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Interactive and common knowledge of information partitions

Working paper

INTERACTIVE AND COMMON KNOWLEDGE OF INFORMATION PARTITIONS

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Abstract

This paper addresses the question whether interactive knowledge and common knowledge of information partitions are additional assumptions in the state-space model of knowledge. Robert Aumann and others have already discussed this issue, but in a way that appears unsatisfactory. This paper provides a thorough answer to the question in four steps. First, it makes clear the methodological nature of the puzzles surrounding interactive and common knowledge of information partitions. Second, it points out two properties – labeled as Substitutivity and Immediacy – that knowledge holds in the state-space model, and that have received little attention in the literature. Third, based on the previous two steps, the paper demonstrates that interactive and common knowledge of information partitions are not additional assumptions of the state-space model of knowledge; this is the main contribution of the work. Finally, the Appendix of the paper offers a critical examination of Aumann's discussion of the issue.

Keywords

Interactive knowledge; Common knowledge; Information partitions; State-space model; Substitutivity; Immediacy; Extension; Intension; Robert Aumann

JEL classification

B40; C70; D80; D82; D83

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INTRODUCTION

Models of economic theory are peopled by agents who take actions on the basis of their knowledge and beliefs about the world, and about each other's knowledge and beliefs. The prevailing formal model for knowledge in contemporary mainstream economics was introduced by Robert Aumann in a seminal paper published in 1976. Aumann's basic idea is that an agent knows an event if, in every state of the world the agent considers possible, that event occurs. This idea is formalized in a set-theoretic setting where the knowledge of an agent becomes an operator K mapping subsets of the space of the states of the world Ω into other subsets of Ω. Aumann's model of knowledge and generalizations of his model have been variously labeled as the event-based approach, possibility correspondence model, semantic formalism, knowledge space, Aumann structures, and state-space model of knowledge. The latter name will be adopted here.

The state-space model makes it possible not only to represent what each agent knows about the world, but also what each agent knows about what other agents know about the world. This kind of knowledge – knowledge of what others know – is called *interactive knowledge*. There are multiple levels of interactive knowledge. Level one is about what an agent, say Ann, knows about what another agent, say Bob, knows. Level two is about what Ann knows about what Bob knows about her knowledge of the world. The staircase of levels of interactive knowledge escalates in the predictable way.

A specific kind of interactive knowledge is *common knowledge*. An event is said to be common knowledge among a group of agents if all know it, all know that all know it, and so on *ad infinitum*. According to this definition, common knowledge involves infinite levels of interactive knowledge.[1] The concept of common knowledge was originally formulated by the philosopher David Lewis (1969). In his 1976 paper Aumann formally restated this concept within the state-space model of knowledge and introduced it into economics.[2] Since then, common knowledge has been recognized as a key notion for economic theory and a ubiquitous assumption in game-theoretic models. In games of complete information, the set of players, the set of strategies, and the payoff functions are assumed to be common knowledge among the players. In games of incomplete information, players typically have prior probability distributions about the unknown variables, and such distributions are usually taken to be

[1] To circumvent the infinitely recursive nature of this definition, a number of alternative characterizations of common knowledge have been proposed. On them, see Geanakoplos (1992, 1994) as well as Vanderschraaf and Sillari (2005). However, these alternative characterizations play no role in the current contribution.

[2] On Lewis' notion of common knowledge and its differences from Aumann's formalization, see Cubitt and Sugden (2003).

common knowledge. Furthermore, some important game-theoretic solution concepts require that each player is rational, and that the rationality of the players is common knowledge among them.[3] Finally, common knowledge of posterior probabilities is essential for the so-called “agreeing to disagree” results, and common knowledge of willingness to trade for no-trade theorems.[4]

Given the importance of common and hence interactive knowledge, it is fundamental to understand clearly whether, by assuming that some element of the model is interactively or commonly known, any additional assumption is implicitly made. In particular, consider Ann and Bob, and assume that Ann knows that Bob knows a certain event E. The question is: to assume this, don’t we need to make the additional assumption that Ann knows how the information is imparted to Bob, that is, that Ann knows Bob’s knowledge operator K_B?

Under certain conditions the way knowledge is imparted to the agents can be modeled also through their so-called *information partitions* $\mathscr{P}$. In this case the state-space model is said to be *partitional*. In partitional state-space models, the above question can be also put as follows: to say that Ann knows that Bob knows E, is it necessary to make the additional assumption that Ann knows Bob’s information partition $\mathscr{P}_B$? More generally: to state that a certain event E is interactively known among a group of agents, is interactive knowledge of their information partitions required as an additional assumption of the model? This is the first question addressed in the present paper.

If we consider the second level of interactive knowledge, as in the statement that Ann knows that Bob knows that she knows E, the question about the interactive knowledge of information partitions comes out again, at level two: to say that Ann knows that Bob knows that she knows E, is it necessary to make the additional assumption that Ann knows that Bob knows her information partition $\mathscr{P}_A$? As we consider higher and higher levels of interactive knowledge, the question about the interactive knowledge of information partitions returns at higher and higher levels, so that when common knowledge is involved the question becomes: to state that a certain event E is common knowledge among a group of agents, is common knowledge of their information partitions required as an additional assumption of the model? This is the second question addressed in the paper.

3 More on this in Brandenburger (1992, 2007); Dekel and Gul (1997); Battigalli and Bonanno (1999).

4 The seminal paper for agreeing to disagree results is, again, Aumann (1976); for no-trade theorems it is Milgrom and Stokey (1982).

Although these two questions have already been discussed in the literature, in my opinion the answers given to them are rather opaque or unsatisfactory.[5] The present paper addresses these questions about interactive and common knowledge of information partitions along original lines, and provides a clear and convincing answer to them.

To attain this goal, the initial step is a methodological distinction between the intuitive and philosophical understanding of knowledge on the one hand, and knowledge as modeled in the state-space formalism through the operator K on the other. The puzzles surrounding interactive and common knowledge of information partitions originate from a confusion between these two understandings, and more precisely from interpreting the operator K as a realistic representation of knowledge in the intuitive and philosophical sense, whereas K is a formal object whose properties are defined by the state-space model.

In effect, K possesses a number of properties that are at odds with both commonsense and the philosophical analysis of knowledge. Some of them, like Necessitation and Monotonicity, have been already discussed in the literature. The present work focuses instead on two other properties of K, that are also quite unrealistic but have attracted less attention. The first one is labeled as *Substitutivity*, and expresses the extensional nature of the operator K. The second one is labeled as *Immediacy*, and emphasizes that in any state of the world, the agent immediately and necessarily knows a number of events in the sense of K. When one appreciates the implications of Substitutivity and Immediacy as properties of K, and recognizes that the knowledge involved in the puzzles concerning interactive and common knowledge of information partitions is the knowledge captured by the operator K, rather than the knowledge of commonsense or philosophy, then the puzzles vanish.

When the puzzles fade away, it is rather simple to demonstrate that the answer to the two questions addressed in the paper is in the negative: neither interactive knowledge nor common knowledge of information partitions is an additional assumption of the state-space model of knowledge.

