PDE models and numerical methods for total value adjustment in European and American options with counterparty risk

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Abstract

Since the last financial crisis, a relevant effort in quantitative finance research concerns the consideration of counterparty risk in financial contracts, specially in the pricing of derivatives. As a consequence of this new ingredient, new models, mathematical tools and numerical methods are required. In the present paper, we mainly consider the problem formulation in terms of partial differential equations (PDEs) models to price the total credit value adjustment (XVA) to be added to the price of the derivative without counterparty risk. Thus, in the case of European options and forward contracts different linear and nonlinear PDEs arise. In the present paper we propose suitable boundary conditions and original numerical methods to solve these PDEs problems. Moreover, for the first time in the literature, we consider XVA associated to American options by the introduction of complementarity problems associated to PDEs, as well as numerical methods to be added in order to solve them. Finally, numerical examples are presented to illustrate the behaviour of the models and numerical method to recover the expected qualitative and quantitative properties of the XVA adjustments in different cases. Also, the first order convergence of the numerical method is illustrated when applied to particular cases in which the analytical expression for the XVA is available.

Keywords: option pricing, counterparty risk, credit value adjustments, (non)linear PDEs, characteristics method, finite elements, augmented Lagrangian active set method

1. Introduction

Since 2007 crisis, when important financial entities went bankrupt, the counterparty risk has become an important ingredient that needs to be taken into account in all financial contracts. It can be described as the risk to each party of a contract that the counterparty will not live up to its contractual obligations. Different institutions and financial analysts consider that the crisis was due to the mistakes made in the financial system, namely in the management of the risk. The complexity of the financial derivatives and the consideration of

a low probability of default were two of the factors that led to the crisis. As a consequence, a review of the counterparty risk consideration has been addressed.

From the point of view of the seller, the risk neutral value of a derivative can be currently adjusted by the following items:

- It is reduced by the existence of funding costs, in the case the latter takes part (FCA).

- It is increased in the case its value produces liquidity for the entity (FBA).

- It is reduced by the necessary costs to compensate the credit risk due to the counterparty (CVA).

- If a bilateral counterparty risk is assumed, the derivative value is increased by its potential benefits due to the issuer probability of default and the issuer has not to face its contractual responsibilities, when those are positive for the issuer (DVA).

- It is increased by the cost of borrowing the collateral (CollVA).

The FCA and the FBA can be merged and the sum of them is known as FVA (funding valued adjustment), understood as the correction to the risk-free price to account for the funding costs. The presence of FVA in the adjustment is reasonable in the case of non-collateralized trades; however when a collateral is posted to fully cover the counterparty risk then the FVA reduces to zero. In this sense, FVA is given by the difference of price between non-collateralized and fully collateralized contracts (see [25]). CVA represents the price to mitigate counterparty credit risk on a trade and the concept was first introduced in [27, 19, 13]. However, as no parts in the contract are risk-free, then DVA is the price of the hedging used to mitigate the own credit risk, so from the other counterparty is understood as a CVA. DVA was first introduced in [13] to account for the presence of two risky counterparties and the consideration of DVA allows to agree on the price by both traders (symmetric prices). However, a long controversy exists about the consideration of DVA and the same happens with FVA (see [16, 17, 21, 7] for different views on FVA).

Thus, including counterparty risk in the pricing of derivatives represents an important change in the existent risk–free pricing models. In particular, in this setting nonlinear partial differential equations (PDE) models can be posed, which have to be mathematically analysed and solved by means of suitable numerical methods. The main goal of the present paper concerns the computing of the European and American options price, accounting for all the associated cash flows that come from the derivative itself, the act of hedging, the default risk management and the funding costs. Following the usual terminology, we will refer to the total value of these adjustments as XVA, which in terms of the previously introduced notations is defined by:

$$\text{XVA} = \text{DVA} - \text{CVA} - \text{FCA} + \text{FBA} + \text{CollVA} = \text{DVA} - \text{CVA} + \text{FVA} + \text{CollVA} \ .$$

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So, we pose the PDE models for the derivative value, $\hat{V}$, from the point of view of the seller, when the trade takes place between two risky counterparties. More precisely, we focus on the case of European and American vanilla options. We use hedging arguments to derive the extensions to the Black–Scholes PDE in the presence of bilateral jump–to–default model and include funding considerations into the financing of the hedge positions.

Actually, nowadays there are three main methodologies to include funding costs, collateral and credit risk in the pricing of derivatives. A first approach, following the seminal papers by [25] and [4] that obtain PDE formulations by means of suitable hedging arguments and the use of Ito lemma for jump-diffusion processes. In [25] funding costs are introduced while in [4] both funding costs and bilateral counterparty credit risk are considered. This approach is also followed in [14] in the more general setting of stochastic spreads, in which three underlying stochastic factors are involved. Moreover, in [14] the solution is also equivalently written in terms of expectations. A second approach follows the initial ideas in [2] to include DVA by means of expectations, next extended to the collateralized, close-out and funding costs in [23]. A third approach is based on backward stochastic differential equations introduced in [10] and [11]. In all previous papers, the case of European derivatives is addressed.

In the present paper we follow the first approach in the line of [4] and propose original numerical methods for solving the PDE models. Thus, after recalling the hedging strategy proposed in this paper of the case of European-style derivatives, different kinds of PDEs arise depending on the assumption of the mark-to-market value at default. Thus, if this mark–to–market value is equal to the riskless derivative then a linear PDE that involves the value of the riskless derivative is obtained. However, if the mark–to–market value is given by the risky derivative, then a nonlinear PDE is obtained. In the linear case, the equivalent expression of the solution in terms of expectations can be solved. In the nonlinear case, this equivalent expression takes the form of a nonlinear integral equation and numerical methods are also required. In the present paper we propose a set of numerical techniques to solve the resulting PDEs for both choices of the mark–to–market at default. For this purpose, we truncate the unbounded asset domain and pose original suitable boundary conditions at the boundaries of the resulting bounded domain, following some ideas in [8] also taken from [12]. After truncation, we propose a time discretization based on the method of characteristics combined with a finite elements discretization in the asset variable. For the case leading to a nonlinear PDE a fixed point iteration algorithm is proposed.

Another original point is the consideration of American-style options. In this case, previously we have to solve numerically the associated obstacle problems as an additional nonlinearity. For this purpose we use an augmented lagrangian active set (ALAS) algorithm proposed in [20], already used in [1] and [6] for problems related to investment valuation and pension plans with early retirement opportunity, respectively.

The plan of the paper is the following. In Section 2, some one stochastic

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factor models in the literature to price European-style options in the presence of counterparty credit risk are described. More precisely, first counterparty credit risk and funding costs are considered, while in a second step the collateral is added to the previous model. In Section 3 we incorporate original models to price American-style options when XVA is considered. Section 4 is devoted to the description of different numerical methods that are proposed to solve the linear and nonlinear PDE models stated in Section 2. Particularly, the domain truncation to pose the PDE problem in a bounded domain requires the consideration of appropriate and original boundary conditions. In Section 5 we present and discuss the numerical results for different examples. Finally, some conclusions are indicated.

2. Mathematical models for pricing European-style options

2.1. Pricing with counterparty credit risk and funding costs

In this section, following [4] we model the derivative value by considering different adjustments on the value of the corresponding risk–free derivative, where risk-free derivative means a derivative without counterparty risk. In particular, bilateral default risk and funding costs are taken into account. More precisely, we consider the following assets associated to the trading [4]:

- Counterparty B zero recovery bond price, P_B, with yield r_{P_B}.

- Counterparty C zero recovery bond price, P_C, with yield r_{P_C}.

- Underlying asset with no default risk.

Due to the involved risks, the stock and the bond prices are modeled as stochastic processes satisfying the following stochastic differential equations (SDEs):

$$
\begin{aligned}
dP_{B_t} &= r_{P_B}(t)P_{B_t}dt - P_{B_t}dJ_t^B \\
dP_{C_t} &= r_{P_C}(t)P_{C_t}dt - P_{C_t}dJ_t^C \\
dS_t &= r_R(t)S_tdt + \sigma(t)S_tdW_t \, ,
\end{aligned}
\tag{1}
$$

where W_t is a Wiener process, and J_t^B and J_t^C are two independent jump processes that change from 0 to 1 on default of B and C, respectively.

Next, we consider a derivative trade where both counterparties, the seller B and the counterparty C, can default. From the point of view of the seller, the value of this derivative at time t is denoted by $\hat{V}_t = \hat{V}(t, S_t, J_t^B, J_t^C)$ and it depends on the spot value of the asset, S_t, and on the default states at time t, J_t^B and J_t^C, of the seller B and counterparty C, respectively. The value of the same derivative when the trade takes place between two default free counterparties is denoted by $V_t = V(t, S_t)$.

Since the trade takes place between defaultable counterparties, we need to incorporate some technical issues around close–outs. In this paper it is assumed

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that the close–out mark–to–market can only take two possible values, namely the value of the risk-free derivative or the one of the defaultable derivative. The value of the defaultable derivative, $\hat{V}(t, S_t, J_t^B, J_t^C)$, includes adjustments, such as CVA, DVA and FCA, into valuation whereas the value of the derivative without default risk, $V(t, S_t)$, does not include any counterparty adjustment. Moreover, we assume a setting such that the function $V(t, S_t)$ can be computed as the solution of a classical Black–Scholes model.

The conditions of the risky value upon default of the issuer or the counterparty are:

- if counterparty B defaults first,

$$\hat{V}(t, S_t, 1, 0) = M^+(t, S_t) + R_B M^-(t, S_t) \tag{2}$$

- if counterparty C defaults first,

$$\hat{V}(t, S_t, 0, 1) = R_C M^+(t, S_t) + M^-(t, S_t), \tag{3}$$

where $R_B \in [0, 1]$ and $R_C \in [0, 1]$ represent the recovery rates on the derivatives position of parties B and C, respectively, and M represents the close–out mark–to–market value.

In order to deduce the value of the credit risky derivative, we hedge the derivative with a self–financing portfolio Π which covers all underlying risk factors of the model. Thus, we have:

$$-\hat{V}_t = \Pi_t.$$

Let r denote the risk–free interest rate, r_F the funding rate from issuer, r_R the rate paid for the underlying asset in a repurchase agreement and $s_F = r_F - r$ the funding cost of the entity. Since P_B and P_C are zero recovery bonds, their spreads are equal to the default intensities λ_B and λ_C, respectively:

$$\lambda_B = r_{P_B} - r, \qquad \lambda_C = r_{P_C} - r. \tag{4}$$

Following [4], we can obtain the PDE that models the value of a European-style derivative including the counterparty risk:

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}\hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} + s_F M^+ \\ \qquad\qquad - \lambda_B(R_B M^- + M^+) - \lambda_C(R_C M^+ + M^-) \\ \hat{V}(T, S) = H(S), \end{cases} \tag{5}$$

where the differential operator $\mathcal{A}$ is given by

$$\mathcal{A}V \equiv \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r_R S \frac{\partial V}{\partial S}, \tag{6}$$

and M refers to the mark–to–market. Moreover $H(S)$ represents the pay–off of the derivative.

