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A unified Bayesian decision theory

Working paper

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Abstract

This paper provides new foundations for Bayesian Decision Theory based on a representation theorem for preferences defined on a set of prospects containing both factual and conditional possibilities. This use of a rich set of prospects not only provides a framework within which the main theoretical claims of Savage, Ramsey, Jeffrey and others can be stated and compared, but also allows for the postulation of an extended Bayesian model of rational belief and desire from which they can be derived as special cases. The main theorem of the paper establishes the existence of a such a Bayesian representation of preferences over conditional prospects i.e. the existence of a pair of real-valued functions respectively measuring the agent's degrees of belief and desire and which satisfy the postulated rationality conditions on partial belief and desire. The representation of partial belief is shown to be unique and that of partial desire, unique up to a linear transformation.

Key Words: Bayesian decision theory; conditionals; probability; desirability; representation theorem

1 Introduction

Bayesian decision theories are formal theories of rational agency: they aim to tell us both what the properties of a rational state of mind are (the theory of pure rationality) and what action it is rational for an agent to perform, given her state of mind (the theory of choice). This unity of theoretical purpose is matched by commonality in the ontological commitments of decision theories, in their representations of the basic objects and relations constitutive of rational agency, and in their commitment to the idea that what matters in choosing an action is the expected benefit of performing them. Indeed such is the extent of this common ground, that it seems natural to speak of versions of Bayesian Decision Theory rather than separate and competing theories of decision making.

Once we look closely at the details of these theories, however, this picture of unity gives way to something altogether more complicated. The theories of Leonard Savage [31] and Richard Jeffrey [18], to take two especially prominent examples, make what appear to be different claims about the nature of rationality. They respectively claim, for instance, that preferences for actions should be determined by the expected utility of their consequences and the conditional expected utility of their consequences given their performance. But as their theories don't represent actions in the same way, it is difficult

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to say how these claims are related and to what extent they conflict. Are Savage and Jeffrey offering rival Bayesian theories of rational agency, different expressions of the same theory or complementary theories applying to different domains? In this paper I show that the last interpretation is the correct one, that both of their accounts amount to restrictions, to a particular domain of prospects, of a more comprehensive Bayesian decision theory.

The demonstration that this is the case depends crucially on the identification of a domain of prospects sufficiently comprehensive that it allows us to express versions of both these theories and, indeed, many others. The set of prospects in question consists not just in the factual possibilities typically countenanced, such as the possibility that it will rain tomorrow or that inflation will exceed 3%, but also conditional possibilities, such as the possibility that if it rains tomorrow, the trip to the seaside will be cancelled, or that if the conflict in Iraq continues, inflation will rise. Various special kinds of conditional prospects will be familiar from other decision theories. They are central to Ramsey's [30] pioneering work, for instance, while Savage's [31] acts may usefully be regarded as conditional prospects of the form 'If the prevailing state of nature is (or belongs to) X, then consequence c will result; if it is not X, then consequence c' will result', with the antecedents of conditionals playing the role of states or events and the consequents playing the role of consequences or outcomes. More recently, the theories of conditional expected utility of Luce and Krantz [26], Fishburn [17] and Joyce [19] posit direct comparisons between act-event pairs which, on the face if it, are conditional prospects of the form 'Action A if event E'.[1] What distinguishes the use made here of conditional prospects is that we explicitly impose an algebraic structure on the set of conditional prospects akin to a logic containing conditional propositions. The resulting algebra of prospects we term a conditional algebra.

The main formal result of the paper is a representation theorem for preferences defined on conditional algebras showing that if they satisfy a number of intuitively plausible rationality assumptions, plus a number of technical conditions, then they may be represented by a pair of functions, respectively measuring the agent's degrees of belief and desire, and displaying a number of important properties. These properties are the contents of some familiar hypotheses; namely:

The Probability Hypothesis Rational degrees of belief in factual possibilities are probabilities.

The SEU Hypothesis Suppose that $\{X_i\}$ is a set of n mutually exclusive and exhaustive possibilities and $\{C_i\}$ a set of arbitrary prospects. Then the desirability of the prospect of C_1 if X_1 is the case, C_2 if X_1 is the case, ..., and C_n if X_n is the case, is the weighted average of the desirabilities of the X_iC_i with the weights being given by the probabilities of the X_i.

The CEU Hypothesis The desirability of the prospect of X is a weighted average of the desirabilities of the different possible ways X could be true, where the weight on each possible way is its conditional probability, given the truth of X.

Adams' Thesis The rational degree of belief for the prospect of Y if X is the conditional probability of Y given X.

The Probability Hypothesis has the widest support of the four and has been defended by, amongst others, Ramsey [30], De Finetti [12], Savage [31], Jeffrey [18] and Anscombe and Aumann [2]. It will be of little surprise to find it here as well. It is important to note, however, that the scope of the claim does not extend to degrees of belief in conditional prospects. In fact, as has been thoroughly demonstrated by the triviality results of David Lewis [24] and others[2], such an extended claim is in contradiction with Adams' Thesis[3], the hypothesis that the degrees of rational belief in conditionals are conditional probabilities. Many philosophers have for this reason regarded Adams' Thesis as comprehensively

[1] Though this interpretation is not without its difficulties: see section 4.
[2] See, for instance, Hajek [20], Edgington [14], Döring [13] and Bradley [7].
[3] So called because it has been most actively championed by Ernst Adams. See Adams [1].

refuted, thereby implicitly taking it to be part of the *concept* of rational partial belief that it should have a standard probabilistic representation. My view is very much to the contrary. Much of the evidence relating to the way in which we reason with conditionals and the conditions under which we are prepared to assert them speaks for the truth of Adams' thesis.[4] This gives us good reason to call into question some of the assumptions implicit in the triviality results: in particular, the assumption that the logic of conditionals is Boolean in character. The significance of this will be apparent later on.

The SEU hypothesis is the core of subjective expected utility theory and has been endorsed in one form or another by a number of decision theorists and especially those working in the tradition of Savage or Ramsey. Note that, as formulated here, the SEU hypothesis does *not* assume that the desirabilities of consequences are independent of the states of the world in which they are realised. For it says that the desirability of the prospect of, say, consequence C if event E is the case, depends on the joint desirability of C and E. So the theories of Ramsey and Savage establish special cases of the SEU hypothesis as stated here, obtained by restricting the hypothesis to the class of prospects formed from conditionals with consequents that are maximally specific. In this case, all relevant utility dependencies between antecedent (state or event) and consequent (consequence or outcome) are automatically captured by virtue of the fact that the latter implies the former. There is a difference here of course: this restriction on the domain is perfectly consistent with Ramsey's theory, but incompatible with another of Savage's assumptions; namely that every possible (total) function from events to consequences belongs to the domain of actions.

Finally the CEU hypothesis is the core of Bolker-Jeffrey conditional subjective expected utility theory (see Jeffrey [18] and Bolker [3] and [4]). As formulated here it is clearly not a competing hypothesis to SEU for they apply to different 'compounds' of prospects. This fleshes out our claim that the theories of Savage and Jeffrey apply to different domains and can, hence, be regarded as elements of a unified theory whose domain encompasses both.

We proceed as follows. In the next section conditional algebras are defined and relevant properties derived. In section 3, Bayesian models of the attitudes of rational agents are characterised axiomatically and it is established that Bayesian utility measures are averaging functions with a particular structure. In section 4, we identify sufficient conditions on a preference relation defined on a conditional algebra of prospects for the existence of Bayesian model, unique up to a choice of scale for the utility function.

2 Conditional Algebras

2.1 Vocabulary

Let $\Gamma = \langle X, \leq \rangle$ be any poset with X a set partially ordered by the relation $\leq$. Let a and b be any elements of the set X. Then, when the relevant objects exist:

1. ab denotes the greatest lower bound on $\{a, b\}$
2. $a \vee b$ denotes the least upper bound on $\{a, b\}$
3. $\neg a$ denotes the (unique) complement of a in X.
4. $\top$ and $\bot$ respectively denote the greatest and least element in X.
5. $a = b$ means $a \leq b$ and $b \leq a$.
6. X' is $X - \{\bot\}$

[4] See, for instance, Over and Evans [28].

A set Y is said to be closed under the operation $\rightarrow$ of **conditionalisation** on a set X iff $\forall(a \in X, \beta \in Y)$, $a \rightarrow \beta \in Y$. The **closure** of a set Y under conditionalisation on X is the set Z such that $a \in X, \beta \in Y \Leftrightarrow a \rightarrow b \in Z$. A subset of X, $A = \{x_1, x_2, ..., x_n\}$ is called a n-fold **partition** of X iff $x_1 \vee x_2 \vee ... \vee x_n = \top$ and $\forall(x_i, x_j \in A)$, $x_i x_j =\perp$. If A is such an n-fold partition of X then for any $y_1, y_2, ..., y_n \in X$ the object $(x_1 \rightarrow y_1)(x_2 \rightarrow y_2)...(x_n \rightarrow y_n)$ is called a **partitioning conditional**.