This conclusion is completely in accord with the view maintained all along by Aumann in the discussion about the status of interactive and common knowledge of information partitions.[6] However, to support this view Aumann used arguments that appear debatable in a number of respects. The Appendix to this paper presents and critiques Aumann's arguments,

[5] Among the contributions to the discussion, see Aumann (1976, 1987, 1989, 1999, 2005); Gilboa (1988); Brandenburger and Dekel (1993); Margalit and Yaari (1996); Hart, Heifetz and Samet (1996); Dekel and Gul (1997); Heifetz and Samet (1998); Heifetz (1999); Fagin, Geanakoplos, Halpern and Vardi (1999); Aumann and Heifetz (2002, Appendix); Cubitt and Sugden (2003, Appendix 2).

[6] See in particular Aumann (1976, p. 1237; 1987, p. 9; 1999, pp. 272–3 , 276–8; 2005, pp. 92–4).

arguing that the case that common and hence interactive knowledge of information partitions "is not an assumption, but a 'theorem', a tautology; it is implicit in the model itself" (Aumann, 1987, p. 9), is better made by bringing into play Substitutivity and Immediacy.

Some final specifications on the scope and style of the current contribution are in order. First, the discussion refers to information partitions, and hence to partitional state-space models, only for the sake of simplicity and because this is the usual approach in the literature. However, this constitutes no loss of generality since neither Substitutivity nor Immediacy depends on the conditions that make the state-space model partitional. Therefore, the arguments made here also hold for non-partitional state-space models.

Second, in the state-space model of knowledge, agents consider possible certain states of the world in Ω and impossible the remaining ones, but they are not endowed with probability distributions that represent their beliefs over Ω. If we first add to the model a probability distribution for each agent, then introduce a belief operator B that says what is the probability assigned by an agent to any given event, and finally redefine knowledge as "belief with probability 1", we obtain a different model that is variously labeled as probabilistic belief space, probabilistic structure or Harsanyi type space. There are a number of analogies between the state-space model and the probabilistic belief space; moreover the issue about interactive and common knowledge of information partitions arising in the former has an analog in the latter. However, the answers to the issue diverge in the two formalisms. This is mainly due to the circumstance that the belief operator B in the probabilistic belief space has certain continuity properties (that derive from the probability measures defining B), whereas the knowledge operator K in the state-space model is not continuous. Now, the present paper deals only with interactive and common knowledge of information partitions in the state-space model, and does not examine the analogous issue in the probabilistic belief space.[7]

Third, the state-space model of knowledge employs set-theoretic tools that are familiar to economists. There is another model of knowledge, mainly elaborated by logicians and philosophers, that employs the language and tools of logics and has been variously called the logic-based approach, the syntactic formalism, Kripke structure or simply epistemic logic.[8] The parallels between the state-space model and the logic-based approach have been explored by, among others, Michael Bacharach (1985) and Aumann himself (1989, 1999). The logic-

[7] On the probabilistic belief space, its relationships to the state-space model, and the issues concerning interactive and common beliefs, see Mertens and Zamir (1985); Monderer and Samet (1989); Brandenburger and Dekel (1993); Heifetz and Samet (1998, 1999a, 1999b); Fagin, Geanakoplos, Halpern and Vardi (1999); Meier (2005); Mariotti, Meier and Piccione (2005).

[8] For a comprehensive presentation of logic-based approach, which can be also extended to deal with probabilistic beliefs, see Fagin, Halpern, Moses and Vardi (1995).

based formalism proved useful to understand the properties of the knowledge operator K and has other nice features, but its language remains unfamiliar to many economists. Therefore, although the logic-based approach is outlined in the Appendix, the focus of the present paper is on the state-space model, and the questions about interactive and common knowledge of information partitions are tackled and answered within this model.

Finally, the key insights of this study are of a methodological and philosophical nature, and the technical results obtained in it are an outcome of these insights. Therefore, the paper is addressed not only to specialists working on the epistemic foundations of game theory, but to a more general audience of economists and philosophers interested in issues concerning knowledge. Since this audience may be not familiar with the state-space model of knowledge, the paper starts with a review of it, and spells out the puzzles about interactive and common knowledge of information partitions by means of a stylized Ann-Bob example.

The paper is organized as follows. Section 1 reviews the state-space model of knowledge and calls attention to Substitutivity and Immediacy as properties of the operator K. Sections 2-6 deal with the question of interactive knowledge of information partitions, while Section 7 addresses the question of common knowledge of information partitions. Section 8 sums up the paper and presents some suggestions for future research. The Appendix offers a critical examination of Aumann's discussion of interactive and common knowledge of information partitions.

1. THE STATE-SPACE MODEL OF KNOWLEDGE

Consider a set Ω whose generic element is ω, and a correspondence $P:\Omega \to 2^{\Omega} \setminus \{\varnothing\}$ that associates to each element $\omega \in \Omega$ a set $P(\omega)$ of elements of Ω (2^{Ω} is the set of all subsets of Ω). Based on P, define an operator $K: 2^{\Omega} \to 2^{\Omega}$ as follows: for every $E \subseteq \Omega$, $K(E) = \{\omega \in \Omega : P(\omega) \subseteq E\}$.[9]

The interpretation of the above set-theoretic structure is the following. Ω is the set of the *possible states of the world*. A state $\omega \in \Omega$ specifies every aspect of the world that is relevant to the situation. Only one state is the true one, but the agent may be uncertain about which one. This uncertainty is modeled by the correspondence P, which associates to each state ω the set of states that the agent regards as *possible* at ω. This is why P is called a *possibility correspondence*.

[9] This review of the state-space model of knowledge is based on Osborne and Rubinstein (1994, Chapter 5); Dekel and Gul (1997); Battigalli and Bonanno (1999); Samuleson (2004). The reader may refer to these works for demonstrations omitted here.

A subset $E \subseteq \Omega$ is called an *event*, and can be thought of as the collection of all states that share a certain feature. For instance, the event "it rains" collects all states $\omega \in \Omega$ characterized by rain. Note that, if $P(\omega) \subseteq E$, in all states the agent regards as possible in ω, the event E occurs.

The operator K is interpreted as a knowledge operator: if $\omega \in K(E)$, then at ω the agent knows that the event E occurs, and this is because in every state the agent regards as possible in ω – that is, in $P(\omega)$ – the event E occurs. Observe that $K(E)$ is itself an event, the event "the agent knows E". As such, $K(E)$ may become the object of further knowledge or uncertainty for another agent.

This set-theoretic formalization makes knowledge easy to handle in economic models, and captures certain features of the intuitive and philosophical understanding of knowledge. In effect, the idea that we know a fact when this fact takes place in any situation we consider possible sounds sensible. On the other hand, the definition of knowledge through the operator K implies some properties of knowledge that appear dubious from the intuitive and philosophical viewpoint. First, it is easy to show that K satisfies the following property, called *Necessitation*:

Necessitation: $K(\Omega) = \Omega$.

Necessitation is interpreted as stating that if an event occurs in all states of the world, then the agent knows it. So, if we think of logical truths as something ingrained in any state of the world, Necessitation says that the agent knows all logical truths. This may seem questionable. Moreover, Dekel, Lipman and Rustichini (1998) have shown that Necessitation is incompatible with a feature of actual knowledge that is relevant for economic analysis, namely that an agent may be unaware of some possible events. A second property satisfied by K is *Conjunction*:

Conjunction: $K(E) \cap K(F) = K(E \cap F)$.

