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Convolutional Codes of Goppa Type*

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Abstract. A new kind of Convolutional Codes generalizing Goppa Codes is proposed. This provides a systematic method for constructing convolutional codes with prefixed properties. In particular, examples of Maximum-Distance Separable (MDS) convolutional codes are obtained.

Keywords: Convolutional Codes, Goppa Codes, MDS Codes, Algebraic Curves, Coherent Sheaves, Finite Fields

1 Introduction

The aim of this paper is to propose a definition of Convolutional Goppa Codes (CGC). This definition will provide an algebraic method for constructing Convolutional Codes with prescribed invariants.

We propose a definition of CGC in terms of families of curves $X \to \mathbb{A}^1$ parametrized by the affine line $\mathbb{A}^1 = \operatorname{Spec} \mathbb{F}_q[z]$ over a finite field $\mathbb{F}_q$. In this setting, the usual definition of a Goppa Code as the code obtained by evaluation of sections at several rational points, is translated as a code obtained by evaluation (of sections of some invertible sheaf over X) along several sections of the fibration $X \to \mathbb{A}^1$.

The paper is organized as follows.

In §2 we offer a summary on Goppa Codes following [5], [8], and using the standard notations of Algebraic Geometry [4].

§3 is devoted to giving the general definition of CGC and gives some general results.

In §4 we study the case of a trivial fibration of projective lines over $\mathbb{A}^1$ and we conclude giving some explicit examples of MDS convolutional codes.

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We freely use the standard notations of abstract Algebraic Geometry as can be found in [4]. After the works of V. Lomadze [6], J. Rosenthal and R. Smarandache [10], [11], there is evidence that the use of methods of Algebraic Geometry can be relevant to the study of Convolutional Codes. This paper is a step in favor of that evidence.

Other algebraic methods for constructing Convolutional Codes have been recently proposed [2], [3].

2 Background on Algebraic Geometry and Goppa Codes

In this Section we summarize the basic definitions about Goppa Codes, constructed using methods of Algebraic Geometry (see [5], [8]).

Let X be a geometrically irreducible, smooth and projective curve over the finite field $\mathbb{F}_q$. Let $p_1, \ldots, p_n$ be n different $\mathbb{F}_q$-rational points of X, and D the divisor $D = p_1 + \cdots + p_n$. Let G be another effective divisor with support disjoint from D. The Goppa code $C(G, D)$ defined by (G, D) is the linear code of length n over $\mathbb{F}_q$ defined as the image of the linear map

$$\begin{aligned} \alpha\colon L(G) &\to \mathbb{F}_q^n \\ f &\mapsto (f(p_1), \ldots, f(p_n)), \end{aligned}$$

where $L(G)$ is the complete linear series defined by G. That is, let $\mathbb{F}_q(X)$ be the field of rational functions over the curve X,

$$L(G) = \left\{ f \in \mathbb{F}_q(X) \text{ such that } \operatorname{Div}(f) + G \geq 0 \right\} .$$

The Goppa code has dimension

$$k = \dim C(G, D) = \dim L(G) - \dim L(G - D) .$$

Let g be the genus of X; if we assume the inequality $2g - 2 < \deg(G) < n$, then one has

$$k = \deg(G) - g + 1 ,$$

and the minimum distance d of $C(G, D)$ satisfies the inequality

$$d \geq n - \deg(G) .$$

Let $\mathcal{O}_X(D)$ be the invertible sheaf on X defined by the divisor D. One has the following exact sequence of sheaves