2.2 Types of Conditional Algebras

The structure $\Psi = \langle Y, \leq, \rightarrow \rangle$ is said to be a **conditional algebra** based on Γ iff Y is closed under conditionalisation on $X \subseteq Y$, $\Gamma = \langle X, \leq, \top, \perp \rangle$ is a Boolean algebra, Ψ is a lattice bounded above and below by $\top$ and $\perp$, and $\forall(a, b, c, d \in X)$:

(C1) $\top \rightarrow b = b$

(C2) $ab \leq ac \Leftrightarrow a \rightarrow b \leq a \rightarrow c$

(M1) $(a \rightarrow b)(a \rightarrow c) \leq a \rightarrow bc$

(J1) $a \rightarrow (b \vee c) \leq (a \rightarrow b) \vee (a \rightarrow c)$

A conditional algebra Ψ is said to be

1. **Indicative** iff $a \rightarrow (b \rightarrow c) = ab \rightarrow c$
2. **Normally bounded above** iff $a \rightarrow \top = \top$ and **below** iff if $a \neq \perp$ then $a \rightarrow \perp = \perp$ (and just normally bounded if both hold)

An indicative conditional algebra that is normally bounded (above and below) is called a **Ramsey algebra**.

Example 1. (Set-valued functions) One very broad-ranging interpretation of $\rightarrow$ is as a restriction operation on the domains of set valued functions. Let Z be any set and Y the set of all functions from Z to $\wp(Z)$. In particular let $X \subseteq Y$ be the set of constant valued, total functions on Z, containing in particular the functions $\top$ and $\perp$ which map all elements of Z respectively to Z and to $\varnothing$. For any $a \in X$ and $f \in Y$, $a \rightarrow f =_{def} f(x)$ if $x \in a(x)$ and is undefined otherwise; hence $a \rightarrow f$ is the restriction of f to $\{x \in Z : x \in a(x)\}$. We write $x \in f$ iff x belongs to the domain of f. Define $f \leq g$ iff $\forall(x \in g)$, $x \in f$ and $f(x) \cap x \subseteq g(x) \cap x$. Then $\langle Y, \leq, \rightarrow \rangle$ is an indicative conditional algebra based on $\langle X, \leq, \top, \perp \rangle$. However it is not normally bounded either above or below.

Example 2. (Conditional sentences) The correct logic and semantics of conditional sentences, both indicative and counterfactual, has been a source of considerable controversy and one that cannot be dispelled by mere algebra. The algebraic framework we develop here can, however, serve to organise the argument in a way that is particularly relevant to our concerns. Let X be a set of non-conditional sentences, $\rightarrow$ a two-place sentential operator and X_X the set of simple conditionals i.e. the set formed from X by closing it under conditionalisation on itself. Let Y be the closure of X_X under negation, disjunction, conjunction and conditionalisation on X. Most accounts of conditionals agree that when the ordering relation $\leq$ is interpreted as logical implication then $\langle Y, \leq, \rightarrow \rangle$ forms a conditional algebra. The exception is Lewis' [25] theory of counterfactuals according to which J1 is not satisfied: it can be the case that in all nearest possible worlds to the actual one in which a is true that $b \vee c$ is true without it being the case that either b is true at all such worlds or that c is true at all such worlds. On the material conditional construal of conditionals, the algebra of conditional sentences is indicative and normally bounded above, but is not normally bounded below. On Stalnaker's [33] theory of conditionals it is normally bounded both above and below but not indicative (though it is consistent with his theory that this principle does hold for indicative, as opposed to counterfactual, conditionals). On the theories of Adams [1] and McGee [27] it is both indicative and normally bounded above and below; hence a Ramsey algebra.

Theorem 1 *Suppose that Ψ is a conditional algebra. Then $\forall(a,b,c \in X)$:*
(i) $a \to b = a \to ab$
(ii) $(a \to b)(a \to c) = a \to bc$
(iii) $(a \to b) \vee (a \to c) = a \to (b \vee c)$
If Ψ is normally bounded above then:
(iv) $a \to a = b \to b$
and if Ψ is normally bounded below then:
(v) If $a \neq \perp$ *then* $a \to \neg b = \neg(a \to b)$

Proof. Note that $a, b, c \in X$, so that the normal Boolean laws apply. (i) Follows immediately from C2 and the fact that $ab = a(ab)$. (ii) By C2, $a \to bc \leq a \to b, a \to c$. But by M1 $(a \to b)(a \to c) \leq a \to bc$. (iii) By C2 and J1. (iv) If Ψ is normally bounded above then by (i) above $a \to a = a \to \top = \top = b \to b$. (v) If Ψ is normally bounded below and $a \neq \perp$ then by (ii) and (iii) $(a \to b)(a \to \neg b) = a \to \perp = \perp$ and $(a \to b) \vee (a \to \neg b) = a \to \top = \top$. So $a \to \neg b = \neg(a \to b)$. ■

Theorem 2 *Let $\Psi = \langle Y, \leq, \to \rangle$ be a normally bounded conditional algebra based on $\Gamma = \langle X, \leq \rangle$ and for any $f \in X$, let X_f be the closure of X under conditionalisation on $\{f\}$. Then the structure $\langle X_f, \leq \rangle$ is a Boolean algebra with upper bound $f \to f$ and lower bound $f \to \neg f$.*

Proof. We prove the theorem by showing that there exists a structure-preserving mapping * from the Boolean algebra $\langle X, \leq, \top, \perp \rangle$ to $\langle X_f, \leq, f \to \top, f \to \perp \rangle$. For all $a \in X$ let $a^* =_{def} f \to a$. Then $\top^* = f \to \top$ and $\perp^* = f \to \perp$. Now by Theorem 1 (iii), $(b \vee c)^* = f \to (b \vee c) = (f \to b) \vee (f \to c) = b^* \vee c^*$ and by Theorem 1 (ii), $(bc)^* = f \to bc = (f \to b)(f \to c) = b^* c^*$. Finally note that $f \to b'$ is the unique complement of $f \to b$, since $(f \to b)(f \to \neg b) = f \to \perp$ and $(f \to b) \vee (f \to \neg b) = f \to \top$. So by Theorem 1 (iii), $(b')^* = f \to \neg b = \neg(f \to b) = \neg(b^*)$. ■

3 Bayesian Models of Rationality

A Bayesian model of a rational agent is a quantitative representation of her state of mind canonically taking the form of a pair of functions respectively measuring her degrees of belief and preference for some class of objects. Our aim here is to axiomatically characterise the class of such models in the case where the agent's partial beliefs and desires are defined over a Ramsey algebra of conditional prospects and to study some of the properties of this class.

The study reveals interesting connections between the various decision theoretic hypotheses that we identified in the introduction to the paper. Bayesianism is, of course, a species of Probabilism: the view that rational degrees of belief are probabilities. But one of its central claims, from Ramsey and Savage onwards, has been that this view about rational belief is derivable from a theory of rational preference or value. The strongest expression of the latter is found in the SEU and CEU hypotheses, but underlying both is a more basic principle; namely that the value of any prospect lies between those of the various disjoint ways it can be realised and can therefore be represented as a weighted average of the values of these ways. More concisely:

Averaging Slogan No prospect is better (worse) than its best (worst) realisation in a set of mutually exclusive and exhaustive prospects

For instance, since the prospect of x can be realised by either of xy or $x\neg y$ being the case, the value of x should lie between those of xy and $x\neg y$. Similarly the prospect that if x then y and if not x then z

can be realised by either xy being the case or $\neg xy$ being the case, so its value should lie between those of xy and $\neg xy$. The rationale in both cases is the same. Any prospect may be realised in a number of different possible ways, but generally one will not be certain as to which of these is the actual one. But at worst it will be realised in the least preferred of the ways and at best by the most preferred of the ways. So a prospect can never be better than the best case scenario or worse than the worst case scenario. And in general one's attitude to the prospect should depend on how likely one thinks the worst and best cases are, given the prospect.

Any function representing the preferences or values of an agent that has this feature, I will dub an averaging function. A Bayesian utility function is an averaging function on prospects with two characteristic features:

(i) The value of a prospect that consists of a number of disjoint conditional prospects is represented as the sum of the values of each of the disjoint conditional prospects.

(ii) The utility value of the conditional prospect of y if x is proportional to the conditional utility of y given x.

This claim about the nature of the Bayesian value function follows from results that we prove below. In essence they establish that if functions P and V respectively represent an agent's degrees of belief and preference and that V is an averaging function that satisfies (i) and (ii), then V satisfies the SEU hypothesis. Moreover, given a couple of additional technical conditions, P and V satisfy not only the Probability hypothesis but also the CEU hypothesis and Adams' Thesis as well. As for the two features themselves, we shall see later on that the former is a consequence of a particular condition of rationality, while the latter is more a consequence of a choice of representational format than anything else.

A final point is worth noting. Since the SEU hypothesis can be shown to imply (i) and (ii) (see below), there is a sense in which it is the strongest of our Bayesian hypotheses. For it follows that if V is an averaging function then, given the extra technical conditions mentioned before, the SEU hypothesis implies the other three.

3.1 The Rationality Axioms

We build up the domains of the representations of rational agents degrees of belief and preference from a basic set $A = \{f, g, h, ...\}$ of non-conditional prospects ordered by a relation $\leq$ expressing logical connections between them. Assume that $\langle A, \leq \rangle$ is a Boolean algebra and let $\Psi = \langle C, \leq, \rightarrow \rangle$ be a Ramsey algebra of prospects based on $\langle A, \leq \rangle$. Let A_A be the closure of A under conditionalisation. Note that in the light of the indicative property of conditionals, A_A is the set of all simple conditional prospects i.e. all prospects in C of the form $f \rightarrow g$, where $f, g \in A$.

