A numerical strategy for telecommunications networks capacity planning under demand and price uncertainty

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Abstract

The massive use of Internet in the last twenty years has created a huge demand for telecommunications networks capacity. In this work, financial option pricing methods are applied to the problem of network investment decision timing. The main innovative aspect is the consideration of two uncertain factors: the capacity demand and the bandwidth price, the evolution of which are modeled by suitable stochastic processes. Thus, we consider the optimal decision problem of upgrading a line in terms of the (highly volatile) uncertain demand for capacity and the price. By using real options pricing methodology, a set of partial differential equation problems are posed and appropriate numerical methods based on characteristics methods combined with finite elements to approximate the solution are proposed. The combination with a dynamic programming strategy gives rise to a global algorithm to help in the decision of optimizing the value of the line.

Keywords: Telecommunications networks planning, real options pricing, numerical methods, decision making

1. Introduction

Financial valuation of telecommunication networks, such as Internet and dark fiber networks, is a relevant current subject, as these kinds of infrastructure are the object of important trading activities and companies have to make large investments on them. When the infrastructure starts to operate, an important question is to price it as an asset that will generate revenues in future dates depending on the future market uncertainty. Different approaches have been made to this interesting problem in a telecommunications networks setting. More specifically, concerning to the network planning problem we address to Kenyon and Cheliotis [8], for example. Unlike the previously described problem, a second very important one concerns to the decision of expanding or not an existing network which is already generating revenues. In this second setting the optimal solution depends on the market demand of capacity and the bandwidth prices. So, this new problem can be related to risk management and investment decision problems. Also time results very relevant, as decisions can be delayed waiting to the arrival of more information.

In both different problems, a real options approach can be considered [1]. For example, in [8], the first setting is addressed and the price of the bandwidth is the only underlying uncertain factor, which is modelled as an stochastic process and the network value is computed by means of a binomial tree technique. More recently, in D'Halluin *et al* [2, 3] the second problem has been formulated by also using the real options theory, when only the uncertainty in market demand is considered. More precisely, the price is given by a deterministic function in terms of time and demand, the latter being the only stochastic factor. Furthermore, a partial differential equation (PDE) formulation combined with a dynamic programming strategy is posed in [2] to solve the network planning capacity problem. In this problem, for a given set of lines with different technical features and already generating revenues, the objective is to find the percentage at which it is optimal to upgrade the line. This percentage is referred to the maximum transmission rate of the line. In [2] it is argued that while telecommunication markets are inefficient the demand is the main uncertain driving factor of the network price, so that the stochastic variation of the bandwidth price can be neglected.

In the present paper, we mainly address the second problem: the optimal decision to expand an existing network that is already producing revenues. In this setting, the main innovation comes from the assumption that the value of the line (that can be understood as the expected revenue for the owner) depends on two underlying stochastic factors: the capacity demand and the bandwidth price, which jointly evolve in time according to their respective stochastic differential equations. In the here presented approach, the model for the stochastic evolution of demand is mainly taken from [2], while the stochastic dynamics of the bandwidth price is adapted from [8]. Additionally, as we have selected a mean reversion process for bandwidth price, several possible cases are considered for the mean reversion value: constant, time dependent or stochastic. Accordingly, we obtain a constant, a time dependent or a stochastic coefficient on the corresponding PDE in two *spatial like* variables. In the latter case, an alternative PDE model with three spatial like variables can be posed. However, the consideration of a stochastic coefficient in the PDE in two variables allows the use of Monte Carlo techniques to obtain the expectation of the line value from the different simulated scenarios of the stochastic coefficient and avoids the building and use of a more complex computational code for the PDE with three spatial variables. For the numerical solution of the involved PDE problems we propose a characteristics method for the time discretization to cope with the convection dominated situations, while the spatial discretization is carried out by a finite elements method. The consideration of upgrading decision can be related to the pricing problem of callable bonds with notice, where the notice date is analogous to the upgrading decision date. In this specific issue, we follow the ideas in [4, 5] for callable bonds. In order to show the good performance of the proposed model and the whole numerical algorithm, several test examples have been considered and some numerical results are presented.

The plan of the paper is the following. In Section 2 the different mathematical models are posed. Section 3 describes the various numerical techniques that have been proposed. In Section 4 several examples with the corresponding numerical results are shown. Finally, some conclusions are indicated in Section 5.

2. Mathematical model

2.1. A one factor model

Following [2], we assume that the time evolution of demand for capacity (measured in megabytes), can be modeled as a stochastic process, Q_t, satisfying the stochastic differential equation:

$$dQ_t = \mu Q_t \, dt + \sigma_Q Q_t \, dZ_t \,,$$

where μ and σ_Q denote the drift (or growth) rate and the instantaneous volatility of the demand, respectively, while Z_t represents a Wiener process.

Assuming that there exists a function V, such that the value of the network is a stochastic process given by $V_t = V(t, Q_t)$ (in monetary units), and using Ito lemma (see [9], for example) plus classical dynamic hedging arguments, the function V is the solution of the following partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{\sigma_Q^2}{2} Q^2 \frac{\partial^2 V}{\partial Q^2} + (\mu - \kappa \sigma_Q) Q \frac{\partial V}{\partial Q} - rV + \mathcal{R}(t, Q) = 0 \,, \tag{1}$$

where $\mathcal{R}$ is a revenue term we will detail later in the two factors case, r is the risk free interest rate and κ represents the market price of risk associated to demand uncertainty. Moreover, we consider that the value V is known at a given finite time horizon (final condition). In our case, we consider an investment time horizon T and assume that $V_T = 0$, so that $V(T, Q) = 0$, which seems an appropriate hypothesis in the rapidly changing telecommunications market. Concerning the boundary conditions, at $Q = 0$ we just pass to the limit in the equation when $Q \to 0$. After time discretization, homogeneous Dirichlet boundary condition naturally arises from the initial condition and the fact that $\mathcal{R}(0, Q) = 0$. For $Q \to \infty$ we assume an homogeneous Neumann boundary condition holds. Other additional issues in the one factor model will be explained in the two factors model described in next section.

2.2. A two factors model

In the present paper we assume that the value of the line depends not only on the capacity demand but also on the bandwidth price. In this new setting we pose $V_t = V(t, Q_t, S_t)$, the additional stochastic factor S_t being the price at time t. In order to pose the stochastic model for prices evolution we mainly follow [8], thus assuming that the logarithmic price, $X_t = \log S_t$, follows the following Orstein-Uhlenbeck process:

$$dX_t = \eta(\overline{X} - X_t)\, dt + \sigma_S dW_t \,,$$

where X_t is therefore a mean reverting Ito process, $\overline{X}$ represents its long term value to which X_t tends for large t, η is the speed of reversion to the long term price $\overline{X}$ and W_t denotes a Wiener process.