According to the two scenarios usually considered for the determination of the derivative mark–to–market value at default, M, two different PDE problems are obtained [4]:

- If $M = \hat{V}$,

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}\hat{V} - r\hat{V} = (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+ + s_F \hat{V}^+ \\ \hat{V}(T, S) = H(S) \,. \end{cases}$$

- If $M = V$,

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -(R_B\lambda_B + \lambda_C)V^- \\ \qquad\qquad\qquad\qquad - (R_C\lambda_C + \lambda_B)V^+ + s_F V^+ \\ \hat{V}(T, S) = H(S) \,. \end{cases}$$

European vanilla call and put options and forwards will be considered.

The derivative value with counterparty risk can be written as:

$$\hat{V} = V + U \,,$$

where U is the total value adjustment (XVA) and the counterparty risk–free value of the derivative, V, satisfies the classical linear Black–Scholes equation:

$$\begin{cases} \partial_t V + \mathcal{A}V - rV = 0 \,, \\ V(T, S) = H(S) \,. \end{cases} \tag{7}$$

Thus, the PDE problems satisfied by U are the following:

- If $M = \hat{V}$, we get a final value nonlinear problem:

$$\begin{cases} \partial_t U + \mathcal{A}U - rU = (1 - R_B)\lambda_B (V + U)^- \\ \qquad\qquad\qquad + (1 - R_C)\lambda_C (V + U)^+ + s_F (V + U)^+ \\ U(T, S) = 0 \,. \end{cases} \tag{8}$$

- If $M = V$, an analogous linear problem is deduced:

$$\begin{cases} \partial_t U + \mathcal{A}U - (r + \lambda_B + \lambda_C)U = (1 - R_B)\lambda_B V^- \\ \qquad\qquad\qquad + (1 - R_C)\lambda_C V^+ + s_F V^+ \\ U(T, S) = 0 \,. \end{cases}$$

In both cases, variable S lies in the unbounded domain $[0, +\infty)$ while $t \in [0, T]$.

2.2. Pricing with counterparty credit risk, funding costs and collateral

Many contracts include the collateralization of an asset. Collateral is a property or other assets that a borrower offers a lender to secure a loan. If the borrower stops making the promised loan payments, the lender can seize the collateral to fully or partly recover its losses.

In this section, mainly following [5], a credit risky collateralized derivative value is modelled in terms of PDEs, so a more generalized framework is studied. For this purpose, we assume an agreement between two risky counterparties B and C, where B is the issuer. As in the previous section, a self–financing hedging portfolio is used. The main difference with respect to the former setting is that in the present one the hedging portfolio hedges out the derivative when the counterparty does not default, whereas in the previous section the hedging portfolio perfectly hedges the derivative.

When the counterparty B defaults, the difference between the hedging portfolio and the short derivative value is known as hedge error.

In a similar way to the previous section, we want to deduce the PDE model for a collateralized derivative. Thus, we need to describe all the items taking part in this new setting. For this purpose, in [5] the authors consider the general case in which B has a portfolio made up of two bonds, P_1 and P_2, with different seniorities and different recoveries, R_1 and R_2, respectively. More precisely, for $R_2 > R_1$:

- P_1 is an issued junior bond with recovery $R_1 \geq 0$ and yield r_1

- P_2 is an issued senior bond with recovery $R_2 > 0$ and yield r_2.

Thus, we assume the price processes satisfy the following SDEs:

$$dS_t = r_R(t)S_t dt + \sigma(t)S_t dW_t \tag{9}$$

$$dP_{C_t} = r_{P_C}(t)P_{C_t} dt - P_{C_t} dJ_t^C \tag{10}$$

$$dP_{1_t} = r_1(t)P_{1_t} dt - (1 - R_1)P_{1_t} dJ_t^B \tag{11}$$

$$dP_{2_t} = r_2(t)P_{2_t} dt - (1 - R_2)P_{2_t} dJ_t^B . \tag{12}$$

The total position, at time t, in the B issued bond is given by

$$P_{B_t} = \alpha_1(t)P_{1_t} + \alpha_2(t)P_{2_t} \tag{13}$$

and the value of P_B in the issuer's default instant is defined as

$$P_{D_t} = \alpha_1(t)R_1 P_{1_t} + \alpha_2(t)R_2 P_{2_t} . \tag{14}$$

The conditions of the collateral derivative value upon default of both counterparties are:

- if B defaults first, then

$$\hat{V}(t, S_t, 1, 0) = g_B(M_t, X_t) = X_t + (M_t - X_t)^+ + R_B(M_t - X_t)^- \tag{15}$$

- if C defaults first, then

$$\hat{V}(t, S_t, 0, 1) = g_C(M_t, X_t) = X_t + (M_t - X_t)^- + R_C(M_t - X_t)^+ , \quad (16)$$

where X_t represents the collateral and M_t is the mark–to–market value. These conditions represent an extension of the ones given in (2–3), which are clearly recovered for $X_t = 0$.

The hedging portfolio built up in this model only hedges out the derivative when the counterparty B does not default, so that, in this case

$$\Pi_t + \hat{V}_t = 0 . \quad (17)$$

When the counterparty B does not default, we have a perfectly hedged portfolio, so that the following funding constraint is obtained:

$$\hat{V}_t + P_{B_t} - X_t = 0 . \quad (18)$$

We can interpret this equation in the following way: if $\hat{V}_t - X_t < 0$, then B bonds are used to fund the difference between the derivative value and the collateral. Conversely, if that difference is positive then they are used to repurchase B issued bonds. Finally, if the risky value is fully hedged by the collateral then the bond position will be reduced to zero. If the collateral is zero, the trade will be financed by B's bonds.

Next, let us consider the case when the counterparty B defaults. In this situation the derivative value is the solution of the final value problem:

$$\begin{cases} \dfrac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = \lambda_B h_e - \lambda_B g_B(M, X) - \lambda_C g_C(M, X) + s_X X \\ \hat{V}(T, S) = H(S) . \end{cases}$$
$$(19)$$

If we compare (19) with the PDE problem (5) obtained in the case without collateral, the two additional terms $\lambda_B h_e$ and $s_X X$ appear. Furthermore, the terms g_B and g_C are now more general.

In addition, in case of counterparty B default a hedge error arises. Nevertheless, while the issuer B is alive, B will incur a cost or gain of size $\lambda_B h_e$ per unit time. In [3] it is proved that this gain is equal to the hedge error, h_e.

As in the case without collateral of previous section, our goal is the computation of the total value adjustment. For this purpose, we write the risky value as the sum of risk–free value, V, and the total value adjustment, U. Depending on the mark–to–market value, we obtain two different equations:

- If $M = \hat{V}$, we get a final value problem governed by a nonlinear PDE:

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - rU = \lambda_B h_e + \lambda_B(1 - R_B)(V + U - X)^- \\ \qquad\qquad\qquad + \lambda_C(1 - R_C)(V + U - X)^+ + s_X X \\ U(T, S) = 0 . \end{cases} \quad (20)$$

- If $M = V$, an analogous linear problem is deduced:

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - (r + \lambda_B + \lambda_C)U = \lambda_B h_e + \lambda_B(1 - R_B)(V - X)^- \\ \qquad\qquad\qquad\qquad\qquad\qquad + \lambda_C(1 - R_C)(V - X)^+ + s_X X \quad (21) \\ U(T, S) = 0 \,. \end{cases}$$

Finally, different assumptions are made on counterparty B bond. As a result, three particular different models can be proposed. Note that the linear versions corresponding to (20) have been proposed in [3].

Collateral model 1: Perfect hedging

If all risks are perfectly hedged, then h_e is reduced to zero; thus we get:

$$h_e = g_B(M_t, X_t) + P_{D_t} - X_t = g_B(M_t, X_t) + \alpha_1(t)R_1 P_{1_t} + \alpha_2(t)R_2 P_{2_t} - X_t = 0 \,. \quad (22)$$

In this case the PDE which models the risky derivative value is reduced to

$$\begin{cases} \dfrac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -\lambda_C g_C(M, X) - \lambda_B g_B(M, X) + s_X X \\ \hat{V}(T, S) = H(S) \,, \end{cases}$$

and the PDEs for the total value adjustment, U, are:

- If $M = \hat{V}$,

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - rU = \lambda_B(1 - R_B)(V + U - X)^- \\ \qquad\qquad\qquad\qquad + \lambda_C(1 - R_C)(V + U - X)^+ + s_X X \\ U(T, S) = 0 \,. \end{cases}$$

- If $M = V$,

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - (r + \lambda_B + \lambda_C)U = \lambda_B(1 - R_B)(V - X)^- \\ \qquad\qquad\qquad\qquad + \lambda_C(1 - R_C)(V - X)^+ + s_X X \\ U(T, S) = 0 \,. \end{cases}$$

Notice that Funding Cost Adjustment vanishes because hedge error is null, so that only CVA, DVA and CollVA are taken into account in the XVA.

Collateral model 2: Two bonds model

In this model, we assume that counterparty B has two bonds. More precisely, a zero recovery bond P_1 and a bond P_2 with recovery R_2. This recovery is equivalent to the recovery rate of counterparty B on a derivative trade, so $R_2 = R_B$. Under this assumption, the corresponding PDE is deduced.

Assuming the funding constraint (18), we write:

$$P_{B_t} = \alpha_1(t)P_{1_t} + \alpha_2(t)P_{2_t} = -(\hat{V}_t - X_t). \tag{23}$$

Now, taking into account this assumption, the general PDE (19) turns into:

$$\begin{cases} \dfrac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = \lambda_B(1 - R_B)(M - X)^+ \\ \qquad\qquad\qquad\qquad\qquad\qquad -\lambda_B g_B(M,X) - \lambda_C g_C(M,X) + s_X X \\ \hat{V}(T,S) = H(S), \end{cases}$$

and the PDE models satisfied by XVA are given by:

- If $M = \hat{V}$,

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - (r + \lambda_B(1 - R_B))U = \lambda_B(1 - R_B)(V - X) \\ \qquad\qquad\qquad\qquad\qquad +\lambda_C(1 - R_C)(V + U - X)^+ + s_X X \\ U(T,S) = 0. \end{cases}$$

- If $M = V$,

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - (r + \lambda_B + \lambda_C)U = \lambda_B(1 - R_B)(V - X) \\ \qquad\qquad\qquad\qquad\qquad +\lambda_C(1 - R_C)(V - X)^+ + s_X X \\ U(T,S) = 0. \end{cases}$$

Collateral model 3: One bond model

Finally, only one bond from B, with recovery rate R_B, is considered so that taking $\alpha_1(t) = 0$ in (23) we set $P_{B_t} = \alpha_2(t)P_{2_t}$.

