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Contact problems with respect to the formation and propagation of heat

Kontaktní problémy se zahrnutím vzniku a šíření tepla

2014

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VŠB - Technická univerzita Ostrava
Fakulta elektrotechniky a informatiky
Katedra aplikované matematiky

Zadání diplomové práce

Student: **Bc. Petr Haškovec**

Studijní program: N2647 Informační a komunikační technologie

Studijní obor: 1103T031 Výpočetní matematika

Téma: Kontaktní problémy se zahrnutím vzniku a šíření tepla
Contact problems with respect to the formation and propagation of heat

Zásady pro vypracování:

V současnosti obsahuje knihovna MatSol vyvíjená na katedře 470 jak schopnost počítat úlohy (nelineární) elasticity, tak úlohy přestupu a šíření tepla. V této práci se budeme věnovat vzniku a přestupu tepla v důsledku kontaktu se třením. Práce naváže na současný stav knihovny MatSol a rozšíří její možnosti.

Seznam doporučené odborné literatury:

S. Hueber, B.I. Wohlmuth: Thermo-mechanical contact problems on non-matching meshes P. Wriggers: Nonlinear Finite Element Methods

další dle pokynů vedoucího diplomové práce

Formální náležitosti a rozsah diplomové práce stanoví pokyny pro vypracování zveřejněné na webových stránkách fakulty.

Vedoucí diplomové práce: **Ing. Oldřich Vlach, Ph.D.**

Datum zadání: 01.09.2013
Datum odevzdání: 07.05.2014

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I declare that I created this thesis myself. I have stated all sources and publications that I had used.

Ostrava, May 4, 2014 ....

This work was supported by the IT4Innovations Centre of Excellence project (CZ.1.05/1.1.00/02.0070), funded by the European Regional Development Fund and the national budget of the Czech Republic via the Research and Development for Innovations Operational Programme, as well as Czech Ministry of Education, Youth and Sports via the project Large Research, Development and Innovations Infrastructures (LM2011033).

Work is partially supported by Grant of SGS No. SP2014/204, VŠB - Technical University of Ostrava, Czech Republic.

I would like to express my thanks to the supervisor of my thesis, Ing. Oldřich Vlach, Ph.D., for explanation and for his help, which was above standard. Furthermore I would like to thank to Ing. Tomáš Brzobohatý, Ph.D. for introduction into MatSol library functioning and to Ing. Alexandros Makropoulos, Ph.D. for advice and suggestions. I appreciate all those who contributed to my work by a suggestion or an explanation.

Abstract

This thesis focuses mainly on the derivation of mathematical model of contact thermodynamics, which can be apply on numerical algorithm. We consider problem, where the thermal expansion affects displacement and vice versa where the displacement of solid influences temperature of the material. This happens especially on the contact boundary of each body, where we consider Coulomb friction, which depends on pressure on contact boundary. Firstly from physical equations we derive weak formulations and discretize them in space and time. We transform this equations into dual problem in Lagrange multipliers, which are more sutiable for numerical computation. In last part we describe the process of finding the numerical solution. At the end reader can see solution of problem based on process derived in this text.

Keywords: contact dynamics problems, minimization problem, Coulomb friction, thermomechanical problems, elasticity, heat transfer

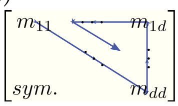
Abstrakt

Hlavním cílem této diplomové práce je odvození matematického modelu termo-dynamického kontaktního problému, který by byl jednoduše aplikovatelný na numerický výpočet. Uvažujeme úlohu ve které je deformace těles ovlivněna vznikem tepla a jeho šířením, a opačně také vliv deformace těles na teplotu materiálu. K tomu dochází především v místě kontaktu, kde uvažujeme Coulombovský model tření závislý na tlaku na kontaktní hranici. Nejprve z fyzikálního modelu odvozujeme slabé formulace, které diskretizujeme v prostoru a čase. Tyto tvary převádíme na duální problém v Lagrangeovských multiplikátorech, které jsou vhodnější pro numerický výpočet. V poslední části popisujeme postup numerického řešení. Na závěr čtenář nalezne řešení úlohy pomocí postupu představeného v tomto textu.