A Bayesian model for a rational agent based on the algebra Ψ is defined as a pair of functions $\langle P, V \rangle$, respectively on A_A and $C^{'}$, and satisfying for all $f, g, h \in A$:

Axioms of Credibility

P0 $P(f) \geq 0$

P1 $P(\top) = 1$

P2 If $fg = \bot$, then $P(f \vee g) = P(f) + P(g)$

P3 $P(f \rightarrow h) = P(h|f)$

Axioms of Desirability

V1 $V(\top) = 0$

V2 If $fg = \bot$, then $V(f \vee g).P(f \vee g) = V(f).P(f) + V(g).P(g)$

V3 $V(f \to h) = V(h|f).P(f)$

V4 If $\{f_i\}$ is an n-fold partition, then $V((f_1 \to g_1)(f_2 \to g_2)...(f_n \to g_n)) = \sum_{i=1}^{n} V(f_i \to g_i)$

Intuitively P and V respectively represent the agent's degrees of belief in, and degrees of desire or preference for, the prospects contained in the underlying conditional algebra. Axioms P0, P1 and P2 are no doubt familiar and require that the restriction of P to the Boolean sub-algebra $\langle A, \leq \rangle$ of non-conditional prospects is formally a probability. V1 and V2 similarly ensure that the restriction of V to non-conditional prospects is formally a desirability.[5] P3 is Adams' Thesis, the aforementioned hypothesis that rational belief in a conditional goes by the conditional probability of its consequent given its antecedent. V3 is the companion axiom to P3: it says that the desirability of a conditional is the conditional desirability of its consequent given the truth of its antecedent, weighted by the probability of its antecedent. Conditional probability and desirability are defined in the usual way by[6]:

Definition 3 *If $P(f) \neq 0$, then:*

$$\begin{aligned} P(h|f) &= \frac{p(fh)}{p(f)} \\ V(h|f) &= V(fh) - V(f) + V(\top) \end{aligned}$$

Note that in the light of this definition, P3 and V3 jointly ensure that P and V are respectively a probability and a desirability function when restricted to the Boolean sub-algebra, A_f, of simple conditional prospects with antecedent f. Conversely if we suppose that P and V *are* respectively a probability and desirability on the sub-algebra A_f, then V3 implies P3.[7]

V4 is an additivity axiom for partitioning conditionals. Its significance is perhaps best appreciated by noting that, in the presence of V1 and V2, axioms V3 and V4 are equivalent to the SEU hypothesis. It follows that the species of Bayesian decision theory characterised here is a proper extension of standard subjective expected utility theory.

Proposition 4 (SEU Hypothesis) *Let P and V respectively measure an agent's degrees of belief and preference. If $\{f_i\}$ is an n-fold partition of A, then for all $g_1...g_n \in A$:*

$$V((f_1 \to g_1)(f_2 \to g_2)...(f_n \to g_n)) = \sum_{i=1}^{n} V(f_i g_i).P(f_i)$$

Theorem 5 *Assume V1. Then:*
(i) V2, V3 and V4 jointly imply the SEU hypothesis.
(ii) The SEU hypothesis implies V3 and V4.

Proof. (i) By V4, $V((f_1 \to g_1)(f_2 \to g_2)...(f_n \to g_n)) = \sum_{i=1}^{n} V(f_i \to g_i)$. But by V3, $V(f_i \to g_i) = V(f_i g_i).P(f_i) - V(f_i).P(f_i)$. So $\sum_{i=1}^{n} V(f_i \to g_i) = \sum_{i=1}^{n} V(f_i g_i).P(f_i) - \sum_{i=1}^{n} P(f_i).P(f_i)$. But by V2, $V(f_1 \vee f_2 \vee ... \vee f_n) = \sum_{i=1}^{n} V(f_i).P(f_i) = 0$ by V1. So $\sum_{i=1}^{n} V(f_i \to g_i) = \sum_{i=1}^{n} V(f_i g_i).P(f_i)$.

[5] See Jeffrey [18]. The normalisation of desirabilities with respect to $\top$ is typically not specified axiomatically in Jeffrey's work, but adopted as a scaling convention. The advantage of having axiom V1 is the symmetry with P1.

[6] See Bradley [8] for the justification of this expression for conditional desirability.

[7] These claims are respectively proved as Theorems 2 and 3 in Bradley [6, pp. 195-198].

(ii) Now assume that the SEU hypothesis is true. Since the algebra of prospects is normally bounded, $V(f \to h) = V(f \to h)(\neg f \to \neg f) = V(fh).P(f) + V(\neg f).P(\neg f)$. In particular, $V(f \to f) = V(f).P(f) + V(\neg f).P(\neg f) = V(\top) = 0$ by V1. Hence $V(f \to h) = V(fh).P(f) - V(f).P(f) = (V(fh) - V(f)).P(f) = V(h|f).P(f)$ as required by V3. Now by the SEU hypothesis:

$$V((f_1 \to g_1)(f_2 \to g_2)...(f_n \to g_n)) = \sum_{i=1}^{n} V(f_i g_i).P(f_i)$$

In particular $V((f_1 \to f_1)(f_2 \to f_2)...(f_n \to f_n)) = \sum_{i=1}^{n} V(f_i).P(f_i) = 0$ in virtue of V1 and the fact that the algebra of prospects is bounded by $(f_i \to f_i) = \top$. So:

$$\begin{aligned}
\sum_{i=1}^{n} V(f_i g_i).P(f_i) &= \sum_{i=1}^{n} V(f_i g_i).P(f_i) - \sum_{i=1}^{n} V(f_i).P(f_i) \\
&= \sum_{i=1}^{n} (V(f_i g_i) - V(f_i)).P(f_i) \\
&= \sum_{i=1}^{n} V(f_i \to g_i)
\end{aligned}$$

■

3.2 Averaging Functions

A real-valued function ϕ on a Ramsey algebra Ψ based on $\langle A, \leq \rangle$ is said to be an **averaging function** iff $\forall f, g, h, i \in A'$:

1. $\phi(fg) \geq \phi(f\neg g) \Longrightarrow \phi(fg) \geq \phi(f) \geq \phi(f\neg g)$, and
2. $\phi(f \to gh) \geq \phi(f \to \neg gi) \Longrightarrow \phi(f \to gh) \geq \phi(f \to (g \to h)(\neg g \to i)) \geq \phi(f \to \neg gh)$.

It is straightforward to establish that any function, V, satisfying the Bayesian rationality axioms is an averaging function in this sense. Much more interesting is the fact that any averaging function that satisfies V3 and V4 also satisfies all the other Bayesian rationality axioms (and hence by Theorem 5(i) above, the SEU hypothesis as well). To prove this we require that two conditions hold; namely:

1. **Non-triviality**: $\forall (f \in A'), \exists (h \in A')$ such that $V(fh) = V(f)$
2. **Mitigation**: $\forall (f, g \in A'), \exists (h \in A')$ such that $V(fh) \approx V(g)$

Neither condition is necessary for the truth of the hypothesis that rational degrees of belief and desire satisfy the Bayesian axioms. And while the non-triviality condition is very weak, the mitigation condition is anything but, requiring in effect that the difference in desirability of any two prospects may be offset (or neutralised) by mitigating conditions that render them equally desirable or undesirable. Nonetheless I do not think the role that either assumption plays in the theorem below is one which is distortive of the results obtained with their help. Their function is technical: they make the result easier to obtain, but do not determine its nature.

Lemma 6 *If $fg = \perp$, then $fh \approx gi \Leftrightarrow (f \vee g) \to fh \approx (f \vee g) \to gi$*

Proof. $fh \approx gi \Leftrightarrow V(fh) \approx V(gi) \Leftrightarrow (V(fh) - V(f \vee g)).P(f \vee g) = (V(gi) - V(f \vee g)).P(f \vee g)$. But since $fg = \perp$, $fh = (f \vee g)fh$ and $gi = (f \vee g)gi$, it follows by V3 that $V((f \vee g) \to fh) = (V(fh) - V(f \vee g)).P(f \vee g)$ and $V((f \vee g) \to gi) = (V(gi) - V(f \vee g)).P(f \vee g)$. Hence $fh \approx gi \Leftrightarrow (f \vee g) \to fh \approx (f \vee g) \to gi$. ■

Theorem 7 *Let P_A be a strictly positive real valued function on A and V a real numbered function on $C^{'}$ that jointly satisfy V3 and V4. Suppose that V satisfies the non-triviality and mitigation conditions. Then if V is an averaging function on $C^{'}$, there exists a function P on A_A which agrees with P_A on A and such that $\langle P, V\rangle$ constitutes a Bayesian model of the agent.*

Proof. We prove the theorem by defining a function P in terms of V and P_A and then proving that P and V jointly satisfy the axioms of Bayesian decision theory. Recall that A_A is the set of all conditional prospects of the form $f \to g$, where $f, g \in A$, and in particular that by property C1, $f = \top \to f$. For all such f and g: (i) If $f \to g = \bot$, $P(f \to g) =_{def} 0$; (ii) If $V(f \to g) = V(\top)$, then $P(f \to g) =_{def} \frac{P_A(fg)}{P_A(f)}$; (iii) Else:

$$P(f \to g) =_{defn} \frac{V(f \to \neg g)}{V(f \to \neg g) - V(f \to g)}$$

Note that by V3, $V(f \to g) \neq V(\top) = V(f \to f)$ iff $V(fg) \neq V(f) \neq V(f\neg g)$, since V is an averaging relation. Hence by V3, $V(f \to g) \neq V(f \to \neg g)$. So $V(f \to \neg g) - V(f \to g) \neq 0$.
P0). If $V(f) = V(\top)$, then it follows from (ii) and the fact that P_A is strictly positive that $P(f) \geq 0$. Else, since V is an averaging relation, $V(\neg f) \geq V(f) \Leftrightarrow V(\neg f) \geq V(\top) \geq V(f)$. So $V(\neg f) - V(f) \geq 0 \Leftrightarrow V(\neg f) \geq 0$. Hence by definition $P(f) \geq 0$.
P1). By the normality of conditional prospects $f \to f = \top$. Then by definition, $P(f \to f) = \frac{P_A(f)}{P_A(f)} = 1$.
V1). By the normality of conditional prospects $f \to f = \top$. But by V3, $V(f \to f) = (V(f) - V(f)).P_A(f) = 0$.
P3). If $V(f \to h) = V(\top)$ then P3 follows by definition. Else by definition and V3:

$$\begin{aligned} P(f \quad \to \quad h) &= \frac{V(f \to \neg h)}{V(f \to \neg h) - V(f \to h)} \\ &= \frac{P_A(f).(V(f\neg h) - V(f))}{P_A(f).(V(f\neg h) - V(fh))} \\ &= \frac{V(f) - V(f\neg h)}{V(fh) - V(f\neg h)} \end{aligned} \qquad (1)$$