Considering this assumption, the following PDE modelling the risky value is obtained:

$$\begin{cases} \dfrac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} - (r + \lambda_B(1 - R_B) + \lambda_C)\hat{V} = \lambda_B(R_B - 1)X - \lambda_C g_C(M,X) + s_X X \\ \hat{V}(T,S) = H(S), \end{cases}$$

and the PDEs for the XVA are:

- If $M = \hat{V}$,

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U - (r + \lambda_B(1 - R_B))U = \lambda_B(1 - R_B)(V - X) \\ \qquad\qquad\qquad\qquad\qquad +\lambda_C(1 - R_C)(V + U - X)^+ + s_X X \\ U(T,S) = 0. \end{cases} \tag{24}$$

- If $M = V$,

$$
\begin{cases}
\dfrac{\partial U}{\partial t} + \mathcal{A}U - (r + \lambda_B(1 - R_B) + \lambda_C)U = \lambda_B(1 - R_B)(V - X) \\
\qquad\qquad\qquad\qquad + \lambda_C(1 - R_C)(V - X)^+ + s_X X \\
U(T, S) = 0\,.
\end{cases}
\tag{25}
$$

We can observe that in the linear problem (25), when $M = V$, if a fully collateralized derivative is considered then only CollVA exists in the adjustment upon risk–neutral value, i.e. CVA, DVA and FCA vanish.

If we analyze the current situation, in which only funding desk can issue bonds in the bank, model 3 results the most realistic one because the trader can not issue bonds in order to raise cash for trade, so that only one bond from B has to be considered.

3. XVA pricing models for American-style options

In this section, we take into account the XVA in the pricing of American options with couterparty risk. So, we will deduce the PDE problem which models the derivative value. Similar techniques as in European options are employed: self–financing portfolios and absence of arbitrage. Without loss of generality, we only deduce the case without collateral.

The same asset trading as in European options is considered, that is: two bonds of counterparties B and C, and the underlying asset with no default risk, the processes of which will be modeled by (1).

Thus, we consider a derivative trade between two default counterparties, the issuer B and the buyer C. From the point of view of the seller the risky derivative value, at time t, is denoted by $\hat{V}(t, S_t, J_t^B, J_t^C)$, where J^B and J^C are the same jump processes defined in the case with European options. The counterparty risk–free American option price is denoted by $V(t, S_t)$, which can be computed using the Black–Scholes complementarity problem for American options (see [29, 30], for example).

Conditions of the defaultable American option price upon the default of different counterparties are given by (2–3). In order to derive the value of the American option with counterparty risk, at time t, we consider the self–financing portfolio Π_t, used in the European option case, which consists of

- $\Delta(t)$ units of the underlying asset S_t,

- $\alpha_B(t)$ units of P_{B_t}, a default risky, zero–recovery, zero–coupon bond of party B

- $\alpha_C(t)$ units of P_{C_t}, an analogous bond for the counterparty C

- $\gamma(t)$, which is made up of a financing amount, the cash needed to buy a position in C's bonds and a repo amount, such that the portfolio value at time t hedges out the value of the derivative contract to the seller. Furthermore, the following issues need to be pointed out:

1. The cost of the portfolio is denoted by γ_P, whereas the amount which is necessary to buy a position in B's bonds or the cash obtained from selling B's bonds is denoted by γ_{P_B}. Thus, the funding account, denoted by γ_F is defined as the difference between the cost of the hedging portfolio and the price of the position in counterparty B's bonds, so $\gamma_F = \gamma_P - \gamma_{P_B}$. The positive amounts in the funding account will be invested at the risk–free rate, r, while lending cash will be done at an unsecured funding rate, r_F.
2. The cash needed to buy a position in C's bonds, or the cash received from selling C's bonds is denoted by γ_{P_C}. This bonds position is used to hedge out the counterparty risk of C. The bonds are placed in a repo agreement, assuming that the repo rate to compute the financing costs is equal to the risk–free rate, r, for the bond (as in [4]).
3. The repo account contains the amount of cash invested or borrowed in order to fund the stock position $\Delta(t)S_t$ through a repurchase agreement, this account is denoted by γ_R.
4. Although γ_P, γ_{P_B} and γ_F depend on t, for simplicity we do not explicit this dependence in the forthcoming expressions.

Thus, the portfolio is given by

$$\Pi_t = \Delta(t)S_t + \alpha_B(t)P_{B_t} + \alpha_C(t)P_{C_t} + \gamma_t \,. \tag{26}$$

As the portfolio is self–financing, its change is given by

$$d\Pi_t = \Delta(t)dS_t + \alpha_B(t)dP_{B_t} + \alpha_C(t)dP_{C_t} + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R)(t)dt \,. \tag{27}$$

In addition, to avoid arbitrage opportunities we introduce the hedging equation:

$$d\Pi_t + d\hat{V}_t \leq 0 \,. \tag{28}$$

The change in the derivative value is obtained by applying Ito's lemma for jump diffusions, so this change is given by:

$$
\begin{aligned}
d\hat{V}_t &= \frac{\partial \hat{V}}{\partial t}dt + \frac{\partial \hat{V}}{\partial S}dS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \hat{V}}{\partial S^2}dt + \Delta\hat{V}_{B_t}dJ_t^B + \Delta\hat{V}_{C_t}dJ_t^C \\
&= \left(\frac{\partial \hat{V}}{\partial t} + r_R\frac{\partial \hat{V}}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \hat{V}}{\partial S^2} \right)dt + \sigma S_t \frac{\partial \hat{V}}{\partial S}dW_t + \Delta\hat{V}_{B_t}dJ_t^B + \Delta\hat{V}_{C_t}dJ_t^C \,.
\end{aligned}
\tag{29}
$$

By replacing the change of the portfolio and the change of the derivative value in (28), we obtain

$$
\begin{aligned}
&\Delta(t)dS_t + \alpha_B(t)dP_{B_t} + \alpha_C(t)dP_{C_t} + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R)dt \\
&\leq -\left(\frac{\partial \hat{V}}{\partial t}dt + \frac{\partial \hat{V}}{\partial S}dS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \hat{V}}{\partial S^2}dt + \Delta\hat{V}_{B_t}dJ_t^B + \Delta\hat{V}_{C_t}dJ_t^C \right),
\end{aligned}
$$

where $\hat{V}$ and all partial derivatives of $\hat{V}$ are evaluated at the point (t, S_t, J_t^B, J_t^C). Moreover, we use the notation

$$\Delta \hat{V}_{B_t} = \hat{V}(t, S_t, 1, 0) - \hat{V}(t, S_t, 0, 0),$$
$$\Delta \hat{V}_{C_t} = \hat{V}(t, S_t, 0, 1) - \hat{V}(t, S_t, 0, 0),$$

which can be computed using the boundary conditions (2) and (3).

Keeping in mind expressions (27) and (29) we deduce the following equation:

$$\Delta(t)dS_t + \alpha_B(t)dP_{B_t} + \alpha_C(t)dP_{C_t} + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R)dt$$
$$\leq -\left(\frac{\partial \hat{V}}{\partial t}dt + \frac{\partial \hat{V}}{\partial S}dS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \hat{V}}{\partial S^2}dt + \Delta \hat{V}_{B_t}dJ_t^B + \Delta \hat{V}_{C_t}dJ_t^C \right). \tag{30}$$

According to the SDEs in (1) we obtain:

$$\Delta(t)dS_t + \alpha_B(t)(r_{P_B}P_{B_t}dt - P_{B_t}dJ_t^B) + \alpha_C(t)(r_{P_C}P_{C_t}dt - P_{C_t}dJ_t^C)$$
$$+ (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R)dt$$
$$\leq -\left(\frac{\partial \hat{V}}{\partial t}dt + \frac{\partial \hat{V}}{\partial S}dS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 \hat{V}}{\partial S^2}dt + \Delta \hat{V}_B dJ_t^B + \Delta \hat{V}_C dJ_t^C \right). \tag{31}$$

Moreover, we choose the following weights:

$$\Delta(t) = -\frac{\partial \hat{V}}{\partial S},$$
$$\alpha_B(t) = \frac{\Delta \hat{V}_{B_t}}{P_{B_t}} = -\frac{\hat{V}_t - (M_t^+ + R_B M_t^-)}{P_{B_t}}, \tag{32}$$
$$\alpha_C(t) = \frac{\Delta \hat{V}_{C_t}}{P_{C_t}} = -\frac{\hat{V}_t - (M_t^- + R_C M_t^+)}{P_{C_t}}$$

in order to remove all risks in the portfolio Π_t. Thus, equation (31) leads to

$$\alpha_B r_{P_B}P_B + \alpha_C r_{P_C}P_C + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R)+$$
$$+ \left(\frac{\partial \hat{V}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} \right) \leq 0. \tag{33}$$

In order to obtain the PDE that models the derivative value, we simplify the following terms

$$\alpha_B r_{P_B}P_B + \alpha_C r_{P_C}P_C + r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R.$$

For this purpose, we consider the equivalences: $\gamma_{P_B} = \alpha_B P_{B_t}$, $\gamma_{P_C} = \alpha_C P_{C_t}$,

$r_F = r + s_F$ and $\gamma_F = \gamma_P - \gamma_{P_B}$, so that

$$\alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R$$
$$= \alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r(\gamma_P - \gamma_{P_B})^+ + r_F(\gamma_P - \gamma_{P_B})^-$$
$$- r\alpha_C P_C - r_R\gamma_R$$
$$= \alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r(\gamma_P - \alpha_B P_B) + s_F(\gamma_P - \alpha_B P_B)^-$$
$$- r\alpha_C P_C - r_R\gamma_R \,.$$

According to the REPO account, we have $\gamma_R = \Delta S$, so that the previous identity becomes:

$$\alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R$$
$$= r\gamma_P + s_F\gamma_F^- - r_R\Delta S + (r_{P_C} - r)\alpha_C P_C + (r_{P_B} - r)\alpha_B P_B \,.$$