The interpretation of Conjunction is that an agent knows that two events occurred if and only if she knows that each of them occurred. Necessitation and Conjunction are the two fundamental properties of the operator K. In fact, it can be shown that K can be derived from a possibility correspondence P if and only if K satisfies Necessitation and Conjunction.

With respect to Necessitation, Conjunction may seem quite innocuous, but in fact it has a strong implication, called *Monotonicity*:

Monotonicity: if $E \subseteq F$ then $K(E) \subseteq K(F)$.

Monotonicity is interpreted as stating that the agent knows the implications of what she knows. This means that if the agent knows the axioms of a mathematical system, she also knows all the theorems that are valid in the system, and this appears at odds with ordinary intuitions about knowledge. Moreover, Dekel, Lipman and Rustichini (1998) have shown also that Monotonicity is incompatible with unawareness, and this independently from Necessitation. Since Necessitation and Monotonicity are properties that the operator K necessarily holds in the state-space model, Dekel, Lipman and Rustichini conclude that this model, at least in its standard form, preclude unawareness.[10]

There are two further properties of K that have attracted less attention in the economic literature and are instead at the core of the present paper. The first one is a special case of Monotonicity when $E = F$, and can be dubbed *Substitutivity*:

Substitutivity: $E = F$ then $K(E) = K(F)$.

Substitutivity states that, if two events collect exactly the same states of the world, when the agent knows one event she also knows the other. So, for instance, in a Euclidean universe if the agent knows that all triangles in front of him are equilateral, he knows also that they are all equiangular. Substitutivity is less demanding than Monotonicity, but still in contrast with our intuition of knowledge as well as with what logicians and philosophers call the intensional character of knowledge. This will be discussed below in Sections 3 and 4.[11] The second property can be called *Immediacy*:

Immediacy: for every $\omega \in \Omega$ and every $E \subseteq \Omega$, if $E \supseteq P(\omega)$ then $\omega \in K(E)$.

From a formal viewpoint, Immediacy is just a restatement of the very definition of K. But this restatement makes clear that at any state ω there is a number of events that the agent knows and cannot avoid knowing, namely the events that are supersets of $P(\omega)$. In a sense, at ω the supersets of $P(\omega)$ make themselves manifest to the agent. By definition of K, the events that are supersets of $P(\omega)$ are none other than the events that the agent knows at ω. What Immediacy emphasizes is that at ω not only can the agent know these events, but in fact she necessarily and immediately knows them: the operator K brushes off any difference between potential knowledge and actual knowledge, and this explains the suggested label.

[10] To accommodate unawareness, some extensions of the standard state-space model have been recently proposed. More on this in Section 8.

[11] In the logic-based approach Substitutivity is usually called the Equivalence Rule. Lismont and Mongin (1994, 2003), as well as Ferrante (1996) have introduced logical models of knowledge where, at least to a certain extent, Monotonicity is replaced with the weaker Equivalence Rule.

In the economic literature, this feature of K has been noticed (and exploited for a number of results) with reference to a particular class of events called *self-evident events* or *truisms*.[12] An event E is said to be self-evident if, for every $\omega \in E$, $P(\omega) \subseteq E$. Therefore, if E is a self-evident event and $\omega \in E$, then it is also the case that $\omega \in K(E)$, i.e. $E \subseteq K(E)$. In words, whenever a self-evident event occurs the agent knows and cannot avoid knowing it. Immediacy highlights that this epiphanic feature of K is not restricted to self-evident events, since in any state ω there is a number of events that make themselves manifest to the agent, namely the supersets of $P(\omega)$.

From a philosophical viewpoint it can be argued that certain events related to sensations (e.g. "I see this object as white") or thoughts (e.g. the Cartesian "I am thinking" or the analytical truth "A is A") are immediately and necessarily known, and that any knowledge ultimately relies on this kind of event. However, in most circumstances knowledge refers to states of affairs that are not immediately and necessarily known, so that Immediacy also appears an unrealistic property of K.

Whereas Necessitation, Monotonicity, Substitutivity and Immediacy may seem too demanding with respect to the intuitive and philosophical understanding of knowledge, there is an almost undisputed property of knowledge that is not captured by the operator K, namely that we cannot know the false. This truth-condition is one of the elements that distinguishes knowledge from belief – a belief can be false – and is embodied in the traditional philosophical definition of knowledge as "justified true belief".[13] In order to warrant the truthfulness of knowledge we need the following assumption on the possibility correspondence P:

P1: for every $\omega \in \Omega$, $\omega \in P(\omega)$.

It is easy to see that P1 implies the so-called *Truth axiom*:

Truth axiom: $K(E) \subseteq E$.

This means that when the agent knows an event E this event actually occurs, that is, her knowledge is true. Another property is usually assumed of P, namely that:

P2: If $\omega' \in P(\omega)$ then $P(\omega') = P(\omega)$.

It can be shown that P2 implies the following two properties of the operator K:

[12] See e.g. Geanakoplos (1992, 1994) and Binmore and Brandenburger (1989).

[13] For an introduction to the definition of knowledge as justified true belief, and the refinements of this definition as a consequence of the so-called Gettier problem, see Steup (2006).

Positive introspection: $K(E) \subseteq K(K(E))$;

Negative introspection: $\Omega \setminus K(E) \subseteq K(\Omega \setminus K(E))$.

Positive introspection says that if the agent knows something, then she knows that she knows it. Negative introspection says that if the agent does not know something, then she knows that she does not know it. Common sense and philosophical considerations suggest that it is unrealistic to assume that actual knowledge satisfies these forms of introspection, and especially the negative one.

It can be proved that P1 and P2 are not only sufficient but also necessary for the operator K to satisfy the Truth axiom, Positive introspection and Negative introspection. However, the arguments made in the current paper do not rely on these latter properties of K, but on Substitutivity and Immediacy, which are independent of P1 and P2. Since P1 and P2 do not play any role for the results obtained in the paper, they can be assumed here without loss of generality. This will make the discussion easier to follow.

When P satisfies P1 and P2, P is called *partitional* because it induces a unique partition $\mathscr{P}$ on Ω, whose elements, or *cells*, are the sets $P(\omega)$.[14] $\mathscr{P}$ is called an *information partition* and expresses the agent's uncertainty about the true state of world as the possibility correspondence P does: when the true state is ω, the agent regards as possible all states that are in the cell $P(\omega)$ of $\mathscr{P}$ containing ω. In fact, there is a one-to-one relationship between information partitions $\mathscr{P}$ and partitional possibility correspondences P, so that the agent's uncertainty and knowledge can be modeled equivalently in either way.

As an illustration of the state-space model of knowledge in the partitional case, suppose that Ann and Bob are interested in a variable v that can take values from 1 to 6, like a die. Each state of the world is characterized by the value taken in it by v, so that there are six possible states: $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$. P_A, the possibility correspondence of Ann, is as follows: $P_A(\omega_1) = P_A(\omega_2) = \{\omega_1, \omega_2\}$, $P_A(\omega_3) = P_A(\omega_4) = P_A(\omega_5) = \{\omega_3, \omega_4, \omega_5\}$, $P_A(\omega_6) = \{\omega_6\}$. This is equivalent to saying that Ann's information partition $\mathscr{P}_A$ is $\mathscr{P}_A = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4, \omega_5\}, \{\omega_6\}\}$. So if $v = 1$, Ann considers possible both $v = 1$ and $v = 2$; if $v = 3$, Ann is uncertain whether $v = 3$, $v = 4$ or $v = 5$, and so on.