Next, in order to obtain the stochastic differential equation satisfied by the real price process, S_t, let f denote the function such that:

$$S_t = f(t, X_t) = \exp(X_t) \,.$$

Thus, applying Ito lemma [9] we obtain:

$$f(t,X_t) - f(s,X_s) = \int_s^t \left[\eta(\overline{X} - X_u)\exp(X_u) + \frac{\sigma_S^2}{2}\exp(X_u) \right] du + \int_s^t \sigma_S \exp(X_u)\, dW_u\,,$$

or, in differential form,

$$dS_t = \left[\eta(\overline{X} - X_t)\exp(X_t) + \frac{\sigma_S^2}{2}\exp(X_t) \right] dt + \sigma_S \exp(X_t)\, dW_t$$

which is equivalent to:

$$dS_t = \left[\left(\eta(\overline{X} - \log S_t) + \frac{\sigma_S^2}{2} \right) S_t \right] dt + \sigma_S S_t\, dW_t\,.$$

Next, we take into account that $V_t = V(t, Q_t, S_t)$ and assume that dW_t and dZ_t are correlated Wiener processes with constant correlation ρ, i.e. $dW_t dZ_t = \rho dt$. Then, Ito lemma for two stochastic factors leads to [9]:

$$\begin{aligned} dV_t = & \left[\frac{\partial V}{\partial t}(t,Q_t,S_t) + \frac{\sigma_Q^2}{2} Q_t^2 \frac{\partial^2 V}{\partial Q^2}(t,Q_t,S_t) + \frac{\sigma_S^2}{2} S_t^2 \frac{\partial^2 V}{\partial S^2}(t,Q_t,S_t) \right. \\ & \left. + \rho\sigma_Q\sigma_S\, Q_t S_t \frac{\partial^2 V}{\partial Q \partial S}(t,Q_t,S_t) \right] dt + \frac{\partial V}{\partial Q}(t,Q_t,S_t)\, dQ_t + \frac{\partial V}{\partial S}(t,Q_t,S_t)\, dS_t \\ = & \left[\frac{\partial V}{\partial t}(t,Q_t,S_t) + \frac{\sigma_Q^2}{2} Q_t^2 \frac{\partial^2 V}{\partial Q^2}(t,Q_t,S_t) + \frac{\sigma_S^2}{2} S_t^2 \frac{\partial^2 V}{\partial S^2}(t,Q_t,S_t) \right. \\ & \left. \qquad + \rho\sigma_Q\sigma_S\, Q_t S_t \frac{\partial^2 V}{\partial Q \partial S}(t,Q_t,S_t) \right] dt \\ & + \left[\mu Q_t \frac{\partial V}{\partial Q}(t,Q_t,S_t) + \left(\eta(\overline{X} - \log S_t) + \frac{\sigma_S^2}{2} \right) S_t \frac{\partial V}{\partial S}(t,Q_t,S_t) \right] dt \\ & + \sigma_S S_t \frac{\partial V}{\partial S}(t,Q_t,S_t)\, dW_t + \sigma_Q Q_t \frac{\partial V}{\partial Q}(t,Q_t,S_t)\, dZ_t\,. \end{aligned}$$

For simplicity, hereafter we remove the subindex t in all stochastic processes.

Next, we build up the portfolio $\Pi = V_1 - \Delta_2 V_2 - \Delta_1 Q$, so that:

$$
\begin{aligned}
d\Pi &= dV_1 - \Delta_2 dV_2 - \Delta_1 dQ \\
&= \left[\frac{\partial V_1}{\partial t} + \frac{\sigma_Q^2}{2} Q^2 \frac{\partial^2 V_1}{\partial Q^2} + \frac{\sigma_S^2}{2} S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho \sigma_Q \sigma_S \, QS \frac{\partial^2 V_1}{\partial Q \partial S} \right. \\
&\qquad \left. + \mu Q \frac{\partial V_1}{\partial Q} + \left(\eta(\overline{X} - \log S) + \frac{\sigma_S^2}{2} \right) S \frac{\partial V_1}{\partial S} \right] dt \\
&+ \sigma_Q Q \frac{\partial V_1}{\partial Q} \, dZ + \sigma_S S \frac{\partial V_1}{\partial S} \, dW \\
&- \Delta_2 \left[\left[\frac{\partial V_2}{\partial t} + \frac{\sigma_Q^2}{2} Q^2 \frac{\partial^2 V_2}{\partial Q^2} + \frac{\sigma_S^2}{2} S^2 \frac{\partial^2 V_2}{\partial S^2} + \rho \sigma_Q \sigma_S \, QS \frac{\partial^2 V_2}{\partial Q \partial S} \right. \right. \\
&\qquad \left. + \mu Q \frac{\partial V_2}{\partial Q} + \left(\eta(\overline{X} - \log S) + \frac{\sigma_S^2}{2} \right) S \frac{\partial V_2}{\partial S} \right] dt \\
&\qquad \left. + \sigma_Q Q \frac{\partial V_2}{\partial Q} \, dZ + \sigma_S S \frac{\partial V_2}{\partial S} \, dW \right] - \Delta_1 \left(\mu Q \, dt + \sigma_Q Q \, dZ \right) .
\end{aligned}
$$

In order to guarantee that the portfolio Π is risk free, we remove the random component of $d\Pi$ with the following choice:

$$
\Delta_1 = \frac{\partial V_1}{\partial Q} - \frac{\partial V_1 / \partial S}{\partial V_2 / \partial S} \frac{\partial V_2}{\partial Q} \qquad \text{and} \qquad \Delta_2 = \frac{\partial V_1 / \partial S}{\partial V_2 / \partial S} .
$$

Thus, using the classical no arbitrage argument, we deduce:

$$
d\Pi = r\Pi \, dt = r \left[V_1 - \frac{\partial V_1 / \partial S}{\partial V_2 / \partial S} V_2 - \left(\frac{\partial V_1}{\partial Q} - \frac{\partial V_1 / \partial S}{\partial V_2 / \partial S} \frac{\partial V_2}{\partial Q} \right) Q \right] dt \, ,
$$

and:

$$
\begin{aligned}
&\frac{1}{\partial V_1 / \partial S} \left[\frac{\partial V_1}{\partial t} + \frac{\sigma_Q^2}{2} Q^2 \frac{\partial^2 V_1}{\partial Q^2} + \frac{\sigma_S^2}{2} S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho \sigma_Q \sigma_S QS \frac{\partial^2 V_1}{\partial Q \partial S} + rQ \frac{\partial V_1}{\partial Q} - rV_1 \right] dt \\
&= \frac{1}{\partial V_2 / \partial S} \left[\frac{\partial V_2}{\partial t} + \frac{\sigma_Q^2}{2} Q^2 \frac{\partial^2 V_2}{\partial Q^2} + \frac{\sigma_S^2}{2} S^2 \frac{\partial^2 V_2}{\partial S^2} + \rho \sigma_Q \sigma_S QS \frac{\partial^2 V_2}{\partial Q \partial S} + rQ \frac{\partial V_2}{\partial Q} - rV_2 \right] dt \, ,
\end{aligned}
$$

so that both members of the equation are independent of i and therefore equal to a function depending on t, Q and S, that we denote by a. Thus, we have:

$$
a(t, Q, S) = \frac{1}{\partial V / \partial S} \left[\frac{\partial V}{\partial t} + \frac{\sigma_Q^2}{2} Q^2 \frac{\partial^2 V}{\partial Q^2} + \frac{\sigma_S^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma_Q \sigma_S QS \frac{\partial^2 V}{\partial Q \partial S} + rQ \frac{\partial V}{\partial Q} - rV \right]
$$