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0 ,$$

where $\mathcal{O}_D \simeq \mathcal{O}_{p_1}/\mathfrak{m}_{p_1} \times \cdots \times \mathcal{O}_{p_n}/\mathfrak{m}_{p_n} \simeq \mathbb{F}_q \times \overset{n)}{\cdots} \times \mathbb{F}_q$. Tensoring the above exact sequence by $\mathcal{O}_X(G)$, one obtains

$$0 \to \mathcal{O}_X(G - D) \to \mathcal{O}_X(G) \to \mathcal{O}_D \to 0 .$$

By taking global sections, we obtain an exact sequence of cohomology

$$\begin{aligned} 0 \to H^0(X, \mathcal{O}_X(G-D)) \ &\to H^0(X, \mathcal{O}_X(G)) \xrightarrow{\alpha} \mathcal{O}_D \to \\ &H^1(X, \mathcal{O}_X(G-D)) \to \\ &\to H^1(X, \mathcal{O}_X(G)) \to 0\,, \end{aligned}$$

where $L(G) = H^0(X, \mathcal{O}_X(G))$ and α is the evaluation map defined above. In the case $2g - 2 < \deg(G) < n$, one has the exact sequence

$$0 \to H^0(X, \mathcal{O}_X(G)) \xrightarrow{\alpha} \mathcal{O}_D \to H^1(X, \mathcal{O}_X(G-D)) \to 0\,. \tag{2.1}$$

Let ω_X be the dualizing sheaf of X, which is isomorphic to the sheaf of regular 1-forms over X; $H^0(X, \omega_X)$ is the $\mathbb{F}_q$-vector space of global regular 1-forms over X, which is of dimension g = genus of X.

By Serre's duality ([4]), there exist canonical isomorphisms of $\mathbb{F}_q$-vector spaces

$$H^1(X, \mathcal{L})^* \simeq H^0(X, \omega_X \otimes \mathcal{L}^{-1})$$

for every invertible sheaf $\mathcal{L}$ on X. Given a divisor D over X, we shall denote by $\Omega(D)$ the vector space $H^0(X, \omega_X \otimes \mathcal{O}_X(-D))$.

The dual Goppa code, $C^*(G, D)$, associated with the Goppa code $C(G, D)$ is defined as the linear code of length n over $\mathbb{F}_q$ given by the image of the linear map

$$\begin{aligned} \alpha^* \colon \Omega(G-D) &\to \mathbb{F}_q^n \\ \eta &\mapsto (\mathrm{Res}_{p_1}(\eta), \dots, \mathrm{Res}_{p_n}(\eta))\,, \end{aligned}$$

Let us take duals in the exact sequence (2.1):

$$0 \to H^1(X, \mathcal{O}_X(G-D))^* \xrightarrow{\beta} \mathcal{O}_D^* \xrightarrow{\alpha^t} H^0(X, \mathcal{O}_X(G))^* \to 0\,.$$

By Serre's duality, one has isomorphisms

$$\begin{aligned} &H^1(X, \mathcal{O}_X(G-D))^* \simeq \Omega(G-D)\,, \\ &H^0(X, \mathcal{O}_X(G))^* \simeq H^1(X, \omega_X \otimes \mathcal{O}_X(-G))\,, \end{aligned}$$

and the above sequence is the cohomology sequence induced by the exact sequence of sheaves

$$0 \to \omega_X(-G) \to \omega_X(D-G) \to \omega_X(D-G) \otimes_{\mathcal{O}_X} \mathcal{O}_D \to 0\,,$$

where we denote $\omega_X(-G) = \omega_X \otimes \mathcal{O}_X(-G)$, and β is precisely the map α^* defining $C^*(G, D)$.

Given a linear series $\Gamma \subseteq H^0(X, \mathcal{O}_X(G))$, that is, a vector subspace defining a family of divisors linearly equivalent to G, we define the Goppa code $C(\Gamma, D)$ associated with Γ and D as the image of the homomorphism $\alpha_{|\Gamma}$:

$$\begin{tikzcd}
H^0(X, \mathcal{O}_X(G)) \arrow[r, "\alpha"] & \mathcal{O}_D \\
\Gamma \arrow[u, phantom, sloped, "\subseteq"] \arrow[ur, "\alpha_{|\Gamma}"'] &
\end{tikzcd}$$

When $\Gamma \subsetneqq H^0(X, \mathcal{O}_X(G))$, we shall say that $C(\Gamma, D)$ is a non-complete Goppa code.