In order to avoid arbitrage opportunities, the hedging portfolio value has to be equal to the derivative value, so that $\gamma_P = -\hat{V}$. Moreover, by considering the expressions in (4) then the previous equation can be further reduced to

$$\alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R$$
$$= -r\hat{V} + s_F\gamma_F^- - r_R\Delta S + \lambda_C\alpha_C P_C + \lambda_B\alpha_B P_B \,.$$

Finally, considering the addends in which $\alpha_B P_{B_t}$ and $\alpha_C P_{C_t}$ take place and expressing them in terms of the mark–to–market value we get

$$\alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R$$
$$= -(r + \lambda_B + \lambda_C)\hat{V} + s_F\gamma_F^- - r_R\Delta S$$
$$+ \lambda_B(M^+ + R_B M^-) + \lambda_C(M^- + R_C M^+) \,.$$

Thus, we introduce the previous expression in (33) to obtain the inequality that models the value of the derivative including the counterparty risk:

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}\hat{V} - r\hat{V} \leq (\lambda_B + \lambda_C)\hat{V} + s_F M^+$$
$$- \lambda_B(M^+ + R_B M^-) - \lambda_C(M^- + R_C M^+), \qquad (34)$$

so the PDE problem which models the American options price in the presence of counterparty risk is the following:

$$\begin{cases} \mathcal{L}_t(\hat{V}) = \partial_t\hat{V} + \mathcal{A}\hat{V} - r\hat{V} - (\lambda_B + \lambda_C)\hat{V} \\ \qquad -s_F M^+ + \lambda_B(R_B M^- + M^+) + \lambda_C(R_C M^+ + M^-) \leq 0 \\ \hat{V}(t, S) \geq H(S) \\ \mathcal{L}_t(\hat{V})\,(\hat{V} - H) = 0 \\ \hat{V}(T, S) = H(S) \end{cases} \qquad (35)$$

where the differential operator $\mathcal{A}$ is defined in (6).

According to the choice of the mark–to–market value, two different obstacle problems are obtained:

- If $M = \hat{V}$, we obtain the following complementarity problem associated to a nonlinear partial differential equation:

$$\begin{cases} \mathcal{L}_1(\hat{V}) = \partial_t \hat{V} + \mathcal{A}\hat{V} - r\hat{V} \\ \qquad -(1-R_B)\lambda_B \hat{V}^- - (1-R_C)\lambda_C \hat{V}^+ - s_F \hat{V}^+ \leq 0 \\ \hat{V}(t,S) \geq H(S) \\ \mathcal{L}_1(\hat{V})\,(\hat{V} - H) = 0 \\ \hat{V}(T,S) = H(S)\,. \end{cases} \tag{36}$$

- If $M = V$, we obtain the following complementarity problem associated to the linear partial differential equation:

$$\begin{cases} \mathcal{L}_2(\hat{V}) = \partial_t \hat{V} + \mathcal{A}\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} \\ \qquad +(R_B\lambda_B - \lambda_C)V^- + (R_C\lambda_C + \lambda_B)V^+ - s_F V^+ \leq 0 \\ \hat{V}(t,S) \geq H(S) \\ \mathcal{L}_2(\hat{V})\,(\hat{V} - H) = 0 \\ \hat{V}(T,S) = H(S)\,. \end{cases} \tag{37}$$

Thus, to compute the price of an American option including counterparty risk, either a nonlinear or a linear complementarity problem has to be solved.

In order to compute the XVA value, previously the Black–Scholes equation for American options without counterparty risk has to be solved. More precisely, the American option price without counterparty risk, V, is solution of the classical problem:

$$\begin{cases} \widetilde{\mathcal{L}}(V) = \partial_t V + \mathcal{A}V - rV \leq 0 \\ V(t,S) \geq H(S) \\ \widetilde{\mathcal{L}}(V)\,(V - H) = 0 \\ V(T,S) = H(S)\,. \end{cases} \tag{38}$$

Finally, the XVA value is obtained after solving the two obstacle problems and is given by $U = \hat{V} - V$.

Remark. *In the case of a collateralized contract, in order to obtain the corresponding obstacle problems the following portfolio has to be considered:*

$$\Pi_t = \Delta(t)S_t + P_{B_t} + \alpha_C(t)P_{C_t} + \gamma(t) - X_t\,,$$

where X_t denotes the amount of collateral, and in this case $\gamma(t)$ consists on an amount of stock position in a repurchase agreement, $\gamma_R(t)$, and the cash amount necessary to purchase $\alpha_C(t)$ bonds of C, γ_{P_C}.

4. Numerical methods

In order to solve the previous models, in this section different numerical methods are proposed. We will mainly focus on nonlinear problems, similar

methods being used in the corresponding linear ones. Moreover, we only develop the problem with collateral, as we can consider the model without collateral as a particular case.

We have developed an approach based on finite elements for spatial discretization. As the initial domain of the problem is unbounded in variable S, a localization procedure to define a suitable bounded domain is required and adequate boundary conditions are deduced and implemented.

We first propose a set of numerical methods to solve PDE problem (20), the solution of which is the adjustment value considering CVA, DVA, FCA and CollVA. Finally, we introduce the Augmented Lagrangian Active Set (ALAS) for obstacle problems in order to solve (36) concerning the case of American options.

In all cases, the change of time variable $\tau = T - t$ is considered in order to write (20) forward in time, so the following initial value problem is obtained:

$$
\begin{cases}
\dfrac{\partial U}{\partial \tau} - \dfrac{\sigma^2}{2}S^2\dfrac{\partial^2 U}{\partial S^2} - r_R S\dfrac{\partial U}{\partial S} + rU = -\lambda_B h_e - (1 - R_B)\lambda_B(V + U - X)^- \\
\qquad\qquad\qquad -(1 - R_C)\lambda_C(V + U - X)^+ - s_X X \\
U(0, S) = 0 \, .
\end{cases}
$$
(39)

Moreover, as we propose to solve (39) by a finite elements method, we write it in divergencial form:

$$
\dfrac{\partial U}{\partial \tau} - \dfrac{\partial}{\partial S}\left(\dfrac{\sigma^2}{2}S^2\dfrac{\partial U}{\partial S}\right) + (\sigma^2 - r_R)S\dfrac{\partial U}{\partial S} + rU = -\lambda_B h_e
$$
$$
- (1 - R_B)\lambda_B(V + U - X)^- - (1 - R_C)\lambda_C(V + U - X)^+ - s_X X \, . \quad (40)
$$

4.1. Characteristics method

Analogously to other advection–diffusion equations, we propose a semi–Lagrangian discretization combined with finite elements. More precisely, for time discretization we use a characteristics method first proposed in financial setting in [28]. For this purpose, we consider the material derivative of function U:

$$
\dfrac{DU}{D\tau} = \dfrac{\partial U}{\partial \tau} + \dfrac{\partial U}{\partial S}\dfrac{dS}{d\tau}
$$

for a given function $S = S(\tau)$. Thus, we can write equation (40) as:

$$
\dfrac{DU}{D\tau} - \dfrac{\sigma^2}{2}\dfrac{\partial}{\partial S}\left(S^2\dfrac{\partial U}{\partial S}\right) + rU = -\lambda_B h_e - (1 - R_B)\lambda_B(V + U - X)^-
$$
$$
- (1 - R_C)\lambda_C(V + U - X)^+ - s_X X \, . \quad (41)
$$

Taking into account that the coefficient of the advective term in (40) is $(\sigma^2 - r_R)S$, hereafter referred as the velocity, we introduce $\Delta\tau > 0$ and $\tau^n = n\Delta\tau$ for $n = 0, 1, 2, \ldots$ and the final value ODE problem:

$$
\begin{cases}
\dfrac{\partial \chi}{\partial \tau} = (\sigma^2 - r_R)\chi(\tau) \\
\chi(\tau^{n+1}) = S \, ,
\end{cases}
$$

the analytical solution of which is:

$$\chi(S, \tau^{n+1}; \tau^n) = S \exp\left((r_R - \sigma^2)\Delta\tau\right).$$

Note that function χ represents the characteristic curve associated to the velocity passing through point S at time τ^{n+1}.

We approximate the material derivative in (41) by a first order quotient, so that equation (41) is approximated by:

$$\frac{U^{n+1} - U^n \circ \chi^n}{\Delta\tau} - \frac{\sigma^2}{2} \frac{\partial}{\partial S}\left(S^2 \frac{\partial U^{n+1}}{\partial S}\right) + rU^{n+1}$$

$$= -\lambda_B h_e - (1 - R_B)\lambda_B (V + U^{n+1} - X)^-$$

$$- (1 - R_C)\lambda_C (V + U^{n+1} - X)^+ - s_X X. \qquad (42)$$

We can evaluate $U^n \circ \chi^n$ at each step of (42) in the mesh points by interpolation.

4.2. Fixed point scheme

In order to solve the nonlinear equation (42) at each iteration of the characteristics method, we propose a fixed point algorithm. Thus, the global scheme can be written in the following way:

Let $N > 1$, $\varepsilon > 0$, U^0 given.

For $n = 0, 1, 2, \ldots$

Let $U^{n+1,0} = U^n$

For $k = 0, 1, 2, \ldots$, we compute $U^{n+1,k+1}$ satisfying:

$$(1 + r\Delta\tau)\, U^{n+1,k+1} - \frac{\sigma^2 \Delta\tau}{2} \frac{\partial}{\partial S}\left(S^2 \frac{\partial U^{n+1,k+1}}{\partial S}\right)$$

$$= U^n \circ \chi^n - \Delta\tau \left[\lambda_B h_e + (1 - R_B)\lambda_B (V^{n+1} + U^{n+1,k} - X)^-\right.$$

$$\left. + (1 - R_C)\lambda_C (V^{n+1} + U^{n+1,k} - X)^+ + s_X X\right] \qquad (43)$$

until $\|U^{n+1,k+1} - U^{n+1,k}\| \leq \varepsilon$.

4.3. Boundary conditions

As previously indicated, we will use a finite elements method to discretize the previous equations and approximate the solution. Thus, we need to truncate the unbounded domain $[0, +\infty)$ into a bounded one, so that the solution is not affected by the truncation in the region of financial interest. We will assume $S \in [0, S_\infty]$, where $S_\infty > 0$ is a large enough value; a typical choice in financial problems is $S_\infty = 4E$ where E represents the strike of the option.

Next, we deduce the boundary conditions from the partial differential equation. More precisely, let us introduce function f, defined by:

$$f(U, V) = \lambda_B h_e + (1 - R_B)\lambda_B (V + U - X)^- + (1 - R_C)\lambda_C (V + U - X)^+ + s_X X,$$

representing the right hand side of (43).