Next, as usual in many financial problems, we choose the following expression for function a in terms of λ, which represents the market price of risk associated to the uncertainty in the bandwidth price:

$$a(t,Q,S) = \sigma_S \lambda(t,Q,S)\, S - \left(\eta(\overline{X} - \log S) + \frac{\sigma_S^2}{2} \right) S$$

By identifying both previous expressions of a, we deduce the following PDE:

$$\frac{\partial V}{\partial t} + \frac{\sigma_Q^2}{2} Q^2 \frac{\partial^2 V}{\partial Q^2} + \frac{\sigma_S^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma_Q \sigma_S S Q \frac{\partial^2 V}{\partial Q \partial S} + rQ \frac{\partial V}{\partial Q} + \left[\eta(\bar{X} - \log S) + \frac{\sigma_S^2}{2} - \sigma_S \lambda \right] S \frac{\partial V}{\partial S} - rV = 0\,.$$

Analogous arguments can be developed in the presence of a revenue term $\mathcal{R}$, thus leading to:

$$\frac{\partial V}{\partial t} + \frac{\sigma_Q^2}{2} Q^2 \frac{\partial^2 V}{\partial Q^2} + \frac{\sigma_S^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma_Q \sigma_S S Q \frac{\partial^2 V}{\partial Q \partial S} + rQ \frac{\partial V}{\partial Q} + \left[\eta(\bar{X} - \log S) + \frac{\sigma_S^2}{2} - \sigma_S \lambda \right] S \frac{\partial V}{\partial S} - rV + \mathcal{R} = 0\,. \quad (2)$$

Although it is not strictly necessary, we introduce a time to horizon T variable, $\tau = T - t$, in order to write (2) forward in time and pose an initial value problem:

$$\frac{\partial V}{\partial \tau} - \frac{\sigma_Q^2}{2} Q^2 \frac{\partial^2 V}{\partial Q^2} - \frac{\sigma_S^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - \rho \sigma_Q \sigma_S S Q \frac{\partial^2 V}{\partial Q \partial S} - rQ \frac{\partial V}{\partial Q} - \left[\eta(\bar{X} - \log S) + \frac{\sigma_S^2}{2} - \sigma_S \lambda \right] S \frac{\partial V}{\partial S} + rV = \mathcal{R} \quad (3)$$

with the initial condition $V(0,Q,S) = 0$.

Adapting some ideas in [2], one possibility for the revenue function is given by

$$\mathcal{R}(\tau,Q,S) = \min(Q,\overline{Q}) D P(\tau,S)\,,$$

where $\overline{Q}$ is the maximum capacity of the line, D is its length and P represents the discounted price, which now depends not only on time as in the one factor case in [2], but also on price S. We choose the expression:

$$P(\tau,S) = S\, e^{-\alpha(T-\tau)}\,,$$

$\alpha > 0$ being a decay rate of the price.

So far we have modelled the evolution of the value of a particular line in terms of demand and price. However, in practice, a telecommunication line can be upgraded by increasing its maximum capacity. For example, let us assume a line with N possible maximum capacities (it can be equivalently understood as N different lines), the values of which are given by $V_i = V_i(t, Q_t, S_t)$ $(i = 1, \dots, N)$. Initially, the manager of the line offers the lowest (and cheapest) maximum capacity. However, as time passes by and the demand of capacity increases, it may be interesting to offer a better maximum capacity in order to give a better (and more expensive) service to the users. This upgrading possibility requires a capital investment that should be balanced by (uncertain) future revenues. So, a decision of investment under uncertainty arises.

The decision on upgrading will be taken periodically at the so called notice dates in the following way [2] . The word *notice* comes from the problem of callable bonds with notice —treated, for example, in [4] and [5], in which the issuer of the bond has the right to call it back at a given call price but having to notice this decision in a previous date (notice date). In telecommunication planning, an analogous situation arises; in this case technological constraints for upgrading require to take the decision in a previous time to actual upgrade date. Thus, at notice date all possible alternative line upgrades are evaluated and the one that leads to the maximum value of the line is chosen. Nevertheless, for technical reasons related to the upgrading procedure, a time delay is in practice needed between the notice (t_n) and the upgrade (t_u) instants. In Section 3.1 concerning to the global algorithm, the way to account this upgrading procedure is detailed in terms of the time variable τ and Figure 1 further illustrates it.

Additionally, for each level of upgrade, the periodic maintenance costs need to be considered. Thus, at fixed maintenance times τ_{m_k} the value of the line i is corrected due to these periodic maintenance costs:

$$V_i(\tau_{m_k}^+, Q, S) = V_i(\tau_{m_k}^-, Q, S) - M_i D \Delta\tau_m \,, \quad i = 1, \dots, N, \quad k = 1, 2, \dots, \tag{4}$$

where M_i is the unitary maintenance cost of line i and $\Delta\tau_m$ is the period between maintenance dates (for example, $\Delta\tau_m = 1/12$ corresponds to monthly payments). The values $\tau_{m_k}^-$ and $\tau_{m_k}^+$ represent the dates immediately before and after τ_{m_k}, respectively. In this case, the analogy in interest rates derivatives pricing appears in jumps in bond values at coupon payment dates [13]. For the numerical computations, the finite differences mesh in time include all the maintenance dates, so that the jump conditions (4) are incorporated at these dates.

Note that initially the previous PDEs are posed in an unbounded domain. In particular, the domain in variable Q is initially unbounded. So, in order to apply numerical methods we need to define an approximated problem in a bounded domain to be denoted by Ω. Typically, this is a characteristic of financial problems, in which localization or truncation of the domain is a common practice, the analysis of the truncation error has been rigorously analyzed for the classical European vanilla option in [6]. As we are solving a PDE for each line, we consider that each (upgrade level of) line has a maximum capacity $\overline{Q}_i$, with $\overline{Q}_i < \overline{Q}_j$ for $i < j$. Let $Q^* = \max_{i=1,\ldots,N} \overline{Q}_i = \overline{Q}_N$.

Concerning the price coordinate, we assume that S is lower bounded by $S_{\min}$ and upper bounded by $S_{\max}$. Thus, we introduce the bounded spatial domain $\Omega = (0, Q_{\max}) \times (S_{\min}, S_{\max})$, with $Q_{\max} = 6Q^*$, so that we can use a unique mesh for the numerical solution in order to compare the different lines. In practice, the value of Q_{max} is selected so that the numerical values of V in the region of financial interest (for example, $Q \leq Q^*$) are not affected by this choice (in our numerical examples we have checked no variation in the numerical results for this region with respect to the alternative choice $Q_{\max} = 7Q^*$).