3 Convolutional Goppa Codes

We shall contruct a kind of convolutional code that generalizes the notion of Goppa codes. These codes will be associated with families of algebraic curves.

Given an algebraic variety S over the field $\mathbb{F}_q$, a family of projective algebraic curves parametrized by S is a morphism of algebraic varieties $\pi : X \to S$, such that π is a projective and flat morphism whose fibres $X_s = \pi^{-1}(s)$ are smooth and geometrically irreducible curves over $\mathbb{F}_q(s)$ (the residue field of $s \in S$).

Let us consider a family of curves $X \overset{\pi}{\to} U$ parametrized by $U = \operatorname{Spec} \mathbb{F}_q[z] = \mathbb{A}^1$. Given a closed point $u \in U$ with residue field $\mathbb{F}_q(u)$, the fibre $X_u = \pi^{-1}(u)$ is a curve over the finite field $\mathbb{F}_q(u)$.

Let p_i, $1 \leq i \leq n$, be n different sections, $p_i : U \to X$, of the projection π. These sections define a Cartier divisor on X:

$$D = p_1(U) + \cdots + p_n(U)\,,$$

which is flat of degree n over the base U ([4]).

Note that given a coherent sheaf $\mathcal{F}$ on X, the cohomology groups $H^i(X, \mathcal{F})$ are finite $\mathbb{F}_q[z]$-modules and $H^i(X, \mathcal{F}) = 0$ for $i \geq 2$ (see [4] III).

Let $\mathcal{L}$ be an invertible sheaf over X. One has an exact sequence of sheaves on X

$$0 \to \mathcal{L}(-D) \to \mathcal{L} \to \mathcal{O}_D \to 0\,, \tag{3.1}$$

(where $\mathcal{L} \otimes \mathcal{O}_D \simeq \mathcal{O}_D$) which induces a long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(X, \mathcal{L}(-D)) \to H^0(X, \mathcal{L}) \overset{\alpha}{\to} H^0(X, \mathcal{O}_D) \to H^1(X, \mathcal{L}(-D)) \\ &\to H^1(X, \mathcal{L}) \to 0\,. \end{aligned} \tag{3.2}$$

Let r be the degree of $\mathcal{L}$ in each fibre of π (which is independent of the fibre) and let g be the genus of any fibre of π (also independent of the fibres).

Proposition 3.1 *Let us assume that* $2g-2 < r$. *Then, one has that* $H^1(X, \mathcal{L}) = 0$ *and* $H^0(X, \mathcal{L})$ *is a free* $\mathbb{F}_q[z]$*-module of rank* $r - g + 1$

Proof. Under the condition $2g - 2 < r$, one has that $H^1(X_u, \mathcal{L}_{|X_u}) = 0$ for every point $u \in U$. Note that $H^i(X, \mathcal{F})\widetilde{\ } = R^i\pi_*\mathcal{F}$ for every coherent sheaf $\mathcal{F}$ on X ([4] III), and applying ([4] III Corollary 12.9) one concludes the proof. □

Under the hypothesis of Proposition 3.1, there exists an exact sequence of $\mathbb{F}_q[z]$-modules

$$0 \to H^0(X, \mathcal{L}(-D)) \to H^0(X, \mathcal{L}) \overset{\alpha}{\to} H^0(X, \mathcal{O}_D) \to H^1(X, \mathcal{L}(-D)) \to 0\,. \tag{3.3}$$

where $H^0(X, \mathcal{O}_D)$ is a free $\mathbb{F}_q[z]$-module of rank n.