The boundary condition at $S = 0$ is obtained just by replacing $S = 0$ in (39). Thus, we obtain the nonlinear ODE:

$$\partial_\tau U + rU = -f(U, V).$$

This equation is discretized by a characteristics (in this case, equivalent to an implicit Euler) method combined with a fixed point scheme:

$$U^{n+1,k+1}(0) - U^n(0) + \Delta\tau\, r\, U^{n+1,k+1}(0) = -\Delta\tau\, f(U^{n+1,k}(0), V^{n+1}(0)),$$

for $k \geq 0$ and $n \geq 0$, so that a nonhomogeneous Dirichlet boundary condition is obtained at each step of the global algorithm:

$$
\begin{aligned}
U^{n+1,k+1}(0) = \frac{1}{1 + r\Delta\tau}\, \big(U^n(0) - \Delta\tau\, [\lambda_B h_e \\
+ (1 - R_B)\lambda_B(V^{n+1}(0) + U^{n+1,k}(0) - X)^- \\
+ (1 - R_C)\lambda_C(V^{n+1}(0) + U^{n+1,k}(0) - X)^+ + s_X X]\big). \quad (44)
\end{aligned}
$$

In order to deduce the boundary condition at $S = S_\infty$, we first multiply equation (39) by S^{-2}. Next, by taking the limit when S tends to infinity the following property is obtained:

$$\lim_{S \to \infty} \frac{\partial^2 U}{\partial S^2} = 0. \quad (45)$$

Then, following [8], when $S \to \infty$ we consider a solution of the form:

$$U = H_0(\tau) + H_1(\tau)S, \quad (46)$$

where $H_0(\tau)$ and $H_1(\tau)$ are constant coefficients with respect to variable S. Next, by assuming $S^2 \dfrac{\partial^2 U}{\partial S^2} \to 0$ when $S \to \infty$ in (39) we have

$$\frac{\partial U}{\partial \tau} - r_R S \frac{\partial U}{\partial S} + rU = -f(U, V), \quad (47)$$

when $S \to \infty$.

Discretizing (47) by the characteristic curve, we have:

$$(1 + r\Delta\tau)\, U^{n+1,k+1} = U^n \circ \chi^n - \Delta\tau\, f(U^{n+1,k}, V^{n+1}) \quad (48)$$

where $\chi^n \equiv \chi(S, \tau^{n+1}; \tau^n)$ is solution of the final value problem

$$
\begin{cases}
\dfrac{d\chi}{d\tau} = -r_R \chi(\tau) \\
\chi(\tau^{n+1}) = S.
\end{cases}
\quad (49)
$$

Thus, the characteristic curve is given by $\chi(S, \tau^{n+1}; \tau^n) = S \exp(r_R \Delta\tau)$.

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Introducing the expression (46) into each fixed point iteration (48), we obtain:

$$(1 + r\Delta\tau)\left(H_0^{n+1,k+1} + H_1^{n+1,k+1} S_\infty\right)$$
$$= U^n \circ \chi^n - \Delta\tau\left[\lambda_B h_e + (1 - R_B)\lambda_B(V^{n+1} + U^{n+1,k} - X)^-\right.$$
$$\left. + (1 - R_C)\lambda_C(V^{n+1} + U^{n+1,k} - X)^+ + s_X X\right]. \tag{50}$$

If we choose $H_0^{n+1,k+1} = 0$, a nonhomogeneous Dirichlet boundary condition is deduced:

$$U^{n+1,k+1}(S_\infty) = H_1^{n+1,k+1} S_\infty$$
$$= \frac{1}{(1 + r\Delta\tau)}\left((U^n \circ \chi^n)(S_\infty)\right.$$
$$- \Delta\tau\left[\lambda_B h_e + (1 - R_B)\lambda_B(V^{n+1}(S_\infty) + U^{n+1,k}(S_\infty) - X)^-\right.$$
$$\left.\left. + (1 - R_C)\lambda_C(V^{n+1}(S_\infty) + U^{n+1,k}(S_\infty) - X)^+ + s_X X\right]\right). \tag{51}$$

Thus, (44) and (51) are evaluated at each iteration of the fixed point algorithm as a previous step to the stating of the linear system of equations issued from the finite elements method.

4.4. Finite elements method

We can now proceed with the spatial discretization. At each time step, $n = 0, 1, 2, \ldots$, and each fixed point iteration, $k = 0, 1, 2, \ldots$, a variational formulation for (43) is posed: find $U^{n+1,k+1} \in H^1(0, S_\infty)$ such that:

$$(1 + r\Delta\tau)\int_0^{S_\infty} U^{n+1,k+1}\varphi dS - \Delta\tau\int_0^{S_\infty}\frac{\partial}{\partial S}\left(\frac{\sigma^2}{2}S^2\frac{\partial U^{n+1,k+1}}{\partial S}\right)\varphi dS$$
$$= \int_0^{S_\infty}(U^n \circ \chi^n)(S)\varphi dS - \Delta\tau\int_0^{S_\infty} f(U^{n+1,k}, V^{n+1})\varphi dS, \quad \forall\varphi \in H_0^1(0, S_\infty),$$

or, after applying Green's theorem,

$$(1 + r\Delta\tau)\int_0^{S_\infty} U^{n+1,k+1}\varphi dS + \Delta\tau\frac{\sigma^2}{2}\int_0^{S_\infty} S^2\frac{\partial U^{n+1,k+1}}{\partial S}\frac{\partial\varphi}{\partial S}dS$$
$$= \int_0^{S_\infty}(U^n \circ \chi^n)(S)\varphi dS - \Delta\tau\int_0^{S_\infty} f(U^{n+1,k}, V^{n+1})\varphi dS, \quad \forall\varphi \in H_0^1(0, S_\infty).$$

For a fixed natural number $M > 0$, we consider a uniform mesh of the computational domain $\Omega = [0, S_\infty]$, the nodes of which are $S_j = j\Delta S$, $j = 0, \ldots M+1$, where $\Delta S = S_\infty/(M + 1)$ denotes the constant mesh step. Associated to this uniform mesh a piecewise linear Lagrange finite elements discretization is considered.

More precisely, we search $U_h^{n+1,k+1} \in W_h$ such that:

$$(1 + r\Delta\tau) \int_0^{S_\infty} U_h^{n+1,k+1} \varphi_h dS + \Delta\tau \frac{\sigma^2}{2} \int_0^{S_\infty} S^2 \frac{\partial U_h^{n+1,k+1}}{\partial S} \frac{\partial \varphi_h}{\partial S} dS$$

$$= \int_0^{S_\infty} (U_h^n \circ \chi^n)(S)\varphi_h dS - \Delta\tau \int_0^{S_\infty} f(U_h^{n+1,k}, V^{n+1})\varphi_h dS, \quad \forall \varphi_h \in W_{h,0},$$

$$(52)$$

where the finite elements spaces are

$$W_h = \{\varphi_h : (0, S_\infty) \to \mathbb{R}/\varphi_h \in \mathcal{C}(0, S_\infty), \varphi_h|_{[S_j, S_{j+1}]} \in \mathcal{P}_1\},$$
$$W_{h,0} = \{\varphi_h \in W_h/\varphi_h(0) = 0, \varphi_h(S_\infty) = 0\},$$

$\mathcal{P}_1$ being the space of polynomials of degree less or equal than one.

The coefficients of the matrix and right hand side vector defining the linear system associated to the fully discretized problem are approximated by adequate quadrature formulae. In particular, Simpson, three nodes Gaussian, midpoint and trapezoidal formulae have been used for the different terms, depending on the degree of the resulting polynomials to be integrated in each term. Finally, the system of linear equations is solved by a partial pivoting LU factorization method.

The value of the derivative without counterparty risk, V^{n+1}, is known at each time step. Actually, it is obtained as the solution to the Black–Scholes equation for options with dividends,

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0 & \text{in} \quad [0, T] \times [0, \infty) \\ V(T, S) = H(S) & S > 0, \end{cases} \quad (53)$$

where $D_0 \equiv r - r_R$. Thus, depending on the type of financial derivative we have different payoff functions. In some cases, the value of the derivative admits an analytical expression. For example, in the three case here treated these expressions come from the wellknown formulae:

- Call option:

$$V(t, S) = S\exp(-D_0(T - t))N(d_1) - E\exp(-r(T - t))N(d_2)$$

- Put option:

$$V(t, S) = E\exp(-r(T - t))N(-d_2) - S\exp(-D_0(T - t))N(-d_1)$$

- Forward:

$$V(t, S) = S\exp\left(\left(\frac{\sigma^2}{4} + \frac{r_R^2}{\sigma^2} - r\right)(T - t)\right)$$

$$- E\exp\left(\left(\sigma^2\left(\frac{r_R}{\sigma^2} - \frac{1}{2}\right)^2 - r\right)(T - t)\right)$$

where:

$$d_1 = \frac{\log(S/E) + (r - D_0 + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_2 = \frac{\log(S/E) + (r - D_0 - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

and $N(x)$ represents the distribution function of the standard $\mathcal{N}(0,1)$ random variable.

4.4.1. Augmented Lagrangian Active Set for American options

All the previous numerical methods have been explained for European options, in which there is no early exercise opportunity. For the pricing of American options, problems (36), (37) and (38) have been previously presented. In these cases, the unknowns V^{n+1} and $\hat{V}^{n+1,k+1}$ satisfy complementarity problems associated to linear and nonlinear partial differential equations. In order to explain their numerical solution, let us first focus on the nonlinear problem for $\hat{V}^{n+1,k+1}$. After a time discretization with characteristic method and a spatial discretization with finite elements, the fully discretized problem can be written in the form:

$$\begin{cases} A_h \hat{V}_h^{n+1,k+1} \geq b_h^{n+1,k+1} \\ \hat{V}_h^{n+1,k+1} \geq \Psi_h \\ \left(A_h \hat{V}_h^{n+1,k+1} - b_h^{n+1,k+1} \right) \left(\hat{V}_h^{n+1,k+1} - \Psi_h \right) = 0 \end{cases} \tag{54}$$

where Ψ_h denotes the discretized exercise value, $H(S)$, which also coincides with the value at maturity.