Klíčová slova: dynamické kontaktní problémy, úlohy minimalizace, Coulombovské tření, deformace, šíření tepla

Notation

d	dimmension $d \in \{2, 3\}$
$\Omega \subset \mathbb{R}^d$	domain
el.	element in discretized domain
dof	degree of freedom
f	scalar values or unknowns (i.e. $f : \Omega \to \mathbb{R}$)
$\underline{v}$	vector values or unknowns (i.e. $\underline{v} : \Omega \to \mathbb{R}^d$)
$\underline{\underline{T}}$	tensor values or unknowns (i.e. $\underline{\underline{T}} : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$)
$\underline{\mathrm{v}}$	algebraic vector values or unknowns (i.e. $\underline{\mathrm{v}} \in \mathbb{R}^n$)
$\underline{\underline{\mathrm{M}}}$	algebraic matrices values or unknowns (i.e. $\underline{\underline{\mathrm{M}}} \in \mathbb{R}^{n\times m}$)
$\underline{\mathrm{SVoi}}\left(\underline{\underline{\mathrm{M}}}_{\mathrm{sym}}\right)$	vectorized values of symmetric matrix or tensor in symmetric Voigt notation:



$$\underline{\mathrm{SVoi}}\left(\underline{\underline{\mathrm{M}}}_{\mathrm{sym}}\right) = \underline{\mathrm{SVoi}}\begin{bmatrix} m_{11} & \cdots & m_{1d} \\ & \ddots & \vdots \\ sym. & & m_{dd} \end{bmatrix}$$

$$= \begin{bmatrix} m_{11} & \cdots & m_{dd} & \frac{1}{\sqrt{2}} m_{d(d-1)} & \cdots & \frac{1}{\sqrt{2}} m_{1d} & \cdots & \frac{1}{\sqrt{2}} m_{12} \cdots \end{bmatrix}^\top$$

Differential operations

$\partial_i f$	partial derivative
$\underline{\nabla}(f) = \nabla f$	gradient of scalar function: $\underline{\nabla}(f) = [\partial_1 f, \dots, \partial_d f]^\top$
$\operatorname{div}(\underline{v})$	divergence of vector field: $\operatorname{div}(\underline{v}) = \underline{\nabla} \cdot \underline{v} = \partial_i v_i = \partial_1 v_1 + \cdots + \partial_d v_d$
$\underline{\operatorname{div}}(\underline{\underline{T}})$	divergence of tensor field:

$$\underline{\operatorname{div}}(\underline{\underline{T}}) = \underline{\underline{T}}\,\underline{\nabla} = [\partial_k T_{1k}, \dots, \partial_k T_{dk}]^\top = \begin{bmatrix} \partial_1 T_{11} & + \cdots + & \partial_d T_{1d} \\ \vdots & \ddots & \vdots \\ \partial_1 T_{d1} & + \cdots + & \partial_d T_{dd} \end{bmatrix}.$$

Specially we have

$$\underline{\mathrm{div}}(f\underline{\underline{\mathrm{Id}}}) = [\partial_k f \mathrm{Id}_{1k}, \ldots, \partial_k f \mathrm{Id}_{dk}]^\top = [\partial_1 f, \ldots, \partial_d f]^\top = \underline{\nabla}(f)$$

$\underline{\underline{\nabla}}(\underline{v})$ gradient of vector function:

$$\underline{\underline{\nabla}}(\underline{v}) = \begin{bmatrix} (\underline{\nabla}(v_1))^\top \\ \vdots \\ (\underline{\nabla}(v_d))^\top \end{bmatrix} = \begin{bmatrix} \partial_1 v_1 & \cdots & \partial_d v_1 \\ \vdots & \ddots & \vdots \\ \partial_1 v_d & \cdots & \partial_d v_d \end{bmatrix}$$

$\underline{\underline{\nabla}}_{\mathrm{sym}}(\underline{v})$ symmetrized gradient of vector function:

$$\underline{\underline{\nabla}}_{\mathrm{sym}}(\underline{v}) = \mathrm{sym}(\underline{\underline{\nabla}}(\underline{v})) = \mathrm{sym}\left(\begin{bmatrix} (\underline{\nabla}(v_1))^\top \\ \vdots \\ (\underline{\nabla}(\nabla v_d))^\top \end{bmatrix} \right) = \begin{bmatrix} \partial_1 v_1 & \cdots & \frac{1}{2}(\partial_d v_1 + \partial_1 v_d) \\ & \ddots & \vdots \\ sym. & & \partial_d v_d \end{bmatrix}$$