Once the bounded domain Ω has been fixed, we consider the following boundary conditions:

$$
\begin{aligned}
V_i(\tau, 0, S) &= 0\,, & A\nabla V_i(\tau, Q_{\max}, S) &= 0\,,\\
A\nabla V_i(\tau, Q, S_{\min}) &= 0\,, & A\nabla V_i(\tau, Q, S_{\max}) &= 0\,,
\end{aligned}
$$

where operator A comes from expressing equation (3) in divergence form and is defined in the next section. Note that boundary conditions at $Q = 0$ and $Q = Q_{\max}$ are in agreement with those ones chosen in the one factor case. Conditions at $S = S_{\min}$ and $S = S_{\max}$ are chosen to minimize the effects of the domain truncation.

2.3. Three factors model

In the previous two factors model, the long term value $\overline{X}$ has been taken as a constant. However, in the more general setting proposed in [8], the long term mean can be considered a stochastic process. Note that the consideration of stochastic mean reversion level in a mean reverting process for the price evolution of commodities has been previously used in [10] and [11], for example. In this latter case, we can assume the following stochastic dynamics for $\overline{X}_t$:

$$d\overline{X}_t = -\nu dt + \overline{\sigma} dB_t \ ,$$

where the constants ν and $\overline{\sigma}$ are the positive instantaneous rate of mean reversion value decrease and the uncertainty of this rate, respectively, and B_t represents a standard Wiener process which is assumed to be uncorrelated with the process W_t. In case $\nu > 0$ we assume that the long term price to which prices are reversing decreases with time. This decline can be motivated by new advances in technology and can be known in terms of t when $\overline{\sigma} = 0$ or stochastic when $\overline{\sigma} > 0$. Note that vanishing some coefficients in the previous stochastic differential equation allows to recover the cases of constant or time dependent long term price. Additional characteristics like regime switching or jumps in prices are out of the scope of the present work.

In this new setting, by using analogous arguments as in the two factors case, a PDE in three spatial like variables can be obtained for a function V such that $V_t = V(t, Q_t, S_t, \overline{X}_t)$. More precisely, this equation can be written in the form

$$\frac{\partial V}{\partial t} + \frac{\sigma_Q^2}{2} Q^2 \frac{\partial^2 V}{\partial Q^2} + \frac{\sigma_S^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\bar{\sigma}^2}{2} \frac{\partial^2 V}{\partial \bar{X}^2} + \rho_1 \sigma_Q \sigma_S Q S \frac{\partial^2 V}{\partial Q \partial S} + \rho_2 \sigma_Q \bar{\sigma} Q \frac{\partial^2 V}{\partial Q \partial \bar{X}}$$
$$+ rQ \frac{\partial V}{\partial Q} + \left(\eta(\bar{X} - \log(S)) + \frac{\sigma_S^2}{2} - \sigma_S \lambda \right) S \frac{\partial V}{\partial S} + r\bar{X} \frac{\partial V}{\partial \bar{X}} - rV + \mathcal{R} = 0,$$

where, as in Kenyon and Cheliotis [8], we have assumed that there is no correlation between bandwidth price S_t and the mean reversion value $\overline{X}_t$. Note that in this modelling approach we assume the possibility of correlation between demand Q_t and the other two stochastic underlying factors, which also seems financially reasonable.

In next section, instead of using numerical methods for the initial-boundary value problem associated to the previous PDE, we will address the three factors model by considering the stochastic parameter $\overline{X}$ in the two factors PDE model, following the dynamics prescribed by the corresponding stochastic differential equation, as detailed in next section.

3. Numerical methods

In this section we describe the set of numerical techniques we propose for solving the two and three factors models. First, we present the global algorithm to account with the upgrading decision. Inside this algorithm we have to solve several PDE problems by appropriate numerical techniques.

3.1. Overall global algorithm

In Figure 1 we consider the time variable τ and the different possibilities of upgrading for each line at the two pairs of notice/upgrade dates when four different lines are available. For example, we show how at the notice date three possible upgrades exist for Line 1 (to Lines 2, 3 or 4), two for Line 2 and just one for Line 3. The decision is taken at notice date and the upgrading starts at the upgrade date. It is important to point out that in terms of (computational) backward times τ (time moves from $\tau = 0$, which corresponds to $t = T$, until $\tau = T$, which corresponds to present date $t = 0$), the upgrade dates (τ_{u_k}) are previous to the corresponding notice ones (τ_{n_k}). Thus, in the outer loop of the algorithm the eventual possible upgrades are taken into account in the following way:

- Between notice time ($\tau_{n_{k+1}}$) and upgrade time (τ_{u_k}), the forward in time PDE (3) is solved for each line i, independently of the other lines.

- Between upgrade time (τ_{u_k}) and notice time (τ_{n_k}), we solve (3) for each line i with initial condition $V_j(\tau_{u_k}, Q)$ (with $j = i, \ldots, N$). Thus, $N \times (N+1)/2$ initial value problems are solved. Next, at notice time τ_{n_k} we take:

  $$V_i(\tau_{n_k}, Q, S) = \max_{j=i,\ldots,N} \{V_j(\tau_{n_k}, Q, S) - K_{i\to j}(\tau_{n_k})\}$$

  for each discrete values (Q, S) associated to mesh points. The resulting j indicates the line to which line i is upgraded for each capacity demand Q, and the time dependent function $K_{i\to j}$ is given by:

  $$K_{i\to j}(\tau) = \overline{K}_{i\to j} \exp(-\alpha(T - \tau)) \,.$$

  In Table 2 a particular choice of constants $\overline{K}_{i\to j}$ is shown.

This procedure is carried on as many times ($k = 1, 2, \ldots$) as pairs of notice/upgrade instants are considered.

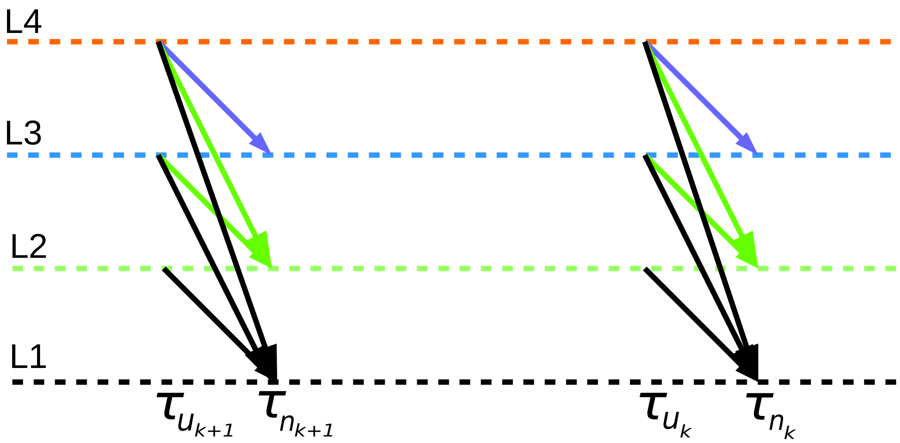

Figure 1: Sketch of possible upgrading with 4 available lines

3.2. PDE numerical solver

At different steps of the global algorithm, we need to solve the set of N PDEs of the form:

$$\frac{\partial V_i}{\partial \tau} - \frac{\sigma_Q^2}{2} Q^2 \frac{\partial^2 V_i}{\partial Q^2} - \frac{\sigma_S^2}{2} S^2 \frac{\partial^2 V_i}{\partial S^2} - \rho \sigma_Q \sigma_S S Q \frac{\partial^2 V_i}{\partial Q \partial S} - r Q \frac{\partial V_i}{\partial Q} \\ - \left[\eta(\bar{X} - \log S) + \frac{\sigma_S^2}{2} - \sigma_S \lambda \right] S \frac{\partial V_i}{\partial S} + r V_i = \mathcal{R} \,, \qquad (5)$$

with the corresponding initial and boundary conditions for $i = 1, \ldots, N$.

As we propose to solve (5) by using a finite element method, we introduce matrix A and vector b given by:

$$A(Q,S) = \begin{pmatrix} \dfrac{\sigma_Q^2}{2}Q^2 & \dfrac{\rho\sigma_Q\sigma_S}{2}QS \\ \dfrac{\rho\sigma_Q\sigma_S}{2}QS & \dfrac{\sigma_S^2}{2}S^2 \end{pmatrix}$$

$$b(Q,S) = \begin{pmatrix} \left(\sigma_Q^2 + \dfrac{\rho\sigma_Q\sigma_S}{2} - r\right) Q \\ \left(-\eta(\overline{X} - \log S - \log S_{\max}) + \dfrac{\sigma_S^2}{2} + \sigma_S\lambda + \dfrac{\rho\sigma_Q\sigma_S}{2}\right) S \end{pmatrix}$$

so that we can write (5) in divergence form:

$$\frac{\partial V_i}{\partial \tau} - \text{div}\ (A(Q,S)\nabla V_i) + b(Q,S)\cdot\nabla V_i + rV_i = \mathcal{R}\,. \tag{6}$$

We note that PDE (6) becomes strongly convection-dominated in certain regions for the set of realistic parameters to be used, that is, the first order derivatives terms are much larger than second order derivatives ones in these regions. This gives rise to large gradients in the solution, as observed in the forthcoming numerical results, so that standard finite differences or finite element discretizations may lead to spurious numerical oscillations which do not appear in the exact solution. In order to avoid this drawback, we propose to use a semi-Lagrangian discretization combined with finite elements. The use of a characteristics method (also known as semilagrangian scheme) has been first introduced in [12] in combination with finite differences methods for vanilla option pricing problems. In other works, the method has been combined with finite elements (also known as Lagrange-Galerkin technique).

In order to describe the numerical solution of the PDE with the proposed method, we first introduce the so called material derivative, which is defined as follows:

$$\frac{DV_i}{D\tau} = \frac{\partial V_i}{\partial \tau} + b(Q,S)\cdot\nabla V_i\,. \tag{7}$$

Thus, equation (6) can be written in the equivalent form:

$$\frac{DV_i}{D\tau} - \text{div}\ (A(Q,S)\nabla V_i) + rV_i = \mathcal{R}_i(\tau^{n+1},Q,S)\,. \tag{8}$$

Then, for a given constant time step $\Delta\tau > 0$, we consider $\tau^n = n\Delta\tau$ ($n = 0,1,\ldots$) and we approximate the material derivative in (8) by the following first order quotient:

$$\frac{V_i^{n+1} - V_i^n \circ \chi^n}{\Delta\tau} - \text{div}\,\left(A(Q,S)\nabla V_i^{n+1}\right) + rV_i^{n+1} = \mathcal{R}_i(\tau^{n+1}, Q, S)\,, \qquad (9)$$

where $V_i^n(\cdot,\cdot) \approx V_i(\tau^n,\cdot,\cdot)$ depends on S and Q, and $\chi^n \equiv \chi((Q,S),\tau^{n+1};\tau^n)$. In order to obtain χ^n, we note that $\chi((Q,S),\tau^{n+1};\tau)$ represents the characteristic curve associated to the vector field b passing through the point (Q,S) at time τ^{n+1}. Thus, the function $\chi = (\chi_1,\chi_2)$ can be obtained as the solution of:

$$\begin{cases} \dfrac{d\chi_1}{d\tau} = \left(\sigma_Q^2 + \dfrac{\rho\sigma_Q\sigma_S}{2} - r\right)\chi_1 \\ \dfrac{d\chi_2}{d\tau} = \left(-\eta(\overline{X} - \log\chi_2 - \log S_{\max}) + \dfrac{\sigma_S^2}{2} + \sigma_S\lambda + \dfrac{\rho\sigma_Q\sigma_S}{2}\right)\chi_2 \end{cases} \qquad (10)$$

with the final conditions $\chi_1(\tau^{n+1}) = Q$ and $\chi_2(\tau^{n+1}) = S$. In this case we can obtain the analytical expression of χ_1, so that χ_1^n is given by :

$$\chi_1^n = \chi_1(Q,\tau^{n+1};\tau^n) = Q\,\exp\left[-(\sigma_Q^2 + \frac{\rho\sigma_Q\sigma_S}{2} - r)\,\Delta\tau\right]\,,$$

while χ_2^n depends on the expression of $\overline{X}$.

As we are generalizing from one to two factors by adding the stochastic model for prices, we distinguish three different cases depending on the assumption on $\overline{X}$: constant, time dependent or stochastic. The constant and time dependent cases can be framed into the two stochastic factors setting, while the third case corresponds to a three factors setting. In our numerical approach based on the solution of a PDE problem in two spatial-like variables, the choice of $\overline{X}$ affects to the kind of coefficients appearing in the PDE (constant, time dependent or stochastic) and particularly to the expression of χ_2^n. Thus, we distinguish three cases:

(a) $\overline{X}$ is constant ($\nu = \overline{\sigma} = 0$): in this case we get

$$\chi_2^n = \frac{1}{S_{\max}}\exp\left[-\frac{\rho\sigma_Q\sigma_S}{2\eta} + \overline{X} - \frac{\sigma_S^2}{2\eta} - \frac{\sigma_S\lambda}{\eta} \right.$$
$$\left. + \exp(-\eta\Delta\tau)\left(\log(SS_{\max}) + \frac{\rho\sigma_S\sigma_Q}{2\eta} - \overline{X} + \frac{\sigma_S^2}{2\eta} + \frac{\sigma_S\lambda}{\eta}\right)\right]$$