Let us now consider the event S "v is not greater than 3". S occurs at states ω_1, ω_2 and ω_3: $S = \{\omega_1, \omega_2, \omega_3\}$. In which states of the world does Ann know S? In ω_1 and ω_2 Ann

[14] A partition of Ω is a collection of nonempty disjoint subsets of Ω whose union is Ω.

knows S since $P_A(\omega_1) = P_A(\omega_2) \subseteq S$. On the contrary, at ω_3 Ann does not know S, since at ω_3 she thinks that $v = 4$ or $v = 5$ could be the case; formally $P_A(\omega_3) \subsetneq S$. Also at ω_4 and ω_5 Ann thinks that v could be greater than 3, whereas at ω_6 she knows that $v = 6$. Therefore, not even at ω_4, ω_5 or ω_6 does Ann know S. In conclusion, $K_A(S) = \{\omega_1, \omega_2\}$. Note that $K_A(S)$ is itself an event: the event that Ann knows that $v \leq 3$.

All this has an intuitive graphical representation. In Figure 1 below, the circular loops stand for the cells of Ann's information partition, whereas the rectangular loop stands for events:

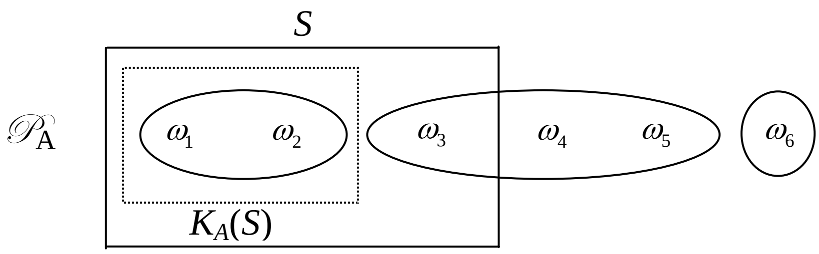

Figure 1

So, if the true state is ω_1 Ann knows S: $\omega_1 \in K_A(S)$. Which other events are known and necessarily known to Ann at ω_1? Immediacy says that Ann necessarily knows all the events that are supersets of $P_A(\omega_1)$. It is easy to show that, besides $P_A(\omega_1)$, S and Ω, there are 13 such supersets in Ω. For instance, two of these supersets are $T = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, which can be interpreted as "$v \leq 4$", and $W = \{\omega_1, \omega_2, \omega_3, \omega_6\}$, which can be interpreted as "$v \leq 3$ or $v = 6$".

2. INTERACTIVE KNOWLEDGE OF INFORMATION PARTITIONS AND PUZZLE 1

So far the state-space formalism was presented for the single-agent case. Yet, this formalism can be easily extended to a multi-agent setting, and used to model what an agent knows about what the other agents know about the world, that is, *interactive knowledge*.

The simplest setting with two agents – Ann and Bob – will be considered here since this makes the discussion simpler without affecting the results of the paper. In this setting, P_i, K_i and $\mathscr{P}_i$, with $i \in \{A, B\}$, are the possibility correspondence, the knowledge operator and the information partition of Ann and Bob, respectively.

Assume now that Bob's information partition is as follows: $\mathscr{P}_B = \{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4, \omega_5\}, \{\omega_6\}\}$. As seen above, the event T "$v \leq 4$" occurs at states ω_1,

ω_2, ω_3 and ω_4: $T = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. It is easy to show that the states of the world where Bob knows that $v \leq 4$ are ω_1, ω_2 and ω_3: $K_B(T) = \{\omega_1, \omega_2, \omega_3\}$. $K_B(T)$ is itself an event, and in our example it happens that the event S, "$v \leq 3$", and the event $K_B(T)$, "Bob knows that $v \leq 4$", occur exactly in the same states of the world: $K_B(T) = S$. This situation is represented in Figure 2:

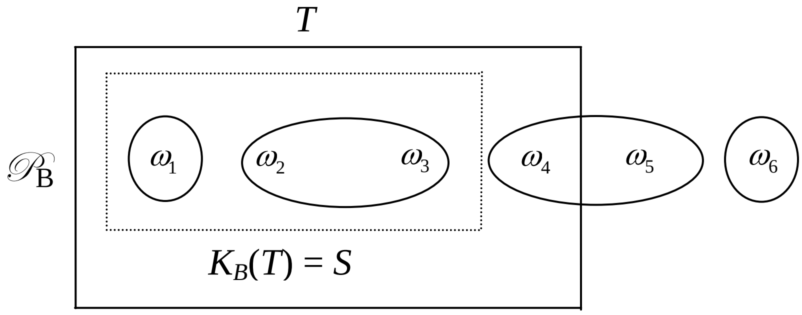

Figure 2

Hence, if the true state is ω_1 Bob knows T: $\omega_1 \in K_B(T)$. At this point, interactive knowledge enters the scene. We can ask whether at ω_1 Ann knows that Bob knows that $v \leq 4$. Since $K_B(T)$ is itself an event, in the state-space formalism the question can be restated as follows: does $\omega_1 \in K_A(K_B(T))$?

The intuitive answer is: it depends on whether Ann knows Bob's information partition (or, to say it another way, on whether Ann knows Bob's possibility correspondence). If at ω_1 Ann knows Bob's information partition, she can draw the same conclusion that we, the external, omniscient model-makers, have drawn: that indeed Bob does know T. On the contrary, if Ann does not know Bob's information partition, she has no clue about what Bob knows. In other words, the intuitive answer is that we need to make the additional assumption that Ann knows Bob's information partition to state that Ann knows that Bob knows T.

However, consider the following objection to this intuitive answer. It was established that at ω_1 Ann knows S (i.e., $\omega_1 \in K_A(S)$), and that the set of states where Bob knows T coincide with S (i.e., $K_B(T) = S$). But if $\omega_1 \in K_A(S)$ and $K_B(T) = S$, by Substitutivity it is also the case that $\omega_1 \in K_A(K_B(T))$, that is, in fact at ω_1 Ann knows that Bob knows that $v \leq 4$. And this is independent of any additional assumption about Ann's knowledge of Bob's information partition.

The intuitive reply to this objection is that if Ann does not know Bob's information partition, she is not aware that $K_B(T) = S$, so that she cannot go from $K_A(S)$ to $K_A(K_B(T))$. In other words, from Ann's subjective viewpoint S and $K_B(T)$ are different events. To say that, for Ann, S is subjectively equivalent to $K_B(T)$, the additional assumption that Ann knows Bob's information partition is indeed necessary.

Which stance is correct? Let us call Puzzle 1 the question whether Ann's knowledge of Bob's information partition is necessary for Ann to know that Bob knows $v \leq 4$. To solve Puzzle 1 an aside on the philosophical notions of extension and intension turns out to be useful.