Now by the mitigation condition there exists an i such that $V(fhi) = V(f\neg h)$. Then by V3, $V(f \to \neg h) = V(f \to hi)$ and since V is an averaging relation, $V(f \to \neg h) = V(f \to ((\neg h \to \neg h)(h \to i))) = V(fh \to i)$ by the indicative and normality properties of conditionals. Now by V3,

$$\begin{aligned} P(f) &= \frac{V(f \to \neg h)}{V(f\neg h) - V(f)} \\ P(fh) &= \frac{V(fh \to i)}{V(fhi) - V(fh)} = \frac{V(f \to \neg h)}{V(f\neg h) - V(fh)} \end{aligned}$$

Hence:

$$\frac{P(fh)}{P(f)} = \frac{V(f\neg h) - V(f)}{V(f\neg h) - V(fh)} = P(f \to h)$$

by equation (1).
P2). Suppose that $V(f) \neq V(g)$. By the complementation property of conditionals $P(\neg((f \vee g) \to f)) = P((f \vee g) \to \neg f) = P((f \vee g) \to g)$ by Theorem 1(i). And by equation (1) above, $P((f \vee g) \to f) + P((f \vee g) \to g)) = \frac{V(f\vee g) - V(g)}{V(f) - V(g)} + \frac{V(f\vee g) - V(f)}{V(g) - V(f)} = 1$. So by P3, $\frac{P(f)}{P(f\vee g)} + \frac{P(g)}{P(f\vee g)} = 1$. Hence $P(f \vee g) = P(f) + P(g)$. Now suppose that $V(f) = V(g)$. Hence since V is an averaging relation, $V(f) = V(f \vee g) = V(g)$. Now by the non-triviality condition there exists $i \in A$ such that $V(gi) \neq V(g)$

and by the mitigation condition there exists an $h \in A$ such that $V(fh) = V(gi)$. Then since V is an averaging relation, $V(fh) = V(fh \vee gi) = V(gi)$ and by Lemma 6, $V((f \to h)(g \to i)) = V((f \vee g) \to gi)$. But by V4 and V3:

$$\begin{aligned}
V((f \quad \to \quad & h)(g \to i)) = V((f \vee g) \to gi) \\
& \Leftrightarrow \quad V(f \to h) + V(g \to i) = V((f \vee g) \to gi) \\
& \Leftrightarrow \quad V(fh).P(f) + V(gi).P(g) - V(f).P(f) - V(g).P(g) = V(fh \vee gh).P(f \vee g) - V(f \vee g).P(f \vee g) \\
& \Leftrightarrow \quad (V(gi) - V(g)).(P(f) + P(g)) = (V(gi) - V(g)).P(f \vee g)
\end{aligned}$$

So $P(f \vee g) = P(f) + P(g)$.
V2). By equation (1) and P3, $P((f \vee g) \to f)) = \frac{V(f \vee g) - V(g)}{V(f) - V(g)} = \frac{P(f)}{p(f \vee g))}$. Hence, $V(f).P(f) + V(g).P(f \vee g) - V(g).P(f) = V(f \vee g).P(f \vee g)$. But by P2, $P(f \vee g) - P(g) = P(f)$. So $V(f \vee g).P(f \vee g) = V(f).P(f) + V(g).P(g)$. ■

4 Preference For Conditional Prospects

We now turn to the question of foundations for Bayesian models of rational agents. As is usual we will introduce a two-place relation $\succsim$ on the set of prospects and give sufficient (and, for the most part, necessary) conditions on this relation for the existence of a Bayesian model that coheres with it. The standard interpretation of $\succsim$ is as a 'as least as preferred' relation on prospects, so that $\alpha \succsim \beta$ expresses the thought that the agent does not prefer that it be the case that β to that it be the case that α. The other common interpretation of $\succsim$ as a choice or revealed preference relation seems doubtful in this context, for the choice behaviour of an agent will only indirectly reflect his or her attitudes to conditional prospects.

The postulation of a preference relation on conditional prospects raises an important interpretative issue, however, concerning how a preference for it to be the case that if f then g rather than the case that if h then i is to be understood. In my opinion it is as a judgement which is made relative to current beliefs about the state of the world and which compares two (epistemically) possible features of it. It it not a comparison between a judgement made about g relative to, or on the assumption that, f and a judgement made about i relative to, or on the assumption that, h. Comparisons of the latter kind are apparently postulated by conditional expected utility theory; at least this is what is suggested by the formalism. Such comparisons may be possible and indeed may depend on preference judgements in some way, but they cannot themselves be preference judgements, simply because the objects 'g relative to f' and 'i relative to h' are not prospects. A topographical analogy may help to clarify the point. There is a difference between comparing how high g will look if you are at f to how high i will look if you are at h and comparing the height of g given that you are at f to the height of i given that you are at h. The former is a judgement about relative heights made from some fixed point, while the latter seems to require that one be able to compare judgements that are made from different points. That no such ability to make comparisons from different point of views is presupposed here is one of the strengths of the theory.

4.1 Axioms of Rational Preference

As before let $\Psi = \langle C', \leq, \to \rangle$ be a Ramsey algebra based on $\Gamma = \langle A, \leq \rangle$ with $C' = \{\alpha, \beta, \gamma, ...\}$ a set of prospects closed under conditionalisation on A. Let $\succsim$ be a two-place relation on C', standardly interpreted as expressing a judgement of preference between prospects. A decision theoretic model of an agent $\langle P, V \rangle$ will be said to represent her preferences just in case the measures of her degrees of belief and desire cohere with her preferences in the sense that for all prospects α and β in the domain of $\succsim$, $V(\alpha) \geq V(\beta) \Leftrightarrow \alpha \succsim \beta$. Our task now is to show under what conditions rationality of preference implies the existence of a decision theoretic model of the agent and to determine to what extent such a model uniquely represents her preferences. The following set of axioms on the preference relation $\succsim$ will be assumed throughout. For all $\alpha, \beta, \gamma \in C^{'}$ and $f, g, h, i \in A$ such that $fg = \bot$:

1. **Transitivity**: $\alpha \succsim \beta$ and $\beta \succsim \gamma$, then $\alpha \succsim \gamma$
2. **Completeness**: $\alpha \succsim \beta$ or $\beta \succsim \alpha$
3. **Independence**: $(f \rightarrow \alpha)(g \rightarrow \beta) \succsim (f \rightarrow \gamma)(g \rightarrow \beta) \Longleftrightarrow f \rightarrow \alpha \succsim f \rightarrow \gamma$
4. **Preference for Conditionals**: $f \rightarrow \alpha \succsim f \rightarrow \beta \Longleftrightarrow f\alpha \succsim f\beta$
5. **Averaging Disjunctions:** $fg \succsim f\neg g \Leftrightarrow fg \succsim f \succsim f\neg g$
6. **Averaging Conditionals**: $f \rightarrow gh \succsim f \rightarrow \neg gi \Leftrightarrow f \rightarrow gh \succsim f \rightarrow (g \rightarrow h)(\neg g \rightarrow i) \succsim f \rightarrow \neg gi$

We will refer to the set of axioms 1 - 6 as the axioms of rational preference, though strictly speaking Completeness is not a principle of rationality. The Transitivity axiom is absolutely standard and the Independence axiom is a version of von Neumann-Morgenstern's eponymous axiom and of Savage's Sure-Thing principle. Neither requires further comment here.[8]

The axiom of Preference for Conditionals is closely related to Savage's D2 whereby he induces a preference ordering over consequences from that over actions, but it does not have the implication (that D2 does) that preferences for consequences are independent of the state in which they are realised. On the contrary, Preference for Conditionals can rightfully be termed a 'state-dependency' assumption for it says that our preference between the prospect of α if f and that β if f should depend, not on our preference between α and β, but that between α and β when co-realised with f.

Finally, the axiom of Averaging Disjunctions is the central axiom of Bolker-Jeffrey decision theory and, like the axiom of Averaging Conditionals, finds its justification in the Averaging Slogan: that no prospect can be better (or worse) than its best (or worse) realisation in a given set of mutually exclusive and exhaustive prospects. In the first case the relevant set of realisations of f is the set $\{fg,\ f\neg g\}$. In the second case the relevant realisations of $f \rightarrow (g \rightarrow h)(\neg g \rightarrow i)$ is the set $\{f \rightarrow gh,\ f \rightarrow \neg gi\}$ since it follows from the former that if f then either gh or $\neg gi$ but not both.