Remark 3.2 Let $\eta \in U$ be the generic point of U, whose residue field is $\mathbb{F}_q(z)$; the fibre $X_\eta = \pi^{-1}(\eta)$ is a smooth, irreducible curve over $\mathbb{F}_q(z)$. Note that $p_1(\eta), \dots, p_n(\eta)$ are n different $\mathbb{F}_q(z)$-rational points of the curve X_η. One then has a canonical decomposition of $H^0(X, \mathcal{O}_D)_\eta$ as a $\mathbb{F}_q(z)$-algebra

$$H^0(X, \mathcal{O}_D)_\eta = \mathbb{F}_q(z) \times \overset{n)}{\cdots} \times \mathbb{F}_q(z)\,.$$

Given a $\mathbb{F}_q[z]$-module M, let us denote by M_η the $\mathbb{F}_q(z)$-vector space

$$M_\eta = M \otimes_{\mathbb{F}_q[z]} \mathbb{F}_q(z)\,.$$

The sequence (3.3) induces an exact sequence of $\mathbb{F}_q(z)$-vector spaces

$$\begin{aligned} 0 \to\ & H^0(X, \mathcal{L}(-D))_\eta \to H^0(X, \mathcal{L})_\eta \overset{\alpha_\eta}{\to} H^0(X, \mathcal{O}_D)_\eta \to \\ & H^1(X, \mathcal{L}(-D))_\eta \to 0\,. \end{aligned} \tag{3.4}$$

Definition 3.3 *The complete convolutional Goppa code associated with $\mathcal{L}$ and D is the image of the homomorphism α_η*

$$\mathcal{C}(\mathcal{L}, D) = \mathcal{I}\mathrm{m}\left(H^0(X, \mathcal{L})_\eta \xrightarrow{\alpha_\eta} H^0(X, \mathcal{O}_D)_\eta \simeq \mathbb{F}_q(z)^n\right)\,.$$

Given a free submodule $\Gamma \subseteq H^0(X, \mathcal{L})$, the convolutional Goppa code associated with Γ and D is the image of $\alpha_{\eta_{|\Gamma_\eta}}$

$$\mathcal{C}(\Gamma, D) = \mathcal{I}\mathrm{m}\left(\Gamma_\eta \xrightarrow{\alpha_\eta} \mathbb{F}_q(z)^n\right)\,.$$

Remark 3.4 We use definition 2.4 of [7] as definition of convolutional codes. Any matrix defining α_η (respectively $\alpha_{\eta_{|\Gamma_\eta}}$) is a generator matrix of rational functions for the code $\mathcal{C}(\mathcal{L}, D)$ (resp. $\mathcal{C}(\Gamma, D)$).

The canonical decomposition $H^0(X, \mathcal{O}_D)_\eta \simeq \mathbb{F}_q(z)^n$ as $\mathbb{F}_q(z)$-algebras does not extend (in general) to a decomposition $H^0(X, \mathcal{O}_D) \simeq \mathbb{F}_q[z]^n$ as rings.

In fact, one has a canonical isomorphism of rings $H^0(X, \mathcal{O}_D) \overset{\phi}{\xrightarrow{\sim}} \mathbb{F}_q[z]^n$ only when $p_1(U), \dots, p_n(U)$ are disjoint sections. However, $H^0(X, \mathcal{O}_D)$ is a free $\mathbb{F}_q[z]$-module; then, there exist (non-canonical) isomorphisms of $\mathbb{F}_q[z]$-modules:

$$H^0(X, \mathcal{O}_D) \overset{\phi}{\xrightarrow{\sim}} \mathbb{F}_q[z] \oplus \overset{n)}{\cdots} \oplus \mathbb{F}_q[z]\,,$$

which are not (in general) isomorphism of rings.

This allows us to give another definition of convolutional Goppa codes, as submodules of a polynomial module [9].