(b) $\overline{X}$ is a deterministic function ($\overline{\sigma} = 0$): in this case $\overline{X}(t)$ satisfies the ODE $d\overline{X}(t) = -\nu\, dt$, the solution of which is $\overline{X}(t) = \overline{X}_0 - \nu t$. Thus $\overline{X}$ is a time dependent function coefficient in the PDE. More precisely, we can obtain

$$\begin{aligned}\chi_2^n = \frac{1}{S_{\max}} \exp \Bigg[& -\frac{\rho\sigma_S\sigma_Q}{2\eta} - \frac{\sigma_S^2}{2\eta} + \overline{X}_0 - \nu\tau^n - \frac{\nu}{\eta} - \frac{\sigma_S\lambda}{\eta} \\ & + \exp(-\eta\Delta\tau)\Bigg(\log(SS_{\max}) + \frac{\sigma_S^2}{2\eta} \\ & + \frac{\rho\sigma_Q\sigma_S}{2\eta} - \overline{X}_0 + \nu\tau^{n+1} + \frac{\sigma_S\lambda}{\eta} + \frac{\nu}{\eta}\Bigg)\Bigg] \,.\end{aligned}$$

(c) $\overline{X}$ is a stochastic process: we assume that $\overline{X}_t$ satisfies the following linear stochastic differential equation

$$d\overline{X_t} = -\nu\, dt + \overline{\sigma} dB_t$$

where ν and ρ are constants and B_t is a Wiener process. Using stochastic calculus we obtain the following analytic expression for the process:

$$\overline{X}_t = \overline{X}_0 - \nu t + \overline{\sigma} B_t \,.$$

In this case, the PDE contains the random coefficient $\overline{X}_t$ with known distribution at each time t. So, we simulate χ_2 by using an explicit Euler method to discretize the second equation in (10) and the simulation of the involved process $B_t \in \mathcal{N}(0,t)$. More precisely, for $\tau^{n,k} \in [\tau^{n+1}, \tau^n]$, $k = 1, \ldots, K$, and each index ℓ associated to each simulation, we compute

$$\begin{aligned}(\chi_2^{n,k+1})_\ell = (\chi_2^{n,k})_\ell + (\tau^{n,k+1} - \tau^{n,k}) & \left[\frac{\sigma_S^2}{2} + \frac{\rho\sigma_S\sigma_Q}{2} + \sigma_S\lambda \right. \\ & \left. - \eta\left(\overline{X}_0 - \nu\tau^{n,k} + \overline{\sigma} B_\ell^{n,k} - \log\left((\chi_2^{n,k})_\ell\right) - \log(S_{\max})\right)\right] (\chi_2^{n,k})_\ell \,,\end{aligned}$$

with $(\chi_2^{n,0})_\ell = (\chi_2^{n-1,K})_\ell$. Therefore, for each value of ℓ and the corresponding path of B^ℓ, we have to solve a particular PDE to provide the value of V^ℓ. Next, in order to compare with the other cases, we compute the solution of this stochastic case as the expected value for all the obtained V^ℓ, that is

$$V = \frac{1}{L}\sum_{\ell=1}^{L} V^\ell \,.$$

Next, for $\ell = 1, \ldots, L$ and each time step $n = 0, 1, \ldots$, a variational formulation for (9) is posed. Moreover, for the spatial discretization of (9) we consider a triangular mesh of the computational domain $\Omega = (0, Q_{\max}) \times (S_{\min}, S_{\max})$ and the associated finite elements space of piecewise linear Lagrange polynomials. More precisely, we search $V_i^{n+1} \in V_h$ such that:

$$\begin{aligned}(1 + r\Delta\tau)\int_{\Omega} V_i^{n+1}\varphi \, dQ \, dS + \Delta\tau \int_{\Omega} A\nabla V_i^{n+1} \, \nabla\varphi \, dQ \, dS \\ = \Delta\tau \int_{\Omega} \mathcal{R} \, \varphi \, dQ \, dS + \int_{\Omega} (V_i^n \circ \chi^n) \, \varphi \, dQ \, dS \\ + \Delta\tau \int_{\partial\Omega} \big((A\nabla V_i^{n+1}) \cdot \mathbf{n}\big) \, \varphi \, d\Gamma \,, \qquad \forall \varphi \in V_h\end{aligned} \tag{11}$$

where the finite element space is:

$$V_h = \Big\{ \varphi : \Omega \to \mathbb{R} \, / \, \varphi \in \mathcal{C}(\Omega) \, , \; \varphi|_{T_k} \in \mathcal{P}_1, \, \forall T_k \in \mathcal{T}_h \Big\} \, ,$$

the parameter h being the mesh step. For simplicity, we have dropped the index h in all the functions appearing in (11). Thus, at each time step the system of linear equations corresponding to the fully discretized problem is solved by a partial pivoting LU factorization method.

4. Numerical examples

4.1. Example 1

In order to validate the proposed numerical methods, we have first compared with different examples in [2], where the one factor model is proposed and numerically solved by means of a finite volume method. Among these examples, we choose the following one that illustrates the good performance of the proposed algorithm in the one factor case. For this purpose, let us consider a line of $D = 550$ miles length, that can be provided with four different maximum capacities (as previously indicated, it can be equivalently understood as four different lines $L1$ to $L4$). Their respective maximum capacities and maintenance costs are indicated in Table 1. Moreover, the upgrading costs from each line to the other ones are shown in Table 2.

We consider a five years time horizon ($T = 5$), with continuous revenues, monthly maintenance and quarterly upgrade decisions. The parameters related to these issues are taken from [2] and summarized in Table 3.

	Maximum capacity (Mbps)	Maintenance costs ($/mile/year)
Line 1 (OC–12)	622	2.4
Line 2 (OC–48)	2488	18.0
Line 3 (OC–192)	9952	48.0
Line 4 (OC–768)	39808	96.0

Table 1: Maximum capacities and maintenance costs for different transmission rates

	L1	L2	L3	L4
L1	–	30 000	80 000	160 000
L2	–	–	80 000	160 000
L3	–	–	–	160 000
L4	–	–	–	–

Table 2: Upgrade costs ($)

Volatility	σ	0.95
Risk-free interest rate	r	0.05
Drift (growth) rate	μ	0.75
Market price of risk	κ	0.10
Current spot price	P	0.90
Decay rate	α	1.40

Table 3: Financial parameters for Example 1

Table 4 shows the percentages of maximum capacity of lines L1, L2 and L3 for which an upgrade to an upper line is convenient in the first two years. Each column shows an increasing percentage with time, which is associated to decreasing opportunities to amortize the line, so that the later the upgrade decision is taken the greater the required demand to obtain enough benefits to be worth to upgrade the line. Results are partly similar to those ones provided in [2]. Note that we only show the upgrade to the immediate upper line for the first two years. The jumping to a second upper line occurs for much larger upgrade percentages, unlike to what happens in [2] where Line 2 (OC-48) is upgraded to Line 4 (OC-768) for a lower percentage of the maximum transmission rate.

After testing the behavior with different numerical parameters, we show the numerical solution for 600 time steps and a uniform mesh with 1000 finite elements.