$\Delta(f)$ Laplace operator of scalar

$$\Delta(f) = \underline{\nabla} \cdot \underline{\nabla}(f) = \partial_{ii} f = \partial_{11} f + \cdots + \partial_{dd} f$$

Binary operations

$\underline{v}\,\underline{w} = v_i w_i \in \mathbb{R}$ scalar product

$\underline{\underline{T}}\,\underline{\underline{U}} = T_{ik} U_{kj} \in \mathbb{R}^{d \times d}$ matrix product

$\underline{\underline{T}} : \underline{\underline{U}} = T_{ij} U_{ij} \in \mathbb{R}$ Frobenius inner product ($\underline{\underline{T}} : \underline{\underline{U}} = \mathrm{tr}(\underline{\underline{T}}^\top \underline{\underline{U}}) = \mathrm{tr}(\underline{\underline{T}}\,\underline{\underline{U}}^\top)$)

Contents

Introduction

The research of solid bodies moves from engineering into computational mechanics. Because of continuing development of computational systems, we are able to easily simulate behavior of body. That can be cheaper and faster in computer, than manufacture thing from the examined material and test it. This is also true in field of contact mechanics.

Contact mechanics is discipline that deals with the analysis of system of deformable bodies. This bodies are exposed to the influences of the real world, for example surface force and heat flux (like in our case), but others e.g. volume forces. To the describtion of the model is needed not only balance equation from continuum mechanics, like in one-body case, but also conditions for describtion of more bodies. Here belongs nonpenetrability condition and friction condition. There are several friction conditions, we are taking into account Coulomb friction, which depends not only on material structure on contact part, but also on the pressure between bodies. This influences formation of heat and motion of body.

Usually these types of problems take into account just one variable heat or elasticity, but not both. This makes these types of problems more complex for analysis. In this thesis we study contact dynamics problem with frictional heating and we are interested in displacement and heat distribution of two bodies in time. We are taking into account heating formed due to deformation and especially from friction on contact part. On the other hand, heating can affect deformations. This work mainly draws from S. Hüeber and B.I. Wohlmuth's paper [1] and monography by Peter Wriggers [2].

In first sections we are introducing balance equations for the linear thermo-elasticity. We have local momentum of balance, given by equation, where beyond common terms is couple-term (i.e. temperature affect deformation and vice versa). In the second part we derive weak formulations of this balance equations. Then apply space discretization and choosing general base functions show, how to assemble mass-, stiffness- and contact boundary matrices, vectors of load forces and contact conditions vector. Next, we discretize this equations in time, using Newmark discretization scheme [7] and backward scheme. In final part of derivation we show, how to get equivalent formulation of problem using minimization of the energy functional.

Practical part of work is especially about extension of the Matlab MatSol library, which is developed by the team of IT4Innovations. About MatSol see [5, 6]. We implemented *MatSol example* in 3D for dynamics contact problem of two bodies. In algorithm are procedures using FETI method, Lumped preconditoning and Mortar technique already included in MatSol, so we don't deal with clarification of these methods. We use one type of geometry, and we are curious about heat transfer between bodies and about formation of heat due to friction between solids. This is demonstrated in different tasks. Solutions obtained in this process is presented at the end of this thesis.

1 Model problem

In this thesis we study two bodies and their deformation and heat transfer. We consider two deformable bodies in their reference configuration $\Omega^i \subset \mathbb{R}^d$, $i \in \{m, s\}$, where $d = 2, 3$ is dimension of problem, m stands for the master body and s for the slave body. We are interested in displacement field $\underline{u}^i(\underline{x}, t)$ and the temperature $\theta^i(\underline{x}, t)$, for $(\underline{x}, t) \in \Omega^i \times (0, T)$, where $(0, T)$ is the given time interval. The boundery $\partial\Omega^i$ is Lipschitz and is divided in this way. For elasticity holds

$$\begin{aligned} \Gamma^i &= \overline{\Gamma}^i_{uD} \cup \overline{\Gamma}^i_{uN} \cup \overline{\Gamma}^i_{C}, \\ \emptyset &= \Gamma^i_{uD} \cap \Gamma^i_{uN} = \Gamma^i_{uN} \cap \Gamma^i_{C} = \Gamma^i_{uD} \cap \Gamma^i_{C}, \end{aligned}$$