3. EXTENSION AND INTENSION

Arguably since Medieval discussions about the status of universals, philosophers have recognized that there is a difference between what a linguistic expression designates and what it means.[15] What a linguistic expression *designates* consists of set of things to which the expression applies, and has been labeled as *denotation* by John Stuart Mill (1843), *reference* by Gottlob Frege (1892), and *extension* by Rudolf Carnap (1947). Carnap's terminology has become standard in contemporary philosophy and will be adopted here. So, for instance, the extension of the term "computer" is the set of existing computers. What a linguistic expression *means* is the notion or idea conveyed by the expression, and has been called *connotation* by Mill, *sense* by Frege, and *intension* by Carnap. The intension of "computer" is the idea of an electronic machine that can store, retrieve, and process data.

Two expressions can have the same extension but different intensions. Frege proposed the example of the morning star, which is the star that can be seen at sunrise and was called Phosphorus by Greek astronomers, and the evening star, the star that appears at sunset and was called Hesperus. The morning star and the evening star have different intensions but the same extension, since both designate the planet Venus. Other expressions with different intensions but equal extension are "51" and "17 × 3", or "equilateral triangle" and "equiangular triangle".

In certain contexts, extensional equality is sufficient to apply the so-called *principle of substitutivity*, according to which equals can be substituted by equals in any statement without modifying the truth-value of the statement. Contexts where substitution of equals requires only extensional equality are called *extensional contexts*. Classical logic, mathematics and standard set theory are typical instances of extensional contexts. Contexts in which intension also matters, and extensional equality alone does not warrant the principle of substitutivity, are called *intensional contexts*. Typical examples of intensional contexts are statements involving verbs of propositional attitude such as "believes", "wants", "knows". For instance,

[15] This section is largely based on Bealer (1998); Christmas (1998); Fitting (2007).

even if Ann knows that the morning star is Venus, she may not know that the evening star is also Venus. Even if Bob knows that the triangle in front of him is equilateral he may not know that it is also equiangular. One could think that the failure of the substitutivity principle in these two examples is due to the fact that the extensional equality among the expressions involved is only accidental, that is, non necessary: equilateral and equiangular triangles coincide in Euclidean geometry but may differ in some non-Euclidean system. Similarly, the evening star and the morning star are the same in the actual astronomical universe, but may be different in another possible universe. In effect, the principle of substitutivity can fail even when extensional equality is necessary, that is, holds in every imaginable universe. For instance, even if it is always the case that $17 \times 3 = 51$, Carl may know that 17×3 is not prime but not know that 51 is not prime.

Logical systems developed for intensional contexts are called *intensional logics*. Even if a number of intensional logics have been proposed in the last 60 years, none of them has gained general acceptance.[16]

4. THE EXTENSIONAL NATURE OF THE OPERATOR K

In the state-space model of knowledge there are not linguistic expressions, but subsets of Ω called events. However, we have seen that events are typically interpreted as set-theoretic images of linguistic expressions like "it rains", "v is not greater than 3", or "Bob knows that v is not greater than 4". According to this interpretation, the extension of an event is the set of states of Ω constituting the event, whereas its intension is identified with the intension of the linguistic expression represented by the event, e.g., the intension of "it rains".

The circumstance that intension matters with respect to knowledge is lost when knowledge is modeled through the operator K. Basically, this is because K is constructed with the tools of standard set theory which is, as noted above, thoroughly extensional. Formally, the extensional nature of K is expressed by its Substitutivity that states that if $E = F$ then $K(E) = K(F)$. Substitutivity says that extensional equality is sufficient to apply the substitutivity principle, and so implies that in the state-space model the contexts involving knowledge are purely extensional.

This is in contrast with the intuitive and philosophical appreciation of the intensional dimension of knowledge, and is at the origin of the puzzle concerning Ann's knowledge of Bob's information partition.

[16] The main systems of intensional logic are those proposed by Carnap (1947); Church (1951); Montague (1960, 1970); Gallin (1975); Zalta (1988).

5. SOLVING PUZZLE 1

Let us return to Puzzle 1: at ω_1, does Ann know that Bob knows that $v \leq 4$, i.e., the event $K_B(T)$, even if Ann does not know Bob's information partition? The intuitive answer was "No": even if at ω_1 Ann knows S, and $K_B(T) = S$, if Ann does not know Bob's information partition, she is not aware that $K_B(T) = S$, so that she cannot go from knowing S to knowing $K_B(T)$.

This answer turns out to be erroneous, and the error derives from a confusion between the formal content of the state-space model of knowledge and its interpretation. As it is quite natural, the interpretation of the model is based on the intuitive and philosophical understanding of knowledge, according to which extensional equality alone is not sufficient for applying the substitutivity principle. Therefore, even if S and $K_B(T)$ are extensionally equal, their intensional difference ("$v \leq 3$" is intensionally different from "Bob knows that $v \leq 4$") does not allow Ann to jump from $K_A(S)$ to $K_A(K_B(T))$.

However, this interpretation ignores that the state-space model of knowledge is a formalism, which as such captures some features of actual knowledge and leaves out others. In particular, as observed in the previous section, the state-space model obliterates the intensional dimension of knowledge. Therefore, within the state-space model the extensional equality of S and $K_B(T)$ is sufficient to apply the principle of substitutivity and to go from $K_A(S)$ to $K_A(K_B(T))$ without any additional assumption. In other words, if Ann knows that $v \leq 3$ in the sense of the operator K, and if the set of states of the world where $v \leq 3$ coincides with the set of the states where Bob knows that $v \leq 4$ in the sense of K, then Ann cannot not know, in the sense of K, that Bob knows that $v \leq 4$. This is a consequence of the way the operator K is built.

The step from $K_A(S)$ to $K_A(K_B(T))$ seems tricky as long as one interprets K according to the ordinary and philosophical understanding of knowledge. If, instead, one thinks of K according to the formal content of the state-space model of knowledge, then Puzzle 1 vanishes.

6. PUZZLE 2

To this line of reasoning one may object that Puzzle 1 and its solution refer to a particular case, namely the one where $K_B(T) = S$ and Substitutivity applies. But consider a different event V such that $K_B(V) \neq S$. For instance, V may be the event "$v \neq 5$", so that $V = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_6\}$ and Bob knows that $v \neq 5$ at ω_1, ω_2, ω_3, and ω_6:

$K_B(V) = \{\omega_1, \omega_2, \omega_3, \omega_6\}$. Here $K_B(V) \neq S$ (in particular, $K_B(V)$ strictly includes S) and the substitutivity principle does not apply. This situation is represented in Figure 3 below:

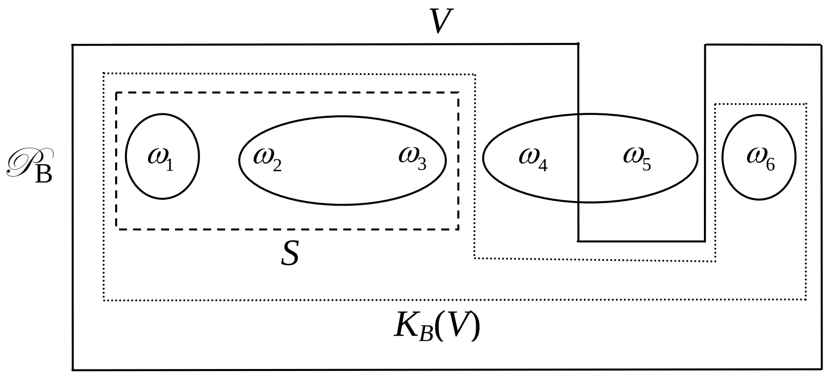

Figure 3

Here if the true state is ω_1 Bob knows that $v \neq 5$. Does Ann also know that Bob knows $v \neq 5$? If so, does this require as an additional assumption that Ann knows Bob's information partition? The latter question will be called Puzzle 2. The difference between Puzzle 1 and Puzzle 2 is that the latter cannot be solved by means of Substitutivity alone.