We now show that, under technical conditions still to be specified, the rationality axioms imply the existence of a numerical representation of rational preference for conditional prospects satisfying V3 and V4, the principles we showed to be jointly equivalent to the SEU hypothesis (in the presence of the other Bayesian axioms). We also show that the representation is unique up to linear transformation. Since the axioms of Averaging Disjunctions and Averaging Conditionals jointly require that the representation be an averaging function, it then follows from Theorem 7 in the previous section that the rationality axioms in conjunction with the mitigation and non-triviality conditions are sufficient for the existence of a Bayesian model of the agent's preferences that is unique up to a choice of scale for the desirability measure.

The solution given here exploits an idea of Ramsey's; namely that conditionals can be used to define the 'sum' of any given pair of prospects and thereby making it possible to apply some standard results in the theory of measurement. The Ramseyian approach that is followed provides a particular elegant solution to the problem of finding a representation of the agent's preferences, but I do not claim that this is the only one. To my mind, it has a couple of particularly attractive features. Firstly, rather than depending on an assumption that preferences are state-independent, the method assumes only that there exists a special class of events - called neutral prospects - which do not affect the agent's attitude to prospects consistent with them. And secondly, the method makes little appeal to expected utility theory to justify its assumptions. The same cannot be said of Savage's P4 or of Bolker's axiom of Impartiality, for instance.

[8] See Broome [11] for a discussion of these principles.

We proceed as follows. In section 4.2 we introduce the concepts of preference neutrality and independence and use them to identify sets of prospects that are equally credible from the agent's point of view. In section 4.3. we use such prospects to define an addition operation on values, where the latter are sets of prospects that are preference ranked together. In section 4.4. we apply a version of Hölder's measurement theorem that is proved by Krantz et al [23] to the structure imposed on the set of values by our addition operation. This allows us to prove the existence of numerical measures of the agent's beliefs and preferences that jointly satisfy V3 and V4.

4.2 Neutral Prospects

Suppose $\phi, \psi \in C'$ and $f \in A$. Then we say that ϕ is **neutral** with respect to ψ iff $\phi\psi \approx \psi$ and that f is **independent** of ϕ iff $\forall \alpha, \beta \in C'$, $\phi(f \to \alpha)(\neg f \to \beta) \approx (f \to \phi\alpha)(\neg f \to \phi\beta)$. Note that since $\phi \approx \alpha\phi \vee (\neg\alpha)\phi$, it follows from the axiom of Averaging Disjunctions that $\alpha\phi \approx \phi \Longleftrightarrow \alpha\phi \approx \phi \approx (\neg\alpha)\phi$. Hence α is neutral with respect to ϕ iff $\neg\alpha$ is. Equally it follows immediately from the definition of independence that if f is independent of ϕ, then so too is $\neg f$. Now let $\Delta p =_{def} \{\phi \in C' : p$ is neutral with respect to $\phi\}$.

Lemma 8 *Suppose that $p, q \in A$ are neutral with respect to α and β and $pq = \bot$. Then $p \to \alpha \approx p \to \beta \Leftrightarrow \alpha \approx \beta$.*

Proof. By the axiom of preference for conditionals $p \to \alpha \approx p \to \beta \Leftrightarrow p\alpha \approx p\beta \Leftrightarrow \alpha \approx \beta$ by the definition of neutrality. (ii) By definition, $p\alpha \approx \alpha \approx q\alpha$. But then by the axiom of Averaging Conditionals, $(p \vee q) \to \alpha \approx (p \to \alpha)(q \to \alpha)$.
■

Suppose that $p, q \in A$ are such that $p \approx \top \approx q$ and $\Delta p \cap \Delta q$ contains elements not ranked with $\top$. Then p and q are said to be **equi-credible** iff $\forall(\alpha, \beta \in \Delta p \cap \Delta q)$, $(p \to \alpha)(\neg p \to \beta) \approx (q \to \alpha)(\neg q \to \beta)$. Let us denote by Π the set of elements $p \in A$, such that p and $\neg p$ are equi-credible. Intuitively, these are the prospects that should be assigned a value of one-half by any measure of the agent's degrees of belief taking values in the $[0, 1]$ interval.

Lemma 9 *If $p, q \in A$ are neutral with respect to ϕ and equi-credible then $p \to \phi \approx q \to \phi$.*

Proof. Since p and q are neutral with respect to ϕ and $\top$, it follows from the definition of equi-credibility and the normality of the conditional algebra that $p \to \phi = (p \to \phi)(\neg p \to \top) \approx (q \to \phi)(\neg q \to \top) \approx q \to \phi$. ■

We now introduce a number of axioms of preference for neutral elements and, in particular, ones which are equi-credible with their negations.

N1 $\forall(\alpha, \beta, \gamma, \delta, \epsilon \in C', f \in A)$ there exists $p, q \in \Pi$ such that $\alpha, \beta, \gamma, \delta, \epsilon \in \Delta p$, Δq and Δpq, and p and q are independent of each other and f.

N2 Suppose that $p, q \in \Pi$. Then $\forall(\alpha, \beta \in \Delta p, \Delta q), (p \to \alpha)(\neg p \to \beta) \approx (q \to \alpha)(\neg q \to \beta)$

N3 $\forall(\alpha, \beta > \top)$ there exists a partition $\{p_1, p_2, ..., p_n\}$ of equi-credible elements such that for some $i \leq n$, $\alpha > p_i \to \beta$

N4 Suppose that $p \in \Pi$ and $\alpha, \beta \in \Delta p$. Then there exists $\gamma, \delta \in C'$ such that $(p \to \alpha)(\neg p \to \beta) \approx (p \to \gamma)(\neg p \to \top)$ and $(p \to \alpha)(\neg p \to \delta) \approx \top$.

N1 ensures the existence of equi-credible and independent prospects and N2 that the definition of equi-credibility is coherent. N3 is an Archimedean axiom. N4 is a domain assumption whose role is to ensure, for any pair of prospects α and β, the existence of a prospect that intuitively has a desirability equal to the sum of that α and β, and of a prospect whose desirability is just the same as that of α but with the opposite sign. Strictly speaking we do not need N4 to apply to all prospects in the domain of the preference ranking and could restrict it to all prospects falling within certain bounds. But this would require us to make use of a much more complicated version of Hölder's measurement theorem and I do not think that the gain in generality justifies the increase in mathematical complexity.[9]

4.3 Addition of Values

An ordering on sets of equally preferred prospects, to be called values, is now derived from the initial preference orderings of the prospects themselves. Let the value of α, denoted by $\boldsymbol{\alpha}$, be defined as the set $\{\phi \in C' : \alpha \approx \phi\}$. Let Θ be the set of all the values of the prospects in C' and define the ordering $\geq$ on Θ by $\boldsymbol{\alpha} \geq \boldsymbol{\beta} \Leftrightarrow_{def} \forall(\alpha \in \boldsymbol{\alpha}, \beta \in \boldsymbol{\beta}), \alpha \succsim \beta$. Next we define an addition operation, $\circ$, on such sets of options with a view to establishing the fact that $\langle \Theta, \geq, \circ \rangle$ is a group.

Definition 10 *For any $\phi \in C'$, let p_ϕ be any element of Π that is neutral with respect to ϕ, α and β. Then $\boldsymbol{\alpha} \circ \boldsymbol{\beta} =_{def} \{\phi \in C' : p_\phi \to \phi \approx (p_\phi \to \alpha)(\neg p_\phi \to \beta)\}$*

Note that the coherence of the definition is ensured by axiom N2, for it follows from the latter that if $p^*_\phi \in \Pi$ is also neutral with respect to ϕ, α and β, that $(p^*_\phi \to \alpha)(\neg p^*_\phi \to \beta) \approx (p_\phi \to \alpha)(\neg p_\phi \to \beta)$.

Theorem 11 *Suppose that $p, q \in \Pi$, $\alpha, \beta \in \Delta p$, and $\gamma, \delta \in \Delta q$. Then $\boldsymbol{\alpha} \circ \boldsymbol{\beta} \geq \boldsymbol{\gamma} \circ \boldsymbol{\delta} \Leftrightarrow (p \to \alpha)(\neg p \to \beta) \succsim (q \to \gamma)(\neg q \to \delta)$*

Proof. By axiom N2, $(p_\phi \to \alpha)(\neg p_\phi \to \beta) \approx (p \to \alpha)(\neg p \to \beta)$ and $(p_\phi \to \gamma)(\neg p_\phi \to \delta) \approx (q \to \gamma)(\neg q \to \delta)$. So by definition, $\boldsymbol{\alpha} \circ \boldsymbol{\beta} \geq \boldsymbol{\gamma} \circ \boldsymbol{\delta}$
$\Leftrightarrow \{\phi \in C' : p_\phi \to \phi \approx (p_\phi \to \alpha)(\neg p_\phi \to \beta)\} \geq \{\phi \in C' : p_\phi \to \phi \approx (p_\phi \to \gamma)(\neg p_\phi \to \delta)\}$
$\Leftrightarrow \{\phi \in C' : p_\phi \to \phi \approx (p \to \alpha)(\neg p \to \beta)\} \geq \{\phi \in C' : p_\phi \to \phi \approx (q \to \gamma)(\neg q \to \delta)\}$
$\Leftrightarrow (p \to \alpha)(\neg p \to \beta) \succsim (p \to \gamma)(\neg p \to \delta)$. ■

Corollary 12 *(i) $\boldsymbol{\alpha} \circ \boldsymbol{\beta} = \boldsymbol{\beta} \circ \boldsymbol{\alpha}$*
(ii) $\boldsymbol{\alpha} \circ \boldsymbol{\beta} \geq \boldsymbol{\alpha} \circ \boldsymbol{\gamma} \Leftrightarrow \boldsymbol{\beta} \geq \boldsymbol{\gamma}$