Following [1], the Augmented Lagrangian Active Set (ALAS) algorithm proposed by [20] has been implemented to solve (54). For this purpose, we introduce a multiplier P_h in order to write (54) in the equivalent form:

$$\begin{cases} A_h \hat{V}_h^{n+1,k+1} + P_h^{n+1,k+1} = b_h^{n+1,k+1} \\ \hat{V}_h^{n+1,k+1} \geq \Psi_h \\ P_h^{n+1,k+1} \leq 0 \\ (\hat{V}_h^{n+1,k+1} - \Psi_h) P_h^{n+1,k+1} = 0 \, . \end{cases} \tag{55}$$

Note that the last equation in (54) and (55) should be understood as componentwise.

ALAS algorithm consists of two steps. The first step decomposes the domain into active (that is, where $P_h^{n+1,k+1} < 0$) and inactive (where $P_h^{n+1,k+1} = 0$) regions. In the second step, a reduced linear system associated to the inactive part is solved.

First, let $\mathcal{N} := \{1, 2, \ldots, N_{dof}\}$ be the set of degrees of freedom. For any descomposition $\mathcal{N} = \mathcal{I} \cup \mathcal{J}$, the principal minor of matrix A_h is denoted by $[A_h]_{\mathcal{I},\mathcal{I}}$, while $[A_h]_{\mathcal{I},\mathcal{J}}$ is the codiagonal block indexed by $\mathcal{I}$ and $\mathcal{J}$. Therefore, for each time step $n + 1$ and each fixed point iteration $k + 1$, ALAS algorithm

computes the decomposition $\mathcal{N} = \mathcal{I}^{n+1,k+1} \cup \mathcal{J}^{n+1,k+1}$ such that $\hat{V}_h^{n+1,k+1}$ and $P_h^{n+1,k+1}$ are the solution of the following system:

$$A_h \hat{V}_h^{n+1,k+1} + P_h^{n+1,k+1} = b_h^{n+1,k+1}$$

$$[P_h^{n+1,k+1}]_j + \beta[\hat{V}_h^{n+1,k+1} - \Psi_h]_j \leq 0, \qquad \forall j \in \mathcal{J}^{n+1,k+1}$$

$$[P_h^{n+1,k+1}]_i = 0, \qquad \forall i \in \mathcal{I}^{n+1,k+1}$$

for a given positive parameter β. In the previous equations, $\mathcal{I}^{n+1,k+1}$ and $\mathcal{J}^{n+1,k+1}$ represent the inactive and the active sets, respectively. Namely, the iterative algorithm builds sequences $\{\hat{V}_{h,m}^{n+1,k+1}\}_m$, $\{P_{h,m}^{n+1,k+1}\}_m$, $\{\mathcal{I}_m^{n+1,k+1}\}_m$ and $\{\mathcal{J}_m^{n+1,k+1}\}_m$ converging to $\hat{V}_h^{n+1,k+1}$, $P_h^{n+1,k+1}$, $\mathcal{I}^{n+1,k+1}$ and $\mathcal{J}^{n+1,k+1}$ through the following steps:

1. Let be $\hat{V}_{h,0}^{n+1,k+1} = \Psi_h$ and $P_{h,0}^{n+1,k+1} = \min\{b_h^{n+1,k+1} - A_h \hat{V}_{h,0}^{n+1,k+1}, 0\} \leq 0$. Choose $\beta > 0$. Set $m = 0$.

2. Compute

$$Q_{h,m}^{n+1,k+1} = \min\{0, P_{h,m}^{n+1,k+1} + \beta(\hat{V}_{h,m}^{n+1,k+1} - \Psi_h)\}$$

$$\mathcal{J}_m^{n+1,k+1} = \{j \in \mathcal{N}, [Q_{h,m}^{n+1,k+1}]_j < 0\}$$

$$\mathcal{I}_m^{n+1,k+1} = \{i \in \mathcal{N}, [Q_{h,m}^{n+1,k+1}]_i = 0\}$$

3. If $m \geq 1$ and $\mathcal{J}_m^{n+1,k+1} = \mathcal{J}_{m-1}^{n+1,k+1}$, then convergence is achieved.

4. Let $\bar{V}$ and $\bar{P}$ be the solution of the linear system:

$$A_h \bar{V} + \bar{P} = b_h$$

$$\bar{P} = 0 \quad \text{on} \quad \mathcal{I}_m^{n+1,k+1} \quad \text{and} \quad \bar{V} = \Psi_h \quad \text{on} \quad \mathcal{J}_m^{n+1,k+1}. \qquad (56)$$

Set $\hat{V}_{h,m+1}^{n+1,k+1} = \bar{V}$, $P_{h,m+1}^{n+1,k+1} = \min\{0, \bar{P}\}$, $m = m + 1$ and go to step 2.

It is important to notice that, instead of solving the full linear system in (56), the following reduced one on the inactive set is solved:

$$[A_h]_{\mathcal{I},\mathcal{I}}[\bar{V}]_{\mathcal{I}} = [b_h]_{\mathcal{I}} - [A_h]_{\mathcal{I},\mathcal{J}}[\Psi]_{\mathcal{J}}$$

$$[\bar{V}]_{\mathcal{J}} = [\Psi]_{\mathcal{J}}$$

$$\bar{P} = b_h - A_h \bar{V}$$

where we have denoted $\mathcal{I} = \mathcal{I}_m^{n+1,k+1}$ and $\mathcal{J} = \mathcal{J}_m^{n+1,k+1}$. Therefore, after applying the ALAS method to problems (36) and (38) or to problems (37) and (38), we can compute the XVA value as $U_h = \hat{V}_h - V_h$.

5. Numerical results

In order to illustrate the good behavior of the proposed numerical strategy, we have first compared the results obtained in specific cases for which an analytical solution is known. Moreover, other examples in which we compute the XVA in different situations are also presented.

In all cases, tests have been performed by using MATLAB on an Intel(R) Xeon(R) CPU E3-1241 3.50GHz computer. In all examples, the elapsed computational time is less than 25 seconds.

5.1. Test 1: Convergence

We first study the error and the order of convergence of the applied numerical methods, for which we take advantage of the analytic solution of the XVA problem in particular cases. For example, we consider a not collateralized call option bought by B, with $M = \hat{V}$ and funding costs. Note that as we consider $s_F = (1 - R_B)\lambda_B$, the analytical expression of the XVA is:

$$U(t, S) = -(1 - \exp(-((1 - R_B)\lambda_B + (1 - R_C)\lambda_C)(T - t)))V(t, S).$$

As we can observe in Table 1, the experimental order of convergence obtained with the discrete norm $L^\infty((0, T) \times L^2([0, S_\infty]))$ is one.

In Figures 1, 2 and 3 we show the XVA value as a percentage of the risk–free value, V. We can observe the relevance of the choice of the mark–to–market value at default (either V or $\hat{V}$), as well as the funding costs. These results correspond to time $t = 0$ and the following set of financial parameters: $\sigma = 0.25$, $r_R = 0.015$, $r = 0.03$ and $R_B = R_C = 0.4$.

Notice that in the four considered cases, with and without funding costs and both possibilities of the mark–to–market value, the value of XVA grows as the default intensity of C increases. Moreover, in the cases which do not consider funding cost the XVA remains constant, independently of the changes of the default intensity of B, λ_B. Nevertheless, when funding costs are considered, the XVA increases with λ_B.

Concerning the fixed–point algorithm (43) introduced in Section 4.2, we have not proved its theoretical convergence. However, convergence is attained in a reduced number of iterations (less than five) in all the experiments for both European and American options. We have used $\epsilon = 10^{-11}$ as the tolerance for the relative quadratic error between two iterations.

5.2. Test 2: European put option

In this example we analyze the time evolution of the CVA and FVA, in terms of the spot value. We have considered the case in which no collateral is posted in the trade.

We assume counterparty B buys a put option from C, the strike depending on the repo rate $(E = 10e^{r_R T})$, and a maturity period of 0.5 years.

Figure 4 shows the total value adjustment for the European put option. The XVA value is negative because it represents the decrease in the risk–free put value due to the probability of default from both counterparties.

Figure 5 shows the credit value adjustment surface for the put option. The function takes negative values, since it represents the amount that B has to charge to C due to C's probability of default. The value is null when the option expires, because at maturity date the exposure at the counterparty default disappears. Furthermore, the absolute value is larger when the put option is in

the money. In this case, B will be interested in exercising and will be (more) exposed to C's default.

Figure 6 represents the funding cost adjustment surface for the same European put option. The value is negative because it represents the funding costs that B charges to C; i.e., B will pay less money to C due to B's incurring in funding cost associated to the financing agreements. So, the FCA increases when the option is in the money, as the funding needed to pay the prime in the money is larger than if the option is out of the money.

5.3. Test 3: European call option and forward including funding costs

Now, according to the counterparties which take part in the agreement, we compare the risk–free value and the risky value considering and not considering funding costs. We have studied the value for an European call option with strike $E = 10e^{r_R T}$ and a maturity time of 3 years; the rest of the input parameters are $\sigma = 0.25$, $R_B = R_C = 0.3$, $r = 0.04$, $\lambda_B = \lambda_C = 0.04$, $r_R = 0.06$ and $s_F = \lambda_B(1 - R_B)$.

On the one hand, if we assume the trade takes place between banks before the crisis, these counterparties are considered to be risk–free. Therefore, no CVA is taken into account and the FCA is negligible; thus the price is equal to the derivative value without counterparty risk.

Let us now assume that counterparty B is a bank, and C is a risky client. Thus, the bank will charge C a credit value adjustment on the trade, i.e., the price B charges to C is equal to the risk–free price plus CVA.

On the other hand, if the trading takes place after the financial crisis, the banks are no more considered parts without counterparty risk (risk–free). Moreover, they charge a prime due to funds lending in the capital market and counterparty B will not be able to fund the premium of the trade at the risk–free rate anymore. This means that B will incur in a funding cost in the agreement. So, the price that B will offer to counterparty C is the risk–free value plus CVA and FCA. These three situations are represented in Figure 7.

A similar test concerning a forward contract has been done. The risk-free value and the risky values (with and without funding costs) are presented in Figure 8(a) for the mark–to–market equal to the risky derivative (non linear model) and in Figure 8(b) for the mark–to–market equal to the risk–free derivative (linear model). We can appreciate that when the forward has a positive value, B has the choice of exercising the contract thus being exposed to C default. On the other hand, if the forward has a negative value, then B may not be interested in exercising the contract, so that all the counterparty risk (from the point of view of B) is included in DVA. As we can observe, the computed results are similar in both cases. So, there is not a big difference in the choice of the mark–to–market close out.