To solve Puzzle 2, note that $\{\omega_1, \omega_2, \omega_3, \omega_6\}$ is an event in the first place, which can be interpreted without any reference to Bob or his knowledge. For instance, at the end of Section 1 we labeled $\{\omega_1, \omega_2, \omega_3, \omega_6\}$ as W and interpreted it as the event such that "$v \leq 3$ or $v = 6$". It was also observed that W is one of the 16 events that, by Immediacy, Ann knows and cannot avoid knowing at ω_1 since $W \supseteq P_A(\omega_1)$. Therefore, $\omega_1 \in K_A(W)$. This situation is represented in Figure 4:

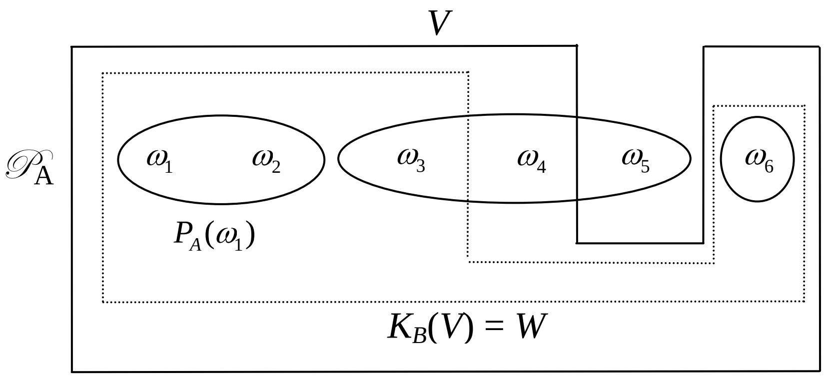

Figure 4

But the event $K_B(V)$, i.e., "Bob knows that $v \neq 5$", and the event W, i.e., "$v \leq 3$ or $v = 6$"

are extensionally equal, so that we are back to the setting of Puzzle 1 where Substitutivity applies: since $\omega_1 \in K_A(W)$ and $K_B(V) = W$, then $\omega_1 \in K_A(K_B(V))$. To put it verbally: yes, at ω_1 Ann knows that Bob knows that $v \neq 5$, and this does not require any additional assumption about Ann's knowledge of Bob's information partition. Substitutivity together with Immediacy are sufficient to do the job. If one does not mix up the formal content of the state-space model of knowledge and an interpretation of it relying on the intuitive and philosophical understanding of knowledge, also Puzzle 2 vanishes.

Notice that Substitutivity and Immediacy do not imply that, if in a certain state Ann and Bob know an event E, it is also the case that each agent knows that the other agent knows E. Think of event $Z = \{\omega_1, \omega_2\}$. At ω_1 both Ann and Bob know Z, but Ann does not know whether Bob knows Z since $K_B(Z) = \{\omega_1\}$ and $P_A(\omega_1) = \{\omega_1, \omega_2\}$, which is not a subset of $K_B(Z)$.

Rather, what Substitutivity and Immediacy imply can be summed up as follows. For any event E and agent i, consider the event $K_i(E)$ associated with E by the operator K_i. $K_i(E)$ can always be relabeled F, where the intension of F does not bear any relationship with agent i's knowledge. If $P_j(\omega) \subseteq F$, by Immediacy agent j knows F at ω. Then, since $F = K_i(E)$, by Substitutivity agent j also knows $K_i(E)$ at ω, that is, agent j knows that agent i knows that E. Such a result does not require any additional assumption about the interactive knowledge of the agents' information partitions. This answers the first question of the paper.

7. COMMON KNOWLEDGE OF INFORMATION PARTITIONS

As said in the Introduction, an event is said to be common knowledge among a group of agents if all know it, all know that all know it, and so on, *ad infinitum*. Within the state-space model of knowledge, an event E is said to be common knowledge between Ann and Bob in the state of the world ω – this is written as $\omega \in CK_{AB}(E)$ – if at ω Ann knows E in the sense of the operator K, Bob knows E in the sense of K, Ann knows that Bob knows E in the sense of K, and so on. Formally, $\omega \in CK_{AB}(E)$ if ω belongs to every set of the infinite sequence $K_A(E)$, $K_B(E)$, $K_A(K_B(E))$, $K_B(K_A(E))$, $K_A(K_B(K_A(E)))$, $K_B(K_A(K_B(E)))$,…

If we look at this definition of common knowledge having in mind the previous discussion, it is natural to ask whether common knowledge of an event requires that the information partitions of the agents are themselves common knowledge. In fact, common knowledge involves infinite levels of interactive knowledge. $K_A(K_B(E))$ and $K_B(K_A(E))$ involve the first level of interactive knowledge, which is the level investigated so far: does Ann (Bob) know that Bob (Ann) knows E? $K_A(K_B(K_A(E)))$ and $K_B(K_A(K_B(E)))$ draw in the second level of interactive knowledge: does Ann (Bob) know that Bob (Ann) knows that she (he) knows E? $K_A(K_B(K_A(K_B(E))))$ and $K_B(K_A(K_B(K_A(E))))$ involve the third level of interactive knowledge, and so on. Each further level of interactive knowledge may seem to require a further level of interactive knowledge of information partitions. So, with reference to the first level one may wonder whether Ann needs to know Bob's information partition to ascertain whether Bob knows E. This question was answered in the negative in the previous section. But one may also wonder whether, at the second level of interactive knowledge, Ann needs to know that Bob knows her information partition to ascertain whether Bob knows that she knows E. At level three, does Ann need to know that Bob knows that she knows his information partition to ascertain whether Bob knows that she knows that he knows E?

Even if things seem to get quickly convoluted here, the basic issue that emerges as we climb the infinite staircase of interactive knowledge is fairly simple. If each further level of interactive knowledge requires a further level of interactive knowledge of information partitions, does common knowledge of an event require as an additional assumption that the information partitions of the agents are themselves common knowledge? This is the second question addressed in the paper.

To answer it, observe first that we can always re-label events $K_A(E)$, $K_B(E)$, $K_A(K_B(E))$, $K_B(K_A(E))$, $K_A(K_B(K_A(E)))$, $K_B(K_A(K_B(E)))$,… in the following recursive way:

$$
\begin{aligned}
&K_A(E) = F^1\\
&K_B(E) = G^1\\
&K_A(K_B(E)) = K_A(G^1) = F^2\\
&K_B(K_A(E)) = K_B(F^1) = G^2\\
&K_A(K_B(K_A(E))) = K_A(G^2) = F^3\\
&K_B(K_A(K_B(E))) = K_B(F^2) = G^3 \ \ldots\ldots
\end{aligned}
$$
[17]

This re-labeling emphasizes that the events related to the agents' interactive knowledge are ultimately just subsets of Ω, which can always be associated with intensions bearing no relationship to the agents' knowledge.