Figure 2 shows the value of each line at the present time ($t = 0$). Clearly, the value of the line increases with demand up to a certain demand level. The region with increasing behavior corresponds to demand values below the maximum capacity of the line ($\overline{Q}_i$). For larger values than $\overline{Q}_i$ the line value tends to a constant, which is also in agreement with the consideration of a Neumann homogeneous boundary condition at $\overline{Q}_{max}$ and with decreasing marginal benefits arguments.

	Line 1		Line 2		Line 3	
Time (τ)	Upgrades to line	Capacity (%)	Upgrades to line	Capacity (%)	Upgrades to line	Capacity (%)
0.00	2	105.4373	3	90.9655	4	72.8566
0.25	2	109.4955	3	92.8415	4	74.5364
0.50	2	117.8417	3	96.6508	4	76.2353
0.75	2	126.4941	3	100.5368	4	78.8195
1.00	2	135.4528	3	106.5093	4	81.4469
1.25	2	149.4652	3	112.6540	4	85.9214
1.50	2	169.2203	3	123.2782	4	90.5157
1.75	2	195.6371	3	134.3809	4	97.1486
2.00	2	235.8365	3	150.7286	4	105.0162

Table 4: Upgrade percentage for different transmission rates

4.2. Example 2

In this example, we consider a situation in which the value of the lines depends on the uncertain capacity demand and bandwidth price. As described in previous section, we pose three possibilities for the long term value $\overline{X}$: constant, time dependent in a deterministic way and stochastic. The complete set of parameters is indicated in Table 5, in particular the values of σ_Q, σ_S, ρ and η are taken from [7]. The values of ν and $\overline{\sigma}$ in Table 5 correspond to the stochastic case. For the time dependent case we use $\overline{\sigma} = 0$ and for the constant case we consider $\nu = \overline{\sigma} = 0$.

Figures 3–5 show the solution for different cases. In particular:

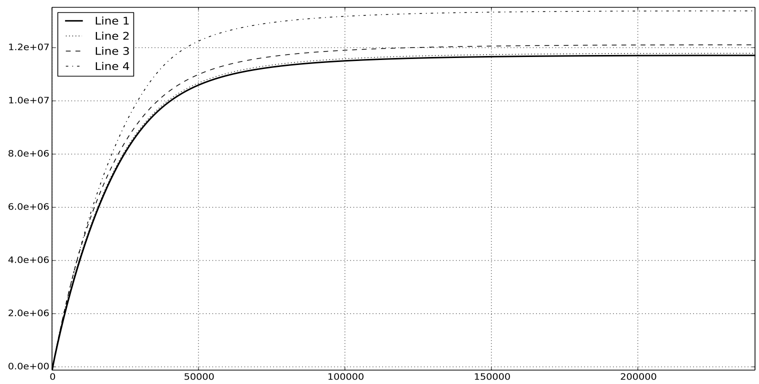

Figure 2: Numerical approximation of the values of the lines (Test 1)

Capacity volatility	σ_Q	0.25
Price volatility	σ_S	0.40
Risk-free interest rate	r	0.02
Decay rate	α	1.40
Correlation	ρ	0.20
Market price of risk	λ	0.10
Lower bound for price	S_{min}	0.5376
Upper bound for price	S_{max}	1.30
Initial X_t value	$\bar{X}_0$	$0.5(\log(S_{min}) + \log(S_{max}))$
Price reversion to trend	η	3
Average logarithmic price drift	$-\nu$	0.8
Average logarithmic price uncertainty	$\overline{\sigma}$	0.3

Table 5: Financial parameters for Example 2

- Figure 3 shows the solution in the constant case
- Figure 4 shows the solution in the deterministic time dependent case
- Figure 5 shows the solution in the stochastic case, obtained with 250 samples.

First, notice that the dependence of the line value on the capacity demand is analogous to the results obtained in the one factor model. Moreover, we can appreciate the influence of the price on the value of the lines, generally increasing their value. This effect cannot be analyzed in the one factor model. Secondly, concerning the influence of the long term price modelling we choose, the cases with time dependent and stochastic long term price are closer each other, the maximum difference being around 0.01%, while a much larger difference with respect to the constant case is observed, with a maximun around 10% at several mesh points. In order to illustrate this issue, for Line 4 in Figure 6 we show the differences in value between the time dependent and constant (left) cases, and between the time dependent and stochastic (right) ones. The behavior of the other lines is very similar to Line 4.

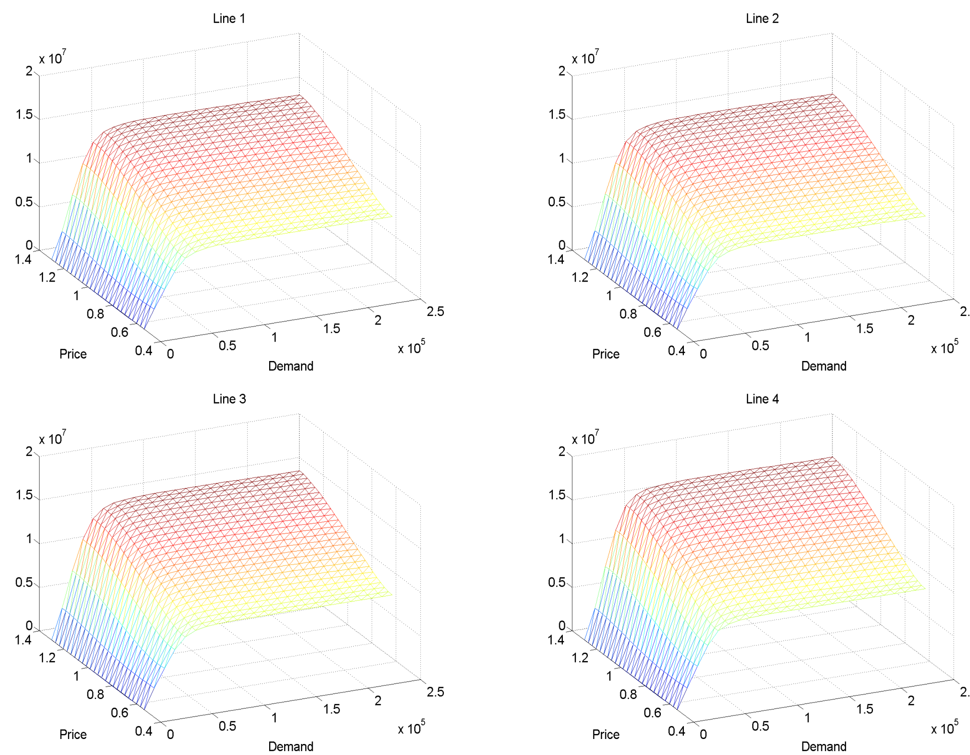

Figure 3: Numerical approximation of the values of the lines with constant $\overline{X}$

5. Conclusions

In the present paper we have presented a new model to price a telecommunication line under uncertainty in price and demand, thus more realistic in certain