5.4. Test 4: Collateralized European options

In this example we study again a European put option bought by B. However, in this example the trading is now on a collateralized derivative and we

use Model 3 of section 2.2. The strike is $E = 10e^{r_R T}$ and the maturity time is equal to 0.5 years. The rest of the parameters are $\sigma = 0.25$, $r_R = 0.06$, $r = 0.04$, $r_C = 0.05$, $R_B = R_C = 0.3$ and $\lambda_B = \lambda_C = 0.04$. Thus, we show in Figure 9 the difference between the fully collateralized and a partially collateralized derivative prices. The difference is positive, because it represents the additional amount that has to be paid by B if the derivative is collateralized. So, this price increases as the collateral is larger, thus the exposure facing C's default is lower. Therefore, the price of a collateralized European put option is larger than the not collateralized one. This difference between both of them is the CollVA.

In Figure 10, the XVA surface is represented when the trading takes place with a collateralized derivative. We show the variation in the XVA value for different collateral values, which are in all cases a percentage of the derivative risk–free value. As expected, if the derivative is not collateralized, $X = 0$ and the XVA value corresponds with the results obtained in Figure 4. Nevertheless, the XVA values decrease when the derivative approaches to the fully collateralized case.

Moreover, we compare the three particular models explained in Section 2.2. Figure 11 represents the computed XVA value according to the different assumptions made about counterparty B's bond. We can observe that for a stock price in the money area, the results obtained using model 2 and model 3 are similar, whereas the XVA is higher in absolute terms if model 1 is employed. In any case, the differences between the models are negligible.

5.5. Test 5: American options

In this example we show the results obtained for American options with the same parameters as in the European case of Test 2 and Test 3. For the ALAS algorithm, we consider $\beta = 10^5$ and the stopping test equal to 10^{-5}, thus obtaining the convergence in two or three iterations. In Figure 12 we have compared the American call option value considering different adjustments upon risk free value. The input parameters are equal to those ones of the analogous example in European options. As in the European call option case, when counterparty B buys a call option, the price that B has to pay by the risk–free derivative is higher than the amount that has to be paid for an option if default risk and funding costs are considered. Moreover, as expected in an option that pays no dividends, risk free value is the same for both options; in other case, when risky values are considered the American option value is larger than the European one, due to the fact that the American option can be exercised before the maturity date.

In Figures 13 and 14, the exercise region for an American put option is represented in white. We can see that in the case with counterparty risk this region is larger than the same area in the case of an American put option without counterparty risk. According to these regions, we can interpret Figure 15, which represents the XVA surface for an American put option. We can observe that the XVA is negative because it represents the discounted value upon risk–free value, due to the risk exposure of counterparty B. Moreover, in terms of absolute value this is larger when the asset value approaches the

exercise area because the buyer B is more interested in exercising the option. Moreover, when the spot price is in the exercise region, the XVA surface tends to zero. This is due to the fact that the risky value and the risk–free value reach the exercise price, so that $\text{XVA} = \hat{V} - V = 0$. Finally, the XVA value is zero at maturity, because the exposure faced by the counterparty has disappeared.

6. Conclusions

In the present paper, we formulate different PDE problems associated to the pricing of total value adjustments (XVA) to be added to the price of the derivative without counterparty risk. For a financial derivative without early exercise opportunity (as European vanilla options or forward contracts) different linear and nonlinear PDEs arise. In the present paper we propose appropriate boundary conditions and numerical methods based on characteristics method (semilagrangian schemes), finite elements and fixed point iteration techniques to solve these PDE problems. The first order convergence of the numerical method is illustrated when applied to particular cases in which the analytical expression for the XVA is available. This is the expected order of convergence to be obtained for the numerical techniques we propose in the linear and nonlinear PDEs. Also, the numerical examples clearly illustrate the good performance of these models and methods for the case of European vanilla options and forward contracts in cases with and without collateral agreements, as different expected financial properties we discuss are recovered. We note that the models and methods can be extended to other European-like derivatives, as well as to the consideration of stochastic default intensities (as proposed in [14]), so that problems in two or three spatial like variables must be solved. This last issue is currently under consideration by the authors.

Furthermore, for first time in the literature, the consideration of the XVA associated to American options is posed. After introducing different complementarity problems to obtain the XVA associated to American options, augmented Lagrangian methods are proposed to tackle the additional inequality constraints involved in the formulation. Numerical examples are presented to illustrate and discuss the behaviour of the models and numerical methods to compute exercise regions with and without counterparty risk, as well as the interpretation of the XVA adjustments that have been obtained. Similar extensions than in the European-style case can be devised.

Aknowledgments

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Table 1: Relative errors in norm $L^\infty((0,T) \times L^2([0,S_\infty]))$, convergence ratios and order. Example with finite elements scheme (Test 1). The input parameters are $E = 15$, $S \in [0, 4E]$, $r = 0.03$, $r_R = 0.015$, $\sigma = 0.25$, $t \in [0,5]$, $\lambda_B = 0.02$, $\lambda_C = 0.05$, $R_B = 0.4$ and $R_C = 0.4$

Time step	Space step	Error	R	Order
400	50	0.02232872	-	-
800	100	0.01192059	1.87312280	0.90544548
1600	200	0.00617545	1.93031711	0.94883787
3200	400	0.00315299	1.95860211	0.96982435
6400	800	0.00160323	1.96665313	0.97574253
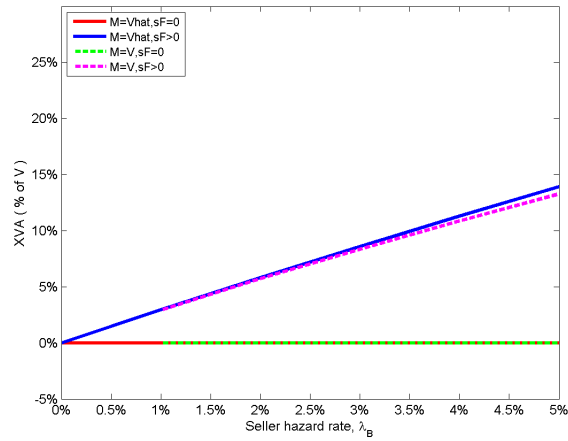
Figure 1: XVA in the cases $M = \hat{V}$ and $M = V$ for counterparty hazard rate, $\lambda_C = 0\%$ (Test 1)
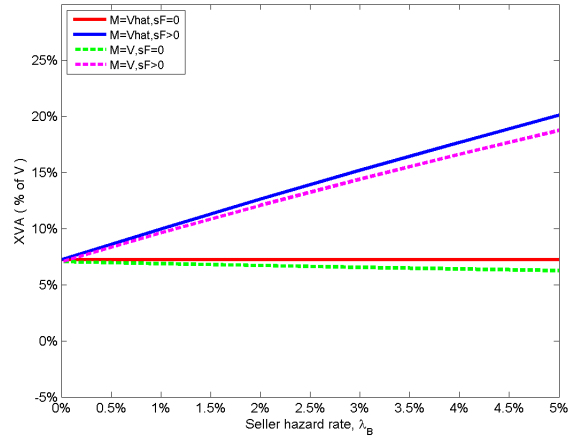
Figure 2: XVA in the cases $M = \hat{V}$ and $M = V$ for counterparty hazard rate, $\lambda_C = 2.5\%$ (Test 1)
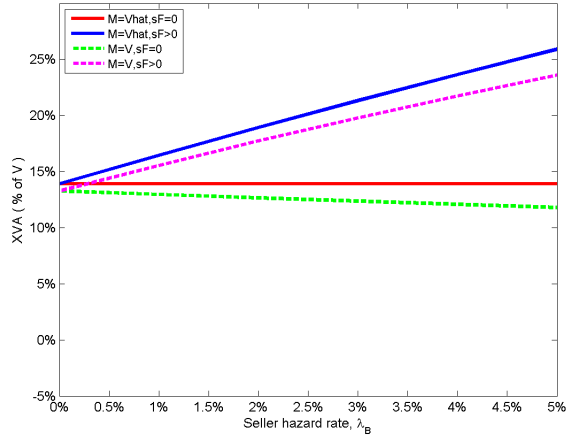
Figure 3: XVA in the cases $M = \hat{V}$ and $M = V$ for counterparty hazard rate, $\lambda_C = 5\%$ (Test 1)
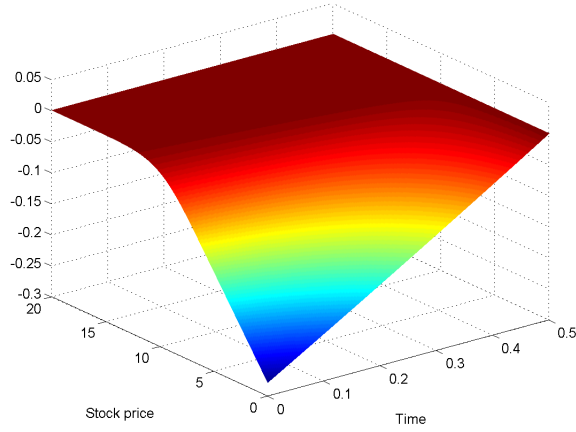
Figure 4: XVA surface for European put option with input arguments (Test 2): $S_0 \in [0, 20]$, $E = 10e^{r_R T}$, $r = 0.04$, $r_R = 0.06$, $\sigma = 0.25$, $T = 0.5$, $r_{P_B} = 0.08$, $r_{P_C} = 0.08$, $R_B = 0.3$ and $R_C = 0.3$
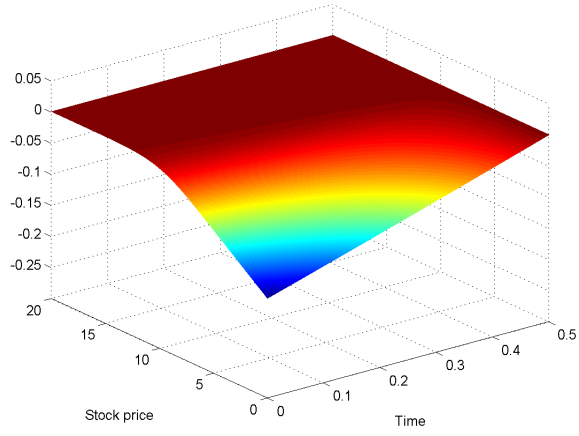
Figure 5: CVA+DVA surface for European put option with input arguments (Test 2): $S_0 \in [0, 20]$, $E = 10e^{r_R T}$, $r = 0.04$, $r_R = 0.06$, $\sigma = 0.25$, $T = 0.5$, $r_{P_B} = 0.08$, $r_{P_C} = 0.08$, $R_B = 0.3$ and $R_C = 0.3$
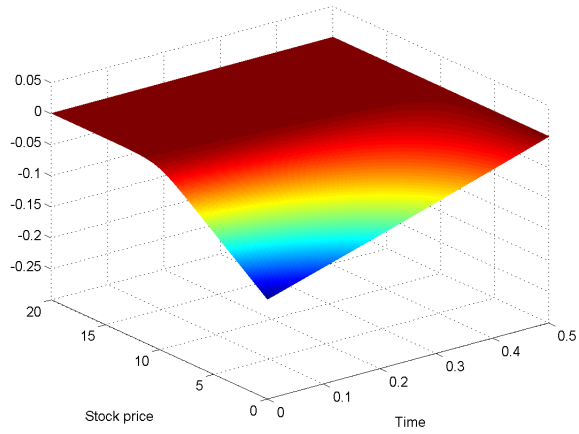
Figure 6: FCA surface for European put option with input arguments (Test 2): $S_0 \in [0, 20]$, $E = 10e^{r_R T}$, $r = 0.04$, $r_R = 0.06$, $\sigma = 0.25$, $T = 0.5$, $r_{P_B} = 0.08$, $r_{P_C} = 0.08$, $s_F = (1 - R_B)\lambda_B$, $R_B = 0.3$ and $R_C = 0.3$
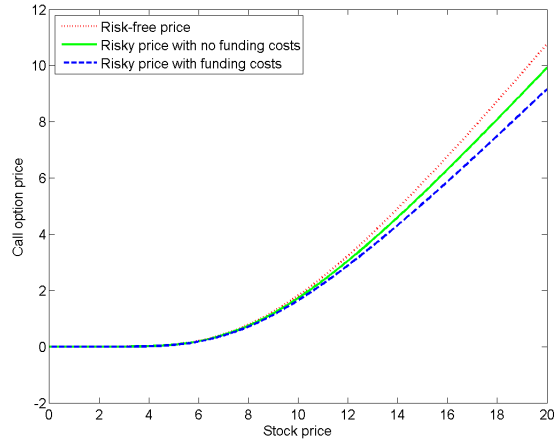
Figure 7: European call option values with CVA and FCA (Test 3)
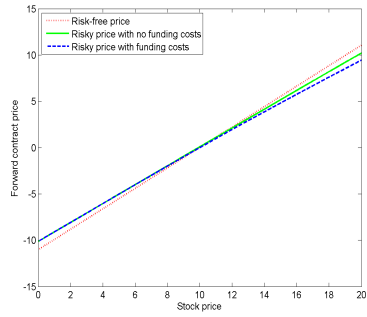
(a) Case with $M = \hat{V}$
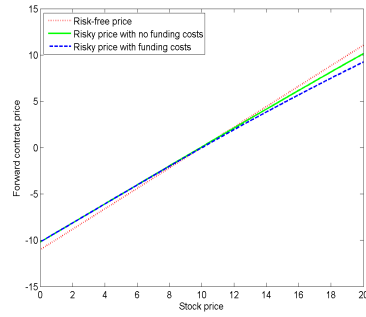
(b) Case with $M = V$