Assume now that $\omega \in CK_{AB}(E)$. From a set-theoretic viewpoint, this simply means that $P_A(\omega)$ and $P_B(\omega)$ are included in E and belong to an infinite sequence of other sets. More

[17] If P1 holds, then $E \supseteq K_A(E) \supseteq K_B(K_A(E)) \supseteq K_A(K_B(K_A(E))) \supseteq\ldots$ and $E \supseteq K_B(E) \supseteq K_A(K_B(E)) \supseteq K_B(K_A(K_B(E))) \supseteq\ldots$ The present discussion is independent of P1, so that the above relationships of inclusion may not hold.

precisely, $P_A(\omega) \subseteq G^l$ for every $l \geq 1$, and $P_B(\omega) \subseteq F^m$ for every $m \geq 1$. In terms of Immediacy, at ω Ann immediately and necessarily knows all the events G^l, and Bob immediately and necessarily knows all the events F^m.

The first level of interactive knowledge is the one already investigated in Puzzle 1 and Puzzle 2. As just noted, by assumption at ω Ann knows and cannot avoid knowing G^1, and Bob knows and cannot avoid knowing F^1. But $G^1 = K_B(E)$ and $F^1 = K_A(E)$, so that Substitutivity applies and each agent knows that the other agent knows E. This is without additional assumptions about the interactive knowledge of their information partitions.

Consider now the second level of interactive knowledge. By assumption, at ω Ann and Bob know and cannot avoid knowing G^2 and F^2, respectively. Since $G^2 = K_B(K_A(E))$ and $F^2 = K_A(K_B(E))$, Substitutivity applies and each agent knows that the other agent knows that he/she knows E. Again, this does not require any additional assumption about the interactive knowledge of the agents' information partitions.

Evidently, the same argument can be used for any higher level of interactive knowledge, so that we can conclude that common knowledge of an event does not require as an additional assumption that the information partitions of the agents are themselves common knowledge. This answers the second question of the paper.

8. CONCLUSION

This paper has shown that interactive and common knowledge of information partitions (or, which is the same for the paper's purposes, of possibility correspondences) are not additional assumptions of the state-space model of knowledge. In fact, they draw from Substitutivity and Immediacy, two properties that the knowledge operator K holds by construction in the state-space formalism, even if the possibility correspondence P is not partitional.

This is not to say that Substitutivity and Immediacy, and hence interactive and common knowledge of information partitions, are realistic features of knowledge. Substitutivity clashes with the intensional character that both commonsense and philosophy attribute to knowledge. Immediacy may hold for specific events related to sensations or thoughts, but it is highly implausible in most circumstances.

In effect, the original insight of the current contribution is the methodological distinction between the formal content of the state-space model of knowledge and its interpretations. More precisely, the current work has argued that the puzzles surrounding interactive and common knowledge of information partitions originate from interpreting the state-space model of knowledge according to the intuitive and philosophical understanding of knowledge rather then to its actual formal content. When one realizes that the knowledge involved in the puzzles is the knowledge captured by the operator K (rather than the knowledge of common-sense or philosophy), the puzzles vanish.

Substitutivity and Immediacy are not the only unrealistic features of K. Necessitation and Monotonicity, two other properties that K necessarily holds in the state-space model, also appear dubious from the intuitive and philosophical viewpoint, not least because they preclude unawareness. In order to accommodate unawareness, some extensions of the standard state-space model have recently been proposed among others by Heifetz, Meier and Schipper (2006), Li (2006) and Galanis (2006). Even if these extensions differ, the basic strategy to generalize the standard model is the same. Whereas in the standard model the state-space Ω is unique, the extensions distinguish among the complete, objective state-space Ω^*, whose elements ω^* specify every aspect of the world that is relevant to the situation from the modeler's perspective, and the subjective spaces Ω_i, whose elements specify every relevant aspect of the world agent i is aware of. Typically, the subjective spaces Ω_i are poorer that the objective space Ω^* (agents are unaware of a number of relevant aspects of the world), and they vary among agents (different agents are unaware of different things).

One may wonder what happens to Substitutivity and Immediacy, and hence to interactive and common knowledge of information partitions, in the extended state-space models. For instance, it could be argued that, even if two events coincide in the objective space Ω^*, they may differ in the subjective space of an agent. In this case Substitutivity may fail and interactive and common knowledge of information partitions may become additional assumptions of the model. This will be left to future research.

APPENDIX: AUMANN'S DISCUSSION OF INTERACTIVE AND COMMON KNOWLEDGE OF INFORMATION PARTITIONS

As mentioned in the Introduction, Aumann has always maintained the view that common knowledge and hence interactive knowledge of information partitions are not additional assumptions of the state-space model, but something built into the model itself. Over time, Aumann used two arguments to make his case. This Appendix offers critical discussions of both arguments and argues that Aumann's view is better supported by bringing into play Substitutivity and Immediacy.

Initially, Aumann (1976, 1987) put forward an argument by contradiction based on the very notion of a state of the world. If the model is well specified, a state of the world is a complete description of every aspect of the world that is relevant to the situation. Therefore, a state of the world contains also a description of the manner in which the information is distributed among the agents, that is, of their information partitions.

Assume now that at state ω Ann has a doubt about Bob's information partition. At ω Ann may think: "I don't know whether Bob's information partition is $\mathscr{P}_B^{'}$ or $\mathscr{P}_B^{''}$". But if this is the case, then the model is ill-specified. In the correct model, in fact, the state ω should be split into two states: ω' where Bob's information partition is $\mathscr{P}_B^{'}$, and ω'' where Bob's information partition is $\mathscr{P}_B^{''}$. Accordingly, the state space Ω should be expanded and Ann's information partitions should be such that she cannot distinguish between ω' and ω''. More generally, if in a state of the world agent i is uncertain about the information partition of agent j, then that state should be broken into different states and Ω should be expanded until the point where all uncertainty of agent i about the information partition of j is eliminated. Therefore, in the correct and complete state space Ω, which is also called *canonical*, each agent knows by construction the information partitions of the other agents.

The problem with this argument is that an arbitrarily large number of state splits may be required to construct the canonical Ω, which hence may have an arbitrarily large number of elements. In effect, Aumann himself (1989, 1999), and with different setting and tools Hart, Heifetz and Samet (1996), Heifetz and Samet (1998), as well as Fagin, Geanakoplos, Halpern and Vardi (1999), have shown that even in the simplest scenario involving the interactive knowledge of two agents about an external variable that can assume only two values, the canonical Ω should have the cardinality of the continuum to exhaust all the uncertainty of the agents.[18]

For the purposes of the present paper, this means that the fact that interactive knowledge and common knowledge of information partitions are implicit in the state-space model of knowledge cannot be proved by exploiting the very notion of state of the world. This negative result leaves another road open, namely that of exploiting the very properties of the knowledge operator K, and this is the road taken in the current contribution. As shown in the paper, the properties of Substitutivity and Immediacy that the knowledge operator K holds by construction, are sufficient to achieve interactive and common knowledge of information partitions, and this in any Ω, not only in the canonical one.