Proof. Let $p \in \Pi$ be neutral with respect to α, β, γ and δ. Then (i) by definition, if $\boldsymbol{\alpha} \circ \boldsymbol{\beta} = \boldsymbol{\beta} \circ \boldsymbol{\alpha}$ then $(p \to \alpha)(\neg p \to \beta) \approx (\neg p \to \alpha)(p \to \beta) = (p \to \beta)(\neg p \to \alpha)$. But by Theorem 11, $(p \to \alpha)(\neg p \to \beta) \approx (p \to \beta)(\neg p \to \alpha) \Leftrightarrow \boldsymbol{\alpha} \circ \boldsymbol{\beta} = \boldsymbol{\beta} \circ \boldsymbol{\alpha}$. (ii) by definition, $\boldsymbol{\alpha} \circ \boldsymbol{\beta} \geq \boldsymbol{\alpha} \circ \boldsymbol{\gamma} \Leftrightarrow (p \to \alpha)(\neg p \to \beta) \succsim (\neg p \to \alpha)(p \to \gamma)$. But by the Independence axiom, $(p \to \alpha)(\neg p \to \beta) \succsim (\neg p \to \alpha)(p \to \gamma) \Leftrightarrow \neg p \to \beta \succsim \neg p \to \gamma \Leftrightarrow \neg p\beta \succsim \neg p\gamma$ by the axiom of Preference for Conditionals. But $\neg p\beta \approx \beta$ and $\neg p\gamma \approx \gamma$. So $\boldsymbol{\alpha} \circ \boldsymbol{\beta} \geq \boldsymbol{\alpha} \circ \boldsymbol{\gamma} \Leftrightarrow \boldsymbol{\beta} \geq \boldsymbol{\gamma}$. ■

Theorem 13 $\boldsymbol{\alpha} \circ \top = \boldsymbol{\alpha}$

Proof. By definition $\boldsymbol{\alpha} \circ \top = \{\phi \in C' : p \to \phi \approx (p \to \alpha)(\neg p \to \top)\} = \{\phi \in C' : p \to \phi \approx p \to \alpha\}$ by the normality of the conditional algebra. But by Lemma 8, $p \to \phi \approx p \to \alpha \Leftrightarrow \phi \approx \alpha$. So $\{\phi \in C' : p \to \phi \approx p \to \alpha\} = \{\phi \in C' : \phi \approx \alpha\} = \boldsymbol{\alpha}$. ■

Definition 14 $-\boldsymbol{\alpha} =_{def} \{\phi \in C' : (p \to \phi)(\neg p \to \alpha) \approx \top\}$

Theorem 15 $\boldsymbol{\alpha} \circ (-\boldsymbol{\alpha}) = \top$

[9] The version we would need to use is that given as Theorem 4 in Krantz et al [23, p. 45].

Proof. Let $\psi \in -\boldsymbol{\alpha}$. Then by definition $\boldsymbol{\alpha} \circ (-\boldsymbol{\alpha}) = \boldsymbol{\alpha} \circ \boldsymbol{\psi} = \boldsymbol{\psi} \circ \boldsymbol{\alpha} = \{\phi \in C' : p \to \phi \approx (p \to \psi)(\neg p \to \alpha)\}$. But by definition of $-\boldsymbol{\alpha}$, $(p \to \psi)(\neg p \to \alpha) \approx \top$. So $\{\phi \in C' : p \to \phi \approx (p \to \psi)(\neg p \to \alpha)\} = \{\phi \in C' : p \to \phi \approx \top\}$. Now by Lemma 8, $p \to \phi \approx \top \approx p \to p \Leftrightarrow \phi \approx p \approx \top$. So $\{\phi \in C' : p \to \phi \approx \top\} = \{\phi \in C' : \phi \approx \top\} = \top$. ■

Theorem 16 $(\mathbf{f} \to \boldsymbol{\alpha})(\neg \mathbf{f} \to \boldsymbol{\beta}) = (\mathbf{f} \to \boldsymbol{\alpha}) \circ (\neg \mathbf{f} \to \boldsymbol{\beta})$

Proof. Let $p \in \Pi$ be neutral with respect to $f \to \alpha$ and $\neg f \to \beta$ and $(f \to \alpha)(\neg f \to \beta)$. Then since p and $\neg p$ are equi-credible, it follows from Corollary 9, that $p \to (\neg f \to \beta) \approx \neg p \to (\neg f \to \beta)$. Hence by repeated application of the import-export condition, $\neg f \to (p \to \beta) \approx \neg f \to (\neg p \to \beta)$ and by the axiom of independence, $(f \to (p \to \alpha))(\neg f \to (p \to \beta)) \approx (f \to (p \to \alpha))(\neg f \to (\neg p \to \beta))$. Then again by repeated application of the import-export condition, $(p \to (f \to \alpha))(p \to (\neg f \to \beta)) \approx (p \to (f \to \alpha))(\neg p \to (\neg f \to \beta))$. But $(p \to (f \to \alpha))(p \to (\neg f \to \beta)) = p \to ((f \to \alpha)(\neg f \to \beta)) = (p \to ((f \to \alpha)(\neg f \to \beta))(\neg p \to T)$. So $(p \to ((f \to \alpha)(\neg f \to \beta))(\neg p \to \top) \approx (p \to (f \to \alpha))(\neg p \to (\neg f \to \beta))$. Hence by Theorem 11, $(\mathbf{f} \to \boldsymbol{\alpha}) \circ (\neg \mathbf{f} \to \boldsymbol{\beta}) = (\mathbf{f} \to \boldsymbol{\alpha})(\neg \mathbf{f} \to \boldsymbol{\beta}) \circ \top = (\mathbf{f} \to \boldsymbol{\alpha})(\neg \mathbf{f} \to \boldsymbol{\beta})$ by Theorem 13. ■

Theorem 17 $(\boldsymbol{\alpha} \circ \boldsymbol{\beta}) \circ \boldsymbol{\gamma} = \boldsymbol{\alpha} \circ (\boldsymbol{\beta} \circ \boldsymbol{\gamma})$

Proof. Suppose that $\phi \in \boldsymbol{\alpha} \circ \boldsymbol{\beta}$, $\psi \in \boldsymbol{\beta} \circ \boldsymbol{\gamma}$, and that $p, q \in \Pi$ are both neutral with respect to α, β and γ, respectively neutral with respect to ψ and ϕ, and independent of one another. Then by axiom N2, $(q \to \alpha)(\neg q \to \beta) \approx (p \to \alpha)(\neg p \to \beta)$ and $q \to \gamma \approx \neg p \to \gamma$. So by substitution of identical values, $(\mathbf{q} \to \boldsymbol{\alpha})(\neg \mathbf{q} \to \boldsymbol{\beta}) \circ \mathbf{q} \to \boldsymbol{\gamma} = (\mathbf{p} \to \boldsymbol{\alpha})(\neg \mathbf{p} \to \boldsymbol{\beta}) \circ \neg \mathbf{p} \to \gamma$. But then by Theorem 11:

$(p \to (q \to \alpha)(\neg q \to \beta))(\neg p \to (q \to \gamma)) \approx (q \to (p \to \alpha)(\neg p \to \beta))(\neg q \to (\neg p \to \gamma)$

$\approx ((pq \to \alpha)(\neg pq \to \beta))(\neg p \neg q \to \gamma)$

$\approx (p \to (q \to \alpha))(\neg p \to (q \to \beta)(\neg q \to \gamma))$

by application of the import-export condition and the commutativity of conditionals. So by Theorem 11,

$(\mathbf{q} \to \alpha)(\neg \mathbf{q} \to \beta) \circ (\mathbf{q} \to \boldsymbol{\gamma}) = (\mathbf{q} \to \alpha) \circ (\mathbf{q} \to \boldsymbol{\beta})(\neg \mathbf{q} \to \boldsymbol{\gamma})$

But by definition of $\boldsymbol{\alpha} \circ \boldsymbol{\beta}$ and $\boldsymbol{\gamma} \circ \boldsymbol{\beta}$, $\neg q \to \phi \approx (q \to \alpha)(\neg q \to \beta)$ and $\neg p \to \psi \approx (p \to \beta)(\neg p \to \gamma)$. So by substitution of identical values,

$(\neg \mathbf{q} \to \boldsymbol{\phi}) \circ (\mathbf{q} \to \boldsymbol{\gamma}) = (\mathbf{q} \to \boldsymbol{\alpha}) \circ (\neg \mathbf{p} \to \boldsymbol{\psi}) = (\mathbf{p} \to \alpha) \circ (\neg \mathbf{p} \to \boldsymbol{\psi})$

since $q \to \alpha \approx p \to \alpha$. And so by Theorem 16, $(\mathbf{q} \to \boldsymbol{\phi})(\neg \mathbf{q} \to \boldsymbol{\gamma}) = (\mathbf{p} \to \boldsymbol{\alpha})(\neg \mathbf{p} \to \boldsymbol{\psi})$. So by Theorem 11, $\boldsymbol{\phi} \circ \boldsymbol{\gamma} = \boldsymbol{\alpha} \circ \boldsymbol{\psi}$. But $\boldsymbol{\phi} = \boldsymbol{\alpha} \circ \boldsymbol{\beta}$ and $\boldsymbol{\psi} = \boldsymbol{\beta} \circ \boldsymbol{\gamma}$. So $(\boldsymbol{\alpha} \circ \boldsymbol{\beta}) \circ \boldsymbol{\gamma} = \boldsymbol{\alpha} \circ (\boldsymbol{\beta} \circ \boldsymbol{\gamma})$. ■

4.4 Additive Functions on Preference

Let $\langle A, \circ \rangle$ be a group with identity $\top$ and let $\geq$ be an ordering relation on A. The triple $\langle A, \geq, \circ \rangle$ is called a **simply ordered group** iff $\forall (a, b, c \in A)$, $a \geq b \Longrightarrow a \circ c \geq b \circ c$ and $c \circ a \geq c \circ b$. Let $\top a =_{def} a$ and, for any $n \in I^+$, let $na =_{def} (n-1)a \circ a$. Then $\langle A, \circ \rangle$ is said to be **Archimedean** iff $a > 0 \Longrightarrow na > b$ for some $n \in I^+$.

