruikt kan worden voor het vinden van de optimale strategie. Voorts laten we zien wanneer het *principe van gladde verbinding* geldt. Dit veelbestudeerde maar niet altijd even goed begrepen principe houdt in dat de waardefunctie van het optimale stop probleem zonder 'knik' in verbinding staat met de uitbetalingsfunctie.

In de laatste twee hoofdstukken bestuderen we *stochastische spelen* voor een spectraal negatief Lévy proces. Dit soort spelen kan men interpreteren als een optimaal stop probleem voor twee spelers. De stochastische spelen in Hoofdstukken 5 en 6 zijn gebaseerd op de eerder genoemde Amerikaanse put-optie en de Russische optie. Deze stochastische spelen werden eerder bestudeerd voor een Brownse beweging, maar de sprongen van het Lévy proces maken het probleem een stuk lastiger. We gebruiken een bijbehorend optimaal stop probleem en ook fluctuatietheorie om de optimale strategieën van beide spelers te vinden. Tevens geven we aan voor welke spectraal negatieve Lévy processen het principe van gladde verbinding geldt. We concluderen dat de oplossing voor een Lévy proces vaak wezenlijk verschilt van die voor een Brownse beweging. Dit kan verklaard worden doordat de sprongen van het Lévy proces plotselingen koerswijzigingen tot gevolg kunnen hebben. Hierdoor wordt het proces in zekere zin minder voorspelbaar en zullen de spelers voorzichtiger worden.

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Curriculum Vitae

Erik Baurdoux was born in Rotterdam, the Netherlands on the 4th of June 1980. In 1998 he obtained his European Baccalaureate at the European School Brussels (1) in Belgium, after which he moved back to the Netherlands to study mathematics at Utrecht University. In 2003 he graduated cum laude with the thesis 'Further Calculations for Israeli Options' and in September of the same year he started his Ph.D. studies at the Mathematical Institute of Utrecht University under the supervision of Prof. dr. ir. E.J. Balder and Dr. A.E. Kyprianou. Following Andreas Kyprianou's departure from Utrecht, Erik was a visiting Ph.D. student at Heriot–Watt University, Edinburgh from May 2005 until August 2006 and at the University of Bath from August 2006 until February 2007. He participated in seminars and conferences in Amsterdam, Antwerp, Bath, Brussels, Chicago, Copenhagen, Edinburgh, Edzell, Eindhoven, Glasgow, Hilversum, Lunteren, Manchester and Marburg. For a year, he organised the Utrecht Stochastics Colloquium and he taught several mathematics tutorials including calculus, analysis, linear algebra and probability theory. In September 2007 Erik will start as a lecturer at the Department of Statistics of the London School of Economics and Political Science. His research on fluctuation theory, optimal stopping problems and stochastic games for Lévy processes has resulted in this thesis.