Given the inconsistency of the argument based on the notion of a state of the world, Aumann (1989, 1999) addresses the question of interactive and common knowledge of information partitions from a new perspective. Aumann examines the relationships between the

[18] In a probabilistic belief space, the problems with the construction of the canonical Ω can be overcome: see Mertens and Zamir (1985) and Brandenburger and Dekel (1993). However, even this positive result is subject to limitations: see Fagin, Geanakoplos, Halpern and Vardi (1999), and Heifetz and Samet (1999b).

state-space model and the logic-based approach to knowledge, and argues that an answer to the question is provided by the latter. To understand Aumann's argument, a brief presentation of the logic-based formalism is necessary.[19]

Its building blocks are a set of letters $\Phi = \{p, q, r, ...\}$, the logical connectives $\wedge$ ("and"), $\vee$ ("or"), $\rightarrow$ ("if...then"), $\neg$ ("not"), and the operators k_i defined for each agent i. Each letter in Φ is interpreted as a primitive proposition describing some natural aspect, i.e., not related to the agents' knowledge, of the world. For instance, predictably, p may stand for "it rains". The set Φ is supposed to be rich enough to leave no relevant natural aspect of the world undescribed. Letters, i.e. primitive propositions, can be composed by means of logical connectives in order to form more complex propositions called *formulas*. For instance, if q means "Ann is happy", $\varphi = p \wedge \neg q$ is a formula that means "it rains and Ann is not happy". Every letter is a formula, so that the set of formulas is a superset of Φ.

The operator k_i applies to formulas and maps them into other formulas, and is interpreted as a knowledge operator. So, for instance, $k_i\varphi$ reads as "agent i knows that it rains and Ann is not happy", where $k_i\varphi$ is also a formula. The logic operator k_i may satisfy certain properties that parallel those of the set-theoretic operator K_i in the state-space model. For instance, it is generally assumed that k_i satisfy Necessitation (if ϕ is a logical truth, then $k_i\phi$ for every agent i) and Conjunction ($k_i\varphi \wedge k_i\phi \rightarrow k_i(\varphi \wedge \phi)$ for any couple of formulas φ and ϕ, and every agent i).

In the state-space model the states of the world are the primitive elements of the analysis. They are conceived as complete descriptions of every aspect of the world that is relevant to the situation, but in fact the state-space formalism lacks a language to describe explicitly such aspects. In the logic-based approach, instead, the states of the world are derived from formulas. More precisely, a state of the world ω is just a list of formulas that specify all circumstances holding at ω and satisfy certain requirements.[20]

In the logic-based framework, what agent i knows at a given state ω is written in the list that defines ω: agent i knows all the formulas φ such that $k_i\varphi$ is in the list. The lists that

[19] The following presentation of the logic-based approach is based on Aumann (1999), and Fagin, Halpern, Moses and Vardi (1995).

[20] First, the list of formulas defining a state of the world should be *closed*: if a formula φ belongs to the list, all formulas implied by φ also belong to the list. Second, the list should be *consistent* (or *coherent*): if a formula φ belongs to the list, its negation ¬φ does not belong to the list. Third, the list should be *complete*: if a formula φ does not belong to the list, its negation ¬φ belongs to the list.

define the states also induce the information partitions of the agents. In fact, two states ω and ω' belong to the same cell of agent i's information partition if in both states agent i knows exactly the same formulas, that is, if for any formula φ, $k_i\varphi \in \omega$ implies $k_i\varphi \in \omega'$ and vice versa. Since the information partitions of the agents are induced by their knowledge operators k_i, in the logic-based approach the issue about the interactive knowledge of information partitions becomes the issue about the interactive knowledge of the operators k_i.

Now, Aumann (1999) argues that interactive knowledge of the operators k_i and hence of the information partitions $\mathscr{P}_i$, is not an additional assumption of the model. His argument is based on a peculiar interpretation of the meaning of knowledge when the knowledge of the operators k_i of the others is involved, that is, when knowledge is interactive. Whereas in a string like $k_i f$, k_i means as usual "agent i knows that f", in a string like $k_j k_i f$, k_j would stand for "agent j knows *what it means* for agent i to know f". To distinguish knowledge as "knowing what it means…" from knowledge as "knowing that", Aumann uses quotation marks when he refers to the former:

> For an individual *j* to "know" k_i means that *j* knows what it means for *i* to know something. […] Suppose, for example, that *f* stands for "it will snow tomorrow". For *j* to know the operator k_i implies that *j* knows that k_if stands for "*i* knows that it will snow tomorrow;" it does *not* imply that *j* knows that *i* knows that it will snow tomorrow […]. In brief, *j*'s knowing the operator k_i means simply that *j* knows what it means for *i* to know something, not that *j* knows anything specific about what *i* knows. (Aumann, 1999, p. 277)

Since "knowing" the knowledge operators of others entails only acquaintance with the dictionary of the model, assuming that each agent "knows" the operators k_i of others constitutes no loss of generality:

> The operator k_i operates on formulas; it takes each formula *f* to another formula. Which other formula? […] Well, it is simply the formula k_if. "Knowing" the operator k_i just means knowing this definition. […] Thus the assertion that each individual "knows" the knowledge operators k_i of all individuals has no real substance; it is part of the framework. If *j* did not "know" the knowledge operators k_i he would be unable to consider formulas in the language. (Aumann, 1999, p. 277)

In a footnote, Aumann acknowledges that his interpretation of interactive knowledge is quite peculiar, and that it is not clear how to deal with the operator "knowing" in the logic-based formalism:

> It should be recognized that "knowledge" […] has a meaning that is somewhat different from that embodied in the [operator] k_i […]; that is why we have been using quotation marks when talking about "knowing" an operator or a partition. That an individual *i* knows […] a formula *f*

> can be embodied in formal statements […] that are well defined within the formal system we have constructed […]. This is not the case for "knowing" an operator or a partition. (Aumann, 1999, p. 277, footnote 17)

Now, Aumann's entire argument appears debatable. From a formal viewpoint, and as Aumann himself notices, the lack of an exact definition of the operator "knowing" makes it difficult to talk rigorously about it. From a more substantial viewpoint, k_j in the string $k_j k_i f$ should *not* be interpreted as "agent j knows *what it means* for agent i to know f". The standard interpretation of that string, i.e., "agent j knows that agent i knows f", is the correct one and this for a very good reason: if interactive knowledge were interpreted as Aumann suggests in the passages quoted above, it would be of little consequence in strategic environments. For instance, we can well imagine that if j knows that i knows that it will snow tomorrow, j rules out the possibility that i takes action a_i, and hence j decides to take action a_j. If, instead, j only knows what it means for i to know that it will snow tomorrow, but j is uncertain whether i knows that it will snow tomorrow, j cannot exclude action a_i by i and maybe j will not take action a_j. In other words, the implications for strategic behavior of "knowing what it means…" are generally weak, and this renders such a notion of interactive knowledge almost useless for game theory and economic applications.

The point made in the present paper is that, in order to show that interactive and common knowledge of information partitions are indeed, as Aumann has always claimed, implicit in the model itself, there is no need to introduce a "knowing what it means…" operator, nor to replace the state-space model of knowledge with the logic-based formalism. Interactive knowledge and common knowledge of information partitions draw from Substitutivity and Immediacy, two properties that the knowledge operator K holds by construction in the state-space model of knowledge.

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