Definition 3.5 *Given a trivialization* $\phi\colon H^0(X, \mathcal{O}_D) \xrightarrow{\sim} \mathbb{F}_q[z]^n$ *as* $\mathbb{F}_q[z]$*-modules, one defines the convolutional Goppa code* $\mathcal{C}(\mathcal{L}, D, \phi)$ *as the image of* $\phi \circ \alpha$

$$H^0(X, \mathcal{L}) \xrightarrow{\alpha} H^0(X, \mathcal{O}_D) \overset{\phi}{\xrightarrow{\sim}} \mathbb{F}_q[z]^n\,.$$

Analogously, one defines the convolutional Goppa code $\mathcal{C}(\Gamma, D, \phi)$.

Let us assume (for the rest of the paper) that the invariants (r, n, g) satisfy the inequality

$$2g - 2 < r < n\,.$$

Proposition 3.6 *Under the above conditions on* (r, n, g)*,* $H^0(X, \mathcal{L}(-D)) = 0$ *and* $H^1(X, \mathcal{L}(-D))$ *is a free* $\mathbb{F}_q[z]$*-module. The following exact sequence is exact*

$$0 \to H^0(X, \mathcal{L}) \xrightarrow{\alpha} H^0(X, \mathcal{O}_D) \to H^1(X, \mathcal{L}(-D)) \to 0\,. \tag{3.5}$$

and remains exact when we take fibres over every point $u \in U$.

Proof. If $2g - 2 < r < n$, $H^0(X_u, \mathcal{L}(-D)_{|X_u}) = 0$ for every point $u \in U$; and applying ([4] III Corollary 12.9) one concludes. □

Corollary 3.7 *The convolutional code* $\mathcal{C}(\mathcal{L}, D, \phi)$ *has dimension* $k = r - g + 1$ *and length n. Every matrix defining* $\phi \circ \alpha$ *is a basic generator matrix [7] for* $\mathcal{C}(\mathcal{L}, D, \phi)$.

Proof. This is a direct consequence of the last statement of Proposition 3.6 and the characterization of basic generator matrices of [7]. □

Let us consider the convolutional Goppa code $\mathcal{C}(\Gamma, D, \phi)$ defined by a submodule $\Gamma \subseteq H^0(X, \mathcal{L})$ and a trivialization ϕ. With the above restrictions, one has:

Proposition 3.8 *Every matrix defining* $\phi \circ \alpha_{|\Gamma}$ *is a basic generator matrix for the code* $\mathcal{C}(\Gamma, D, \phi)$ *if and only if* $H^0(X, \mathcal{L})/\Gamma$ *is a torsion-free* $\mathbb{F}_q[z]$*-module.*

Proof. The sequence (3.5) induces a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & & \\
0 \arrow[r] & \Gamma \arrow[r, "\alpha_{|\Gamma}"] \arrow[d] & H^0(X, \mathcal{O}_D) \arrow[r] \arrow[d, equal] & H^1(X, \Gamma) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(X, \mathcal{L}) \arrow[r] \arrow[d] & H^0(X, \mathcal{O}_D) \arrow[r] & H^1(X, \mathcal{L}(-D)) \arrow[r] \arrow[d] & 0 \\
 & H^0(X, \mathcal{L})/\Gamma & & 0 &
\end{tikzcd}$$

Then, the kernel of $H^1(X, \Gamma) \to H^1(X, \mathcal{L}(-D))$ is isomorphic to $H^0(X, \mathcal{L})/\Gamma$ and $H^1(X, \mathcal{L}(-D))$ is free. This implies that the torsion elements of $H^1(X, \Gamma)$ are contained in $H^0(X, \mathcal{L})/\Gamma$, from which one concludes the proof. □

The above results allow us to construct basic generator matrices for the codes $\mathcal{C}(\Gamma, D, \phi)$. If $p_1(U), \dots, p_n(U)$ are disjoint sections and ϕ the canonical trivialization, this gives us a basic generator matrix for $\mathcal{C}(\Gamma, D)$. However, in general the codes $\mathcal{C}(\Gamma, D)$ and $\mathcal{C}(\Gamma, D, \phi)$ are different.