Figure 8: Forward values with CVA and FCA (Test 3)

32
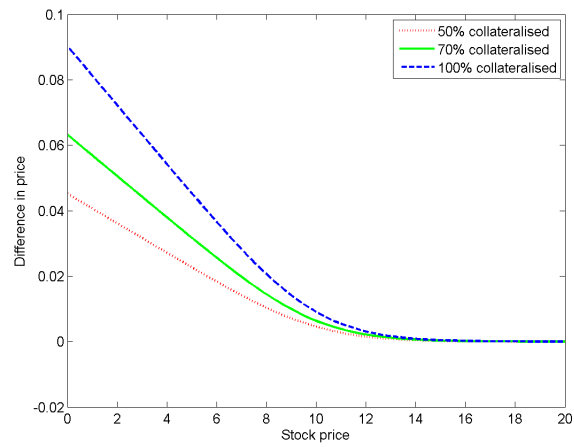
Figure 9: Collateral Value adjustment for different amount of collateral (Test 4)
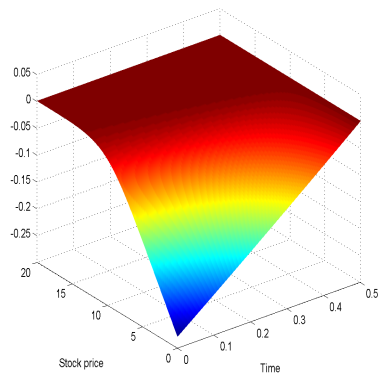
(a) Collateral $= 0$
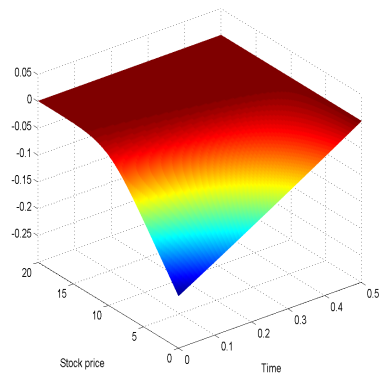
(b) Collateral $= 0.33$V
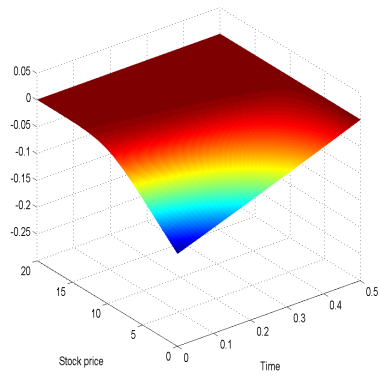
(c) Collateral $= 0.66$V
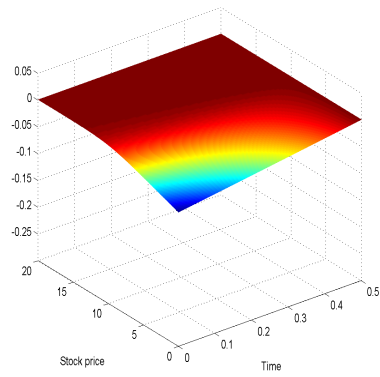
(d) Collateral $= $V

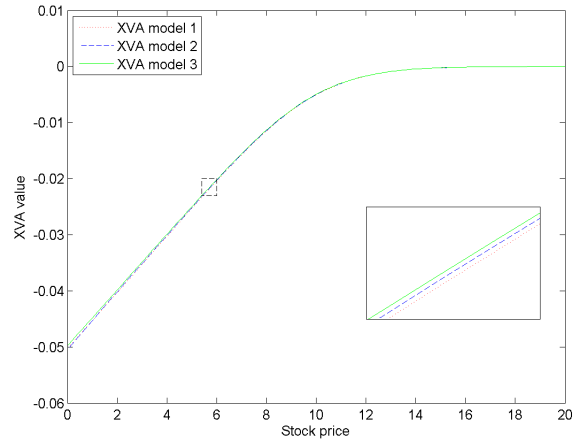Figure 10: XVA surfaces for different collateral values (Test 4)
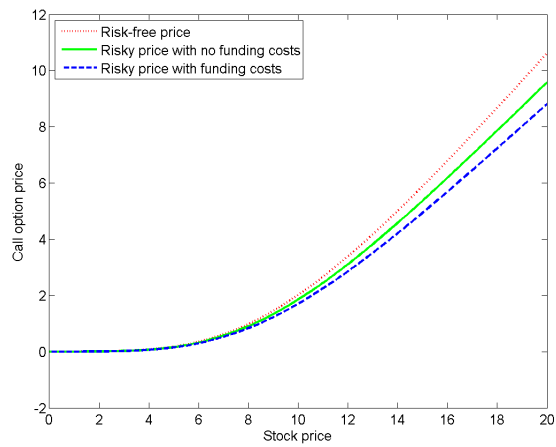
Figure 11: Total Value Adjustment according to the different collateral models (Test 4)
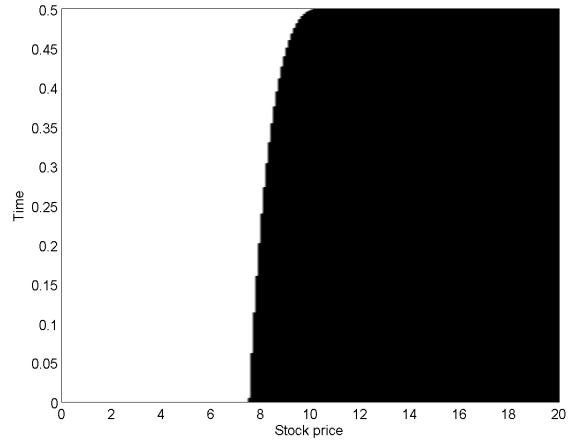

Figure 12: American call option value (Test 5)

Figure 13: Exercise region (white) for an American put option without counterparty risk (Test 5)
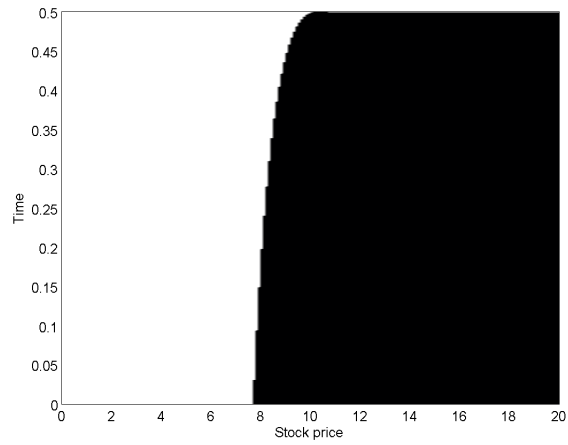

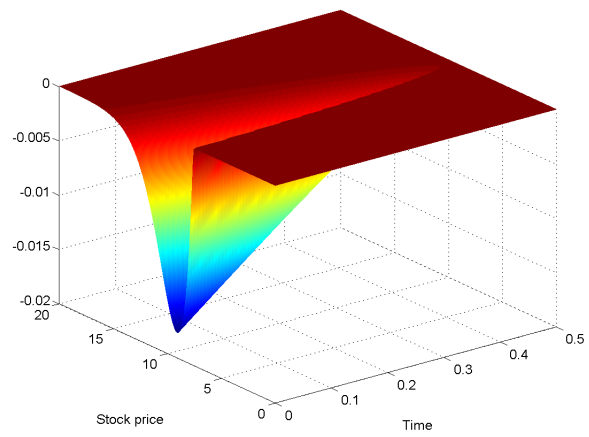Figure 14: Exercise region (white) for an American put option with counterparty risk (Test 5)

Figure 15: Total Value Adjustment surface for American put option (Test 5)