Theorem 18 *(Hölder's Theorem) Let $\langle A, \geq, \circ \rangle$ be an Archimedean simply ordered group. Then $\langle A, \geq, \circ, 0 \rangle$ is isomorphic to a subgroup of $\langle \Re, \geq, +, 0 \rangle$ and if ϕ and ϕ' are any isomorphisms, $\phi = \alpha \phi'$ for some $\alpha > 0$.*[10]

Lemma 19 *(Archimedean) Suppose $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Theta$, $\boldsymbol{\alpha} \geq \top$ and that $\{p_1, ..., p_n\}$ is a partition of equi-probable propositions such that $\alpha \succsim p_1 \to \beta$. Let $1.\boldsymbol{\alpha} =_{def} \boldsymbol{\alpha}$ and, for any $n \in I^+$, let $n\boldsymbol{\alpha} =_{def} (n-1)\boldsymbol{\alpha} \circ \boldsymbol{\alpha}$. Then $n\boldsymbol{\alpha} \succsim \boldsymbol{\beta}$.*

[10] Proved as Theorem 5 in Krantz et al [23, p.53].

Proof. By Lemma 9, $p_1 \rightarrow \beta \approx p_2 \rightarrow \beta \approx ... \approx p_n \rightarrow \beta$. So by Lemma 6, for $i, j \leq n$, $(p_i \rightarrow \beta)(p_j \rightarrow \beta) \approx (p_i \vee p_j) \rightarrow \beta$. Hence $(p_1 \rightarrow \beta)(p_2 \rightarrow \beta)...(p_n \rightarrow \beta) \approx \top \rightarrow \beta \approx \beta$. Now we prove the theorem by induction on n. It is immediately true for $n = 1$. Now assume it is true for $n = k$. Then $k\boldsymbol{\alpha} \geq (\mathbf{p}_1 \vee ... \vee \mathbf{p}_k) \rightarrow \boldsymbol{\beta}$. Now by Corollary 12 (ii), since $\boldsymbol{\alpha} \geq \mathbf{p}_{k+1} \rightarrow \boldsymbol{\beta}$, $k\boldsymbol{\alpha} \circ \boldsymbol{\alpha} \geq (\mathbf{p}_1 \vee ... \vee \mathbf{p}_k) \rightarrow \boldsymbol{\beta} \circ \mathbf{p}_{k+1} \rightarrow \boldsymbol{\beta}$. But by Theorem 16, $(\mathbf{p}_1 \vee ... \vee \mathbf{p}_k) \rightarrow \boldsymbol{\beta} \circ \mathbf{p}_{k+1} \rightarrow \boldsymbol{\beta} = ((\mathbf{p}_1 \vee ... \vee \mathbf{p}_k) \rightarrow \boldsymbol{\beta})(\mathbf{p}_{k+1} \rightarrow \boldsymbol{\beta}) = (\mathbf{p_1} \vee ... \vee \mathbf{p_{k+1}}) \rightarrow \boldsymbol{\beta} = \boldsymbol{\beta}$, by Lemma 6. ■

Theorem 20 *There exists a function U on Θ such that $\forall(\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Theta)$*
(i) $U(\boldsymbol{\alpha}) \geq U(\boldsymbol{\beta}) \Leftrightarrow \boldsymbol{\alpha} \geq \boldsymbol{\beta}$
(ii) $U(\top) = 0$
(iii) $U(\boldsymbol{\alpha} \circ \boldsymbol{\beta}) = U(\boldsymbol{\alpha}) + U(\boldsymbol{\beta})$
Furthermore, if U' is another function on Θ satisfying (i) - (iii), then $U' = aU$, for some $a > 0$.

Proof. We prove this theorem by showing that $\langle \Theta, \geq, \circ \rangle$ is a simply ordered group. By N4, $\boldsymbol{\alpha} \circ \boldsymbol{\beta}$ belongs to Θ. By Theorem 17, $\circ$ is associative; by Corollary 12 (i), it is commutative. By Theorem 13 the group identity, $\top$, exists and by N4 and Theorem 15, additive inverses exist. So $\langle \Theta, \circ \rangle$ is a group.

Now we show that Θ is simply ordered by $\geq$. Suppose $\alpha, \beta, \gamma \in C'$, that $\alpha \succsim \beta$ and that p is neutral with respect to α, β, $\phi \in \boldsymbol{\alpha} \circ \boldsymbol{\gamma}$ and $\psi \in \boldsymbol{\beta} \circ \boldsymbol{\gamma}$. Then by the definition of neutrality and the axiom of conditional preference, $\phi \succsim \psi \Leftrightarrow p\varphi \succsim p\psi \Leftrightarrow p \rightarrow \varphi \succsim p \rightarrow \psi$. But by definition of neutrality and the axiom of independence $p \rightarrow \varphi \approx (p \rightarrow \alpha)(\neg p \rightarrow \gamma) \succsim (p \rightarrow \beta)(\neg p \rightarrow \gamma) \approx p \rightarrow \psi$. So $p \rightarrow \phi \succsim p \rightarrow \psi$ and, hence, by Lemma 8, $\phi \succsim \psi$. It follows that $\alpha \circ \gamma \geq \beta \circ \gamma$. Hence $\langle \Theta, \geq, \circ \rangle$ is a simply ordered group. But by Lemma 19, it is Archimedean. The existence of the postulated function U on Θ now follows immediately from Theorem 18. ■

Lemma 21 *(i)* $\mathbf{f} \rightarrow \boldsymbol{\phi} = (\mathbf{f} \rightarrow \boldsymbol{\alpha}) \circ (\neg \mathbf{f} \rightarrow \boldsymbol{\beta}) \Leftrightarrow \mathbf{f}\boldsymbol{\phi} \circ \mathbf{f} = \mathbf{f}\boldsymbol{\alpha} \circ \mathbf{f}\boldsymbol{\beta}$
(ii) $\mathbf{f} \rightarrow \boldsymbol{\phi} = -(\mathbf{f} \rightarrow \boldsymbol{\alpha}) \Leftrightarrow \mathbf{f}\boldsymbol{\phi} \circ \mathbf{f}\boldsymbol{\alpha} = \mathbf{f} \circ \mathbf{f}$

Proof. (i) Assume that $\phi \in C'$ is such that $f \rightarrow \phi \in (\mathbf{f} \rightarrow \boldsymbol{\alpha}) \circ (\mathbf{f} \rightarrow \boldsymbol{\beta})$. Let $p \in \Pi$ by such that $f\alpha, f\beta, f\phi, f \in \Delta p$ and independent of f. Then by definition and by virtue of the fact that $\neg p \rightarrow (f \rightarrow f) = \neg p \rightarrow \top = \top$:

$$
\begin{array}{lll}
(p \rightarrow & (f \rightarrow \alpha))(\neg p \rightarrow (f \rightarrow \beta)) \approx p \rightarrow (f \rightarrow \phi) = (p \rightarrow (f \rightarrow \phi))(\neg p \rightarrow (f \rightarrow f)) & \\
\Leftrightarrow & (pf \rightarrow \alpha)(\neg pf \rightarrow \beta)) \approx (pf \rightarrow \phi)(\neg pf \rightarrow f)) & \text{[indicative property]} \\
\Leftrightarrow & f \rightarrow ((p \rightarrow \alpha)(\neg p \rightarrow \beta)) \approx f \rightarrow ((p \rightarrow \phi)(\neg p \rightarrow f)) & \text{[indicative property]} \\
\Leftrightarrow & f(p \rightarrow \alpha)(\neg p \rightarrow \beta)) \approx f(p \rightarrow \phi)(\neg p \rightarrow f)) & \text{[by the axiom of preference for conditionals]} \\
\Leftrightarrow & (p \rightarrow f\alpha)(\neg p \rightarrow f\beta)) \approx (p \rightarrow f\phi)(\neg p \rightarrow f)) & \text{[by the independence of } p \text{ from } f] \\
\Leftrightarrow & (\mathbf{f}\boldsymbol{\alpha} \circ \mathbf{f}\boldsymbol{\beta}) = \mathbf{f}\boldsymbol{\phi} \circ \mathbf{f} & \text{[by Theorem 11]}
\end{array}
$$