Let us describe a geometric way to obtain a basic generator matrix for $\mathcal{C}(\mathcal{L}, D)$ and $\mathcal{C}(\Gamma, D)$.

Assume that the curves $p_1(U), \dots, p_n(U)$ meet transversally at some points, and let $\bar{X}$ be the blowing-up [4] of X at these points. One has morphisms

$$\begin{tikzcd}
\bar{X} \arrow[r, "\beta"] \arrow[dr, "\bar{\pi}=\pi\circ\beta"'] & X \arrow[d, "\pi"] \\
 & U
\end{tikzcd}$$

such that the proper transform of D under π is a divisor $\bar{D} \subset \bar{X}$ satisfying

$$\bar{D} = p_1(U) \amalg \cdots \amalg p_n(U) \xrightarrow{\beta} D\,,$$

and one has a canonical homomorphism of rings

$$0 \to \mathcal{O}_D \to \beta_* \mathcal{O}_{\bar{D}}$$

which induces

$$0 \to \pi_* \mathcal{O}_D \xrightarrow{\beta} \bar{\pi}_* \mathcal{O}_{\bar{D}} \simeq \widetilde{\mathbb{F}_q[z]}^n\,,$$

where $\bar{\pi}_* \mathcal{O}_{\bar{D}} \simeq \widetilde{\mathbb{F}_q[z]}^n$ is the canonical isomorphism of sheaves of rings.

$\beta^* \mathcal{L}$ is an invertible sheaf on $\bar{X}$ and there exists a canonical homomorphism

$$\beta^* \mathcal{L} \to \mathcal{O}_{\bar{D}} \to 0\,,$$

whose kernel is $(\beta^* \mathcal{L})(-\bar{D})$. We have also an injective homomorphism

$$0 \to \mathcal{L} \to \beta_* \beta^* \mathcal{L}\,,$$

and taking global sections one obtains

$$0 \to H^0(X,\mathcal{L}) \xrightarrow{\gamma} H^0(X,\beta_*\beta^*\mathcal{L}) \xrightarrow{\mu} \mathbb{F}_q[z]^n\,.$$

The image of μ is precisely a free submodule of $\mathbb{F}_q[z]^n$ that defines a basic generator matrix for $\mathcal{C}(\mathcal{L}, D)$.

Let us consider the sequence of homomorphisms

$$0 \to H^0(X,\mathcal{L}) \xrightarrow{\alpha} H^0(X,\mathcal{O}_D) \overset{\beta}{\hookrightarrow} H^0(X,\mathcal{O}_{\bar{D}}) = \mathbb{F}_q[z]^n\,.$$

$\beta\circ\alpha$ is not in general a basic matrix, since $H^0(X,\mathcal{O}_{\bar{D}})/H^0(X,\mathcal{O}_D)$ has torsion. Let us define

$$\bar{H}^0(X,\mathcal{L}) = \{p \in \mathbb{F}_q[z]^n \text{ such that } \lambda p \in H^0(X,\mathcal{L}) \text{ for some } \lambda \in \mathbb{F}_q[z]\}\,.$$

$\bar{H}^0(X,\mathcal{L})/H^0(X,\mathcal{L})$ is a torsion module and $\mathbb{F}_q[z]^n/\bar{H}^0(X,\mathcal{L})$ is torsion-free. Then, every matrix defining the homomorphism $\bar{H}^0(X,\mathcal{L}) \hookrightarrow \mathbb{F}_q[z]^n$ is a basic generator matrix for $\mathcal{C}(\mathcal{L}, D)$.

This is an algebraic-geometric interpretation of Forney's construction of the basic matrices of a convolutional code [1].