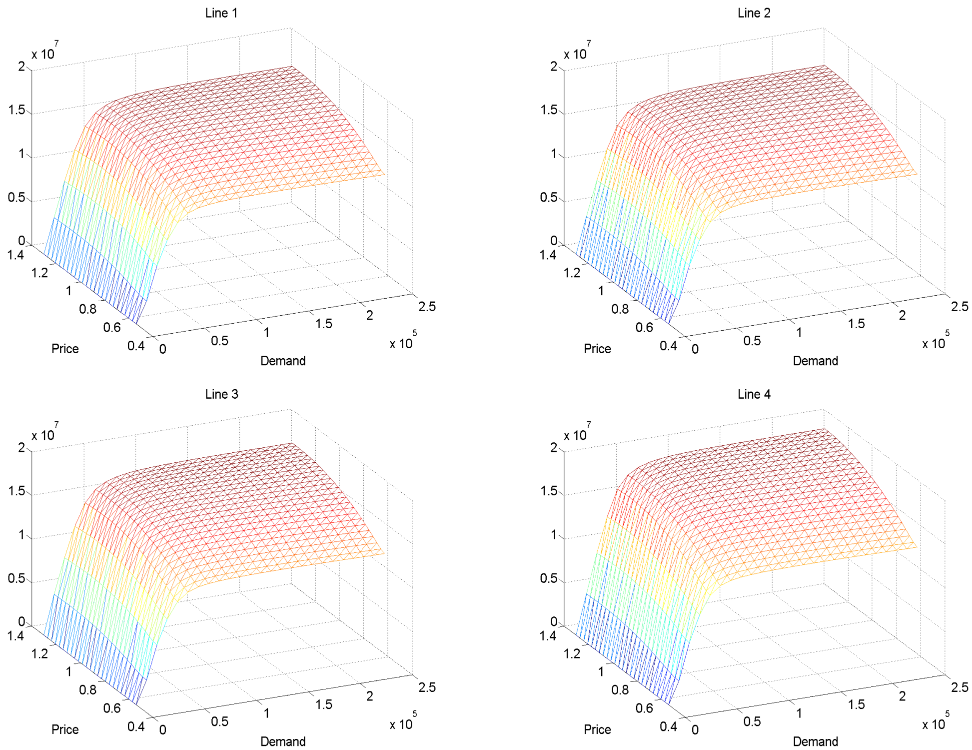

Figure 4: Numerical approximation of the values of the lines with deterministic time dependent $\overline{X}$

situations than the only consideration of uncertainty in the demand. Furthermore, we have proposed a set of suitable numerical methods to solve the resulting Black–Scholes type equation, that we have combined with a dynamic programming strategy to optimize the network management. Up to our knowledge, the joint consideration of two uncertain factors has not been previously addressed in the literature.

Clearly, the modelling can be extended to consider jumps in the demand and/or the prices, thus leading to partial integro-differential equations (PIDEs) instead of PDEs, so that additional numerical methods are required to treat the nonlocal terms. Also, as proposed in [7], a regime switching in the price process can be incorporated. Additionally, the calibration or estimation of the involved parameters requires a further study. In the one factor case, the estimation of parameters is carefully addressed in [2].

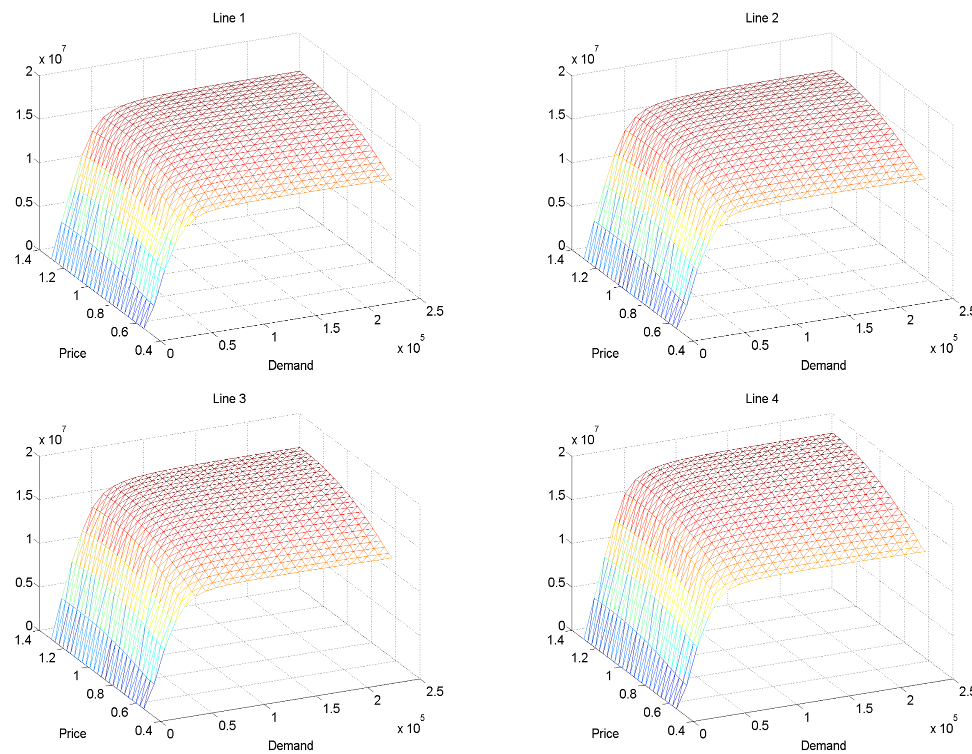

Figure 5: Numerical approximation of the values of the lines with stochastic $\overline{X}$

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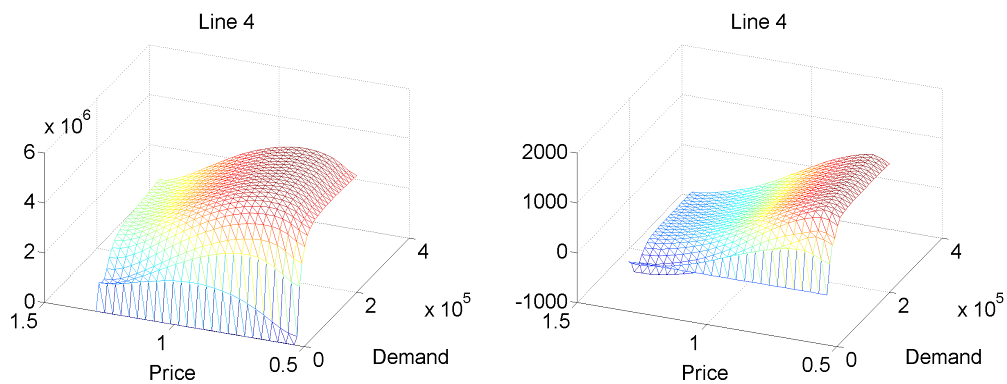

Figure 6: Differences in computed values of the Line 4 between the time dependent and constant cases (left) and between time dependent and stochastic cases (right)

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