(ii) Assume that $\phi \in C'$ is such that $f \rightarrow \phi \in -(\mathbf{f} \rightarrow \boldsymbol{\alpha})$. Let $p \in \Pi$ by such that $f\alpha, f\phi, f \in \Delta p$ and independent of f. Then by definition:

$$
\begin{array}{lll}
(p \rightarrow & (f \rightarrow \phi))(\neg p \rightarrow (f \rightarrow \alpha)) \approx (p \rightarrow (f \rightarrow f))(\neg p \rightarrow (f \rightarrow f)) \ = \top & \\
\Leftrightarrow & (pf \rightarrow \phi)(\neg pf \rightarrow \alpha)) \approx (pf \rightarrow f)(\neg pf \rightarrow f) & \text{[indicative property]} \\
\Leftrightarrow & f \rightarrow ((p \rightarrow \phi)(\neg p \rightarrow \alpha)) \approx f \rightarrow ((p \rightarrow f)(\neg p \rightarrow f)) & \text{[indicative property]} \\
\Leftrightarrow & f(p \rightarrow \phi)(\neg p \rightarrow \alpha) \approx f(p \rightarrow f)(\neg p \rightarrow f)) & \text{[by the axiom of preference for conditionals]} \\
\Leftrightarrow & (p \rightarrow f\phi)(\neg p \rightarrow f\alpha)) \approx (p \rightarrow f)(\neg p \rightarrow f) & \text{[by the independence of } p \text{ from } f] \\
\Leftrightarrow & \mathbf{f}\boldsymbol{\phi} \circ (-\mathbf{f}\boldsymbol{\alpha}) = \mathbf{f} \circ \mathbf{f} & \text{[by Theorem 11]}
\end{array}
$$

■

Theorem 22 *For all $f \in A$, let $C_f^{'}$ be the set defined by: if $\alpha \in C^{'}$ then $f \to \alpha \in C_f^{'}$. Let Θ_f be the corresponding set of the values of prospects in $C_f^{'}$. Then there exists a function on Θ_f, $U_f(\cdot) =_{def} U(f\cdot) - U(f)$, such that $\forall(\mathbf{f} \to \boldsymbol{\alpha}, \mathbf{f} \to \boldsymbol{\beta} \in \Theta_f)$*
(i) $U_f(\mathbf{f} \to \boldsymbol{\alpha}) \geq U_f(\mathbf{f} \to \boldsymbol{\beta}) \Leftrightarrow \mathbf{f} \to \boldsymbol{\alpha} \geq \mathbf{f} \to \boldsymbol{\beta}$
(ii) $U_f(\mathbf{f} \to \mathbf{f}) = 0$
(iii) $U_f((\mathbf{f} \to \boldsymbol{\alpha}) \circ (\mathbf{f} \to \boldsymbol{\beta})) = U_f(\mathbf{f} \to \boldsymbol{\alpha}) + U_f(\mathbf{f} \to \boldsymbol{\beta})$
Furthermore, if U_f' is another function on Θ_f satisfying (i) - (iii), then $U_f' = a_f U_f$, for some $a_f > 0$.

Proof. We begin by constructing a function U_f on Θ_f from the function U on Θ whose existence was established in Theorem 20 and showing that it satisfies conditions (i) - (iii). Let $U_f(\mathbf{f} \to \boldsymbol{\alpha}) =_{def} U(\mathbf{f}\boldsymbol{\alpha}) - U(\mathbf{f})$. Then by construction:

(i) $U_f(\mathbf{f} \to \boldsymbol{\alpha}) \geq U_f(\mathbf{f} \to \boldsymbol{\beta})$

$$\begin{aligned}
&\Leftrightarrow \quad U(\mathbf{f}\boldsymbol{\alpha}) - U(\mathbf{f}) \geq U(\mathbf{f}\boldsymbol{\beta}) - U(\mathbf{f}) \\
&\Leftrightarrow \quad U(\mathbf{f}\boldsymbol{\alpha}) \geq U(\mathbf{f}\boldsymbol{\beta}) \\
&\Leftrightarrow \quad \mathbf{f}\boldsymbol{\alpha} \geq \mathbf{f}\boldsymbol{\beta} \qquad \text{[Theorem 20]} \\
&\Leftrightarrow \quad \mathbf{f} \to \boldsymbol{\alpha} \geq \mathbf{f} \to \boldsymbol{\beta} \quad \text{[Axiom of Preference for Conditionals]}
\end{aligned}$$

(ii) $U_f(\mathbf{f} \to \mathbf{f}) = U(\mathbf{f}) - U(\mathbf{f}) = 0$.

(iii) By Lemma 21 (i) $U_f((\mathbf{f} \to \boldsymbol{\alpha}) \circ (\mathbf{f} \to \boldsymbol{\beta})) = U_f((\mathbf{f} \to \boldsymbol{\varphi})$ where φ is such that $(\mathbf{f}\boldsymbol{\alpha} \circ \mathbf{f}\boldsymbol{\beta}) = \mathbf{f}\boldsymbol{\phi} \circ \mathbf{f}$. By Theorem 20, $U(\mathbf{f}\boldsymbol{\phi}) + U(\mathbf{f}) = U(\mathbf{f}\boldsymbol{\alpha}) + U(\mathbf{f}\boldsymbol{\beta})$. So

$$\begin{aligned}
U_f(\mathbf{f} \to \boldsymbol{\varphi}) &= U(\mathbf{f}\boldsymbol{\varphi}) - U(\mathbf{f}) \\
&= U(\mathbf{f}\boldsymbol{\alpha}) + U(\mathbf{f}\boldsymbol{\beta}) - U(\mathbf{f}) - U(\mathbf{f}) \\
&= U_f((\mathbf{f} \to \boldsymbol{\alpha}) + U_f(\mathbf{f} \to \boldsymbol{\beta})
\end{aligned}$$

Let $\geq_f$ be the restriction of $\geq$ to Θ_f. We now prove that $\langle \Theta_f, \geq_f, \circ \rangle$ is a simply ordered group. We have already shown that $\circ$ is associative and commutative. Since $f \to f = \top$ the group identity exists. By the axiom of mitigation there exists a prospect ϕ such that $f\phi \in (\mathbf{f}\boldsymbol{\alpha} \circ \mathbf{f}\boldsymbol{\beta}) \circ (-\mathbf{f})$. So by Lemma 21(i), $f \to \phi \in (\mathbf{f} \to \boldsymbol{\alpha}) \circ (\mathbf{f} \to \boldsymbol{\beta})$; hence $(\mathbf{f} \to \boldsymbol{\alpha}) \circ (\mathbf{f} \to \boldsymbol{\beta})$ has an element in $C_f^{'}$. Similarly by the axiom of mitigation that there exists a ϕ such that $f\phi \in (\mathbf{f} \circ \mathbf{f}) \circ (-\mathbf{f}\boldsymbol{\alpha})$. So by Lemma 21(ii), $f \to \phi \in -(\mathbf{f} \to \boldsymbol{\alpha})$; hence $C_f^{'}$ contains additive inverses. So $\langle \Theta_f, \circ \rangle$ is a group. That it is simply ordered by $\geq_f$ follows immediately from the fact that $\geq$ simply orders all of Θ. So by application of Theorem 18, since *both* U_f and U are functions on Θ_f that satisfy conditions (i)-(iii), it follows that there exists a unique real number $a_f > 0$ such that for all $\boldsymbol{\alpha} \in \Theta$, $U(\mathbf{f} \to \boldsymbol{\alpha}) = a_f U_f(\mathbf{f} \to \boldsymbol{\alpha}) = a_f U(\mathbf{f}\boldsymbol{\alpha}) - a_f U(\mathbf{f})$.
■

Corollary 23 *There exists a real-valued function V on C and a strictly positive real-valued function P_A on A that jointly satisfy V3 and V4.*

Proof. We begin by defining V on $C^{'}$ in terms of U. For all $\alpha \in C^{'}$, let $V(\alpha) = U(\boldsymbol{\alpha})$. It follows by Theorems 16 and 20 that $\forall(\alpha, \beta \in C^{'})$, $V((f \to \alpha)(\neg f \to \beta)) = V(f \to \alpha) + V(\neg f \to \beta)$. Let $\{f_i\}$ be an n-fold partition of prospects and $\{g_i\}$ be a set of prospects. We now prove by induction on n that V satisfies V4. It is immediately true for $n = 1$. Assume true for $n = k$. Now by indicative property of conditionals :

$$\begin{aligned}
V((f_1 \quad &\to \quad g_1)(f_2 \to g_2)...(f_{k+1} \to g_{k+1})) \\
&= \quad V(((f_1 \vee f_2... \vee f_k) \to ((f_1 \to g_1)...(f_k \to g_k))(f_{k+1} \to g_{k+1}) \\
&= \quad V((f_1 \vee f_2... \vee f_k) \to ((f_1 \to g_1)...(f_k \to g_k)) + V(f_{k+1} \to g_{k+1})
\end{aligned}$$

by virtue of the fact that that $\neg(f_1 \vee f_2 ... \vee f_n) = f_{k+1}$. Hence, since the hypothesis holds for $n = k$,

$$V((f_1 \to g_1)(f_2 \to g_2)...(f_{k+1} \to g_{k+1})) = \sum_{1}^{k+1} V(f_i \to g_i)$$

Hence V satisfies V4. We now define, for all $f \in A$, $P_A(f)$ as the unique real number $a_f > 0$ such that for all $\boldsymbol{\alpha} \in \Theta$, $U(\mathbf{f} \to \boldsymbol{\alpha}) = a_f U_f(\mathbf{f} \to \boldsymbol{\alpha}) = a_f U(\mathbf{f}\boldsymbol{\alpha}) - a_f U(\mathbf{f})$, whose existence was established by Theorem 22. Hence by definition, $V(f \to \alpha) = P_A(f).V(f\alpha) - P_A(f).V(f) = (V(f\alpha) - V(f)).P_A(f)$. So V and P_A jointly satisfy V3. ■

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