4 Convolutional Goppa Codes associated with the projective line

Let $\mathbb{P}^1 = \operatorname{Proj}\mathbb{F}_q[x_0, x_1]$ be the projective line over $\mathbb{F}_q$, and

$$X = \mathbb{P}^1 \times U \xrightarrow{\pi} U = \operatorname{Spec}\mathbb{F}_q[z]$$

the trivial fibration. Let us denote by $t = x_1/x_0$ the affine coordinate in $\mathbb{P}^1$, and by p_∞ its infinity point. Let us consider the following n different sections of π

$$p_i\colon U \to \mathbb{P}^1 \times U$$

defined in the coordinates (t, z) by

$$p_i(z) = (\alpha_i z + \beta_i, z)\,,\ \alpha_i, \beta_i \in \mathbb{F}_q\,.$$

Let $D = p_1(U) + \cdots + p_n(U)$ and let $\mathcal{L}$ be the invertible sheaf on X

$$\mathcal{L} = \pi_1^*\mathcal{O}_{\mathbb{P}^1}(rp_\infty) \otimes_{\mathbb{F}_q} \mathcal{O}_U\,,\ r < n\,,$$

The exact sequence (3.5) is in this case:

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{L}) \arrow[r, "\alpha"] \arrow[d, equal] & H^0(X,\mathcal{O}_D) \arrow[r] \arrow[d, equal] & H^1(X,\mathcal{L}(-D)) \arrow[r] & 0\,. \\
 & H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(rp_\infty)) \otimes \mathbb{F}_q[z] \arrow[r, "\alpha"] & \mathbb{F}_q[z]^n & &
\end{tikzcd}$$

Taking the fibres over the generic point η, and the canonical trivialization $(\pi_*\mathcal{O}_D)_\eta \simeq \mathbb{F}_q(z)^n$, the homomorphism α_η is the evaluation map at the points $p_1(\eta), \dots, p_n(\eta)$

$$\begin{aligned}&\alpha_\eta\colon H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(rp_\infty)) \otimes_{\mathbb{F}_q} \mathbb{F}_q(z) \to \mathbb{F}_q(z)^n\\ &\alpha_\eta(t^j) = \big(t^j(p_1(\eta)), \dots, t^j(p_n(\eta))\big) = \big((\alpha_1 z + \beta_1)^j, \dots, (\alpha_n z + \beta_n)^j\big),\end{aligned}$$

where $\{1, t, \dots, t^r\}$ is the "canonical" basis of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(rp_\infty))$ in the affine coordinate t. The convolutional code $\mathcal{C}(\mathcal{L}, D)$ is a kind of *generalized Reed-Solomon (RS) code* (for $z = 0$ we obtain a classical RS-code).

Let $\Gamma \subseteq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(rp_\infty))$ be the linear subspace generated by $\{t^s, \dots, t^r\}$. The convolutional Goppa code $\mathcal{C}(\Gamma, D)$ is the image of the homomorphism

$$\begin{aligned}\alpha_\eta\colon \Gamma \otimes_{\mathbb{F}_q} \mathbb{F}_q(z) &\to \mathbb{F}_q(z)^n\\ t^j &\longmapsto \alpha_\eta(t^j)\,, \quad \text{for } s \le j \le r.\end{aligned}$$

In this case $H^0(X, \mathcal{L})/\Gamma \simeq \big(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(rp_\infty))/\Gamma\big) \otimes_{\mathbb{F}_q} \mathbb{F}_q[z]$ is torsion-free. Then, by Proposition 3.8 every matrix defining

$$\alpha\colon \Gamma \otimes_{\mathbb{F}_q} \mathbb{F}_q[z] \to H^0(X, \mathcal{O}_D)$$

is a basic generator matrix. To compute a matrix for α explicitly, we need to fix an isomorphism of $\mathbb{F}_q[z]$-modules

$$H^0(X, \mathcal{O}_D) \xrightarrow{\phi} \mathbb{F}_q[z]^n,$$

and this gives a generator matrix for $\mathcal{C}(\Gamma, D, \phi)$. However, it would be desirable to compute basic matrices for the codes $\mathcal{C}(\Gamma, D)$. We shall do this in general in a forthcoming paper. Here we shall offer some explicit examples.

Example 4.1 Let $a, b \in \mathbb{F}_q$ be two different non-zero elements, and

$$p_i(z) = (a^{i-1}z + b^{i-1}, z)\,, i = 1, \dots, n\,, \text{ with } n < q.$$

The evaluation map α_η over Γ is defined by the matrix

$$\begin{pmatrix} (z+1)^s & (az+b)^s & (a^2z+b^2)^s & \dots & (a^{n-1}z+b^{n-1})^s \\ (z+1)^{s+1} & (az+b)^{s+1} & (a^2z+b^2)^{s+1} & \dots & (a^{n-1}z+b^{n-1})^{s+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (z+1)^r & (az+b)^r & (a^2z+b^2)^r & \dots & (a^{n-1}z+b^{n-1})^r \end{pmatrix}. \tag{4.1}$$

This matrix is a generator matrix for the code $\mathcal{C}(\Gamma, D)$. Using this construction we can give concrete examples of CGC of dimension $k = r - s + 1$ that are Maximum-Distance Separable (MDS) convolutional codes, i.e., whose *free distance* d attains the generalized Singleton bound $d \le (n-k)(\lfloor \delta/k \rfloor + 1) + \delta + 1$, where δ is the degree of the code. ([10] Th. 2.2 and Definition 2.5).

- If $s = r$, the convolutional Goppa code $\mathcal{C}(\Gamma, D)$ has dimension 1, degree r, and (4.1) is a *canonical* (reduced and basic [7]) generator matrix. We can list a few examples, where k/n, δ and d are respectively the rate, the degree and the free distance of the code.

field	*canonical generator matrix*	k/n	δ	d
$\mathbb{F}_3 = \{0, 1, 2\}$	$\begin{pmatrix} z+1 & z+2 \end{pmatrix}$	1/2	1	4
$\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$ where $\alpha^2 + \alpha + 1 = 0$	$\begin{pmatrix} z+1 & z+\alpha & z+\alpha^2 \end{pmatrix}$	1/3	1	6
$\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$	$\begin{pmatrix} (z+1)^2 & (z+2)^2 & (z+4)^2 \end{pmatrix}$	1/3	2	9

In these examples the sections $p_1, \dots, p_n$ are disjoint, such that $\mathcal{C}(\Gamma, D) = \mathcal{C}(\Gamma, D, \phi)$, where $\phi\colon H^0(X, \mathcal{O}_D) \overset{\sim}{\to} \mathbb{F}_q[z]^n$ is the corresponding canonical trivialization.

- If $s < r$, let us take $a \in \mathbb{F}_q$ as a primitive element.

  Now, the matrix (4.1) is reduced, since the matrix of highest-degree terms in each row is a Vandermonde matrix of rank k. The sections $p_1, \dots, p_n$ are not disjoint, but in some cases the matrix (4.1) is actually basic and we do not have to find an isomorphism of $\mathbb{F}_q[z]$-modules, $\phi\colon H^0(X, \mathcal{O}_D) \overset{\sim}{\to} \mathbb{F}_q[z]^n$, in order to compute a basic generator matrix for the code $\mathcal{C}(\Gamma, D)$.

  We present two examples of this situation.

field	*canonical generator matrix*	k/n	δ	d
$\mathbb{F}_4$	$\begin{pmatrix} 1 & 1 & 1 \\ z+1 & \alpha z+\alpha^2 & \alpha^2 z+\alpha \end{pmatrix}$	2/3	1	3
$\mathbb{F}_5$	$\begin{pmatrix} z+1 & 2z+3 & 4z+4 & 3z+2 \\ (z+1)^2 & (2z+3)^2 & (4z+4)^2 & (3z+2)^2 \end{pmatrix}$	1/2	3	8

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