and for heat transfer holds

$$\begin{aligned} \Gamma^i &= \overline{\Gamma}^i_{\theta D} \cup \overline{\Gamma}^i_{\theta N} \cup \overline{\Gamma}^i_{C}, \\ \emptyset &= \Gamma^i_{\theta D} \cap \Gamma^i_{\theta N} = \Gamma^i_{\theta N} \cap \Gamma^i_{C} = \Gamma^i_{\theta D} \cap \Gamma^i_{C}. \end{aligned}$$

Moreover $\overline{\Gamma}^i_{\theta D} \cup \overline{\Gamma}^i_{\theta N} = \overline{\Gamma}^i_{\mathrm{u}D} \cup \overline{\Gamma}^i_{uN}$. The portion $\Gamma^i_{\cdot D}$ represents the part of the boundary where displacement or temperature are prescribed. $\Gamma^i_{\cdot N}$ is the part, where tractions or heat flow are prescribed. Finally Γ^i_C is the part of possible contact between master and slave bodies which is equal for elasticity and heat problem. Inside area Ω^i can be prescribed body force for elastictiy or heat source for heat transfer problem.

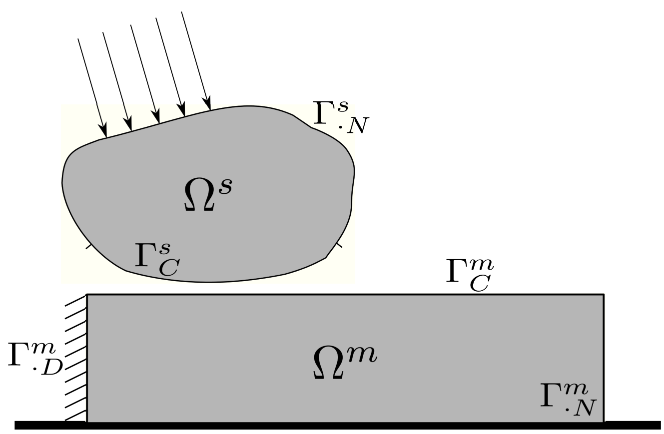


Figure 1.1: Master and slave body.

2 Strong formulation

In this part we show separately elasticity problem with given boundary and initial conditions and similary heat transfer problem with given conditions. How to obtain local momentum of balance equations can be seen in [3, 4]. For both cases let $\Omega^i \subset \mathbb{R}^d$ be a bounded domain with the Lipschity boundary described in previous section.

2.1 Elasticity

For $i \in \{m, s\}$ we have local momentum of balance equation with Dirichlet and Neumann boundary conditions and initial conditons for each body. We are looking for
$\underline{u}(\underline{x}, t) : \Omega^i \times (0, T) \to \mathbb{R}^d$ which is called strong solution of a contact problem and it satisfies following five equations:

$$\varrho^i \underline{\ddot{u}}^i - \underline{\mathrm{div}}(\underline{\underline{P}}^i(\underline{u}^i, \theta^i)) = \underline{f}^i, \qquad \text{in } \Omega^i \times (0, T), \tag{2.1}$$

$$\underline{u}^i = \underline{u}^i_D, \qquad \text{on } \Gamma^i_{uD} \times (0, T), \tag{2.2}$$

$$\underline{\underline{\sigma}}^i(\underline{u}^i)\underline{n}^i_0 = \underline{p}^i_N, \qquad \text{on } \Gamma^i_{uN} \times (0, T), \tag{2.3}$$

$$\underline{\dot{u}}^i = \underline{\dot{u}}^i_0, \quad \underline{u}^i = \underline{u}^i_0, \qquad \text{in } \Omega^i \times \{0\}, \tag{2.4}$$

where ρ is density of material, $\underline{n}_0$ is outward unit normal vector in reference configuration on Γ^i, and the first Piola-Kirchhoff stress tensor is

$$\underline{\underline{P}}^i(\underline{u}^i, \theta^i) = \underline{\underline{\sigma}}^i(\underline{u}^i) + dK^i\alpha^i(\theta^i - \theta_0)\underline{\underline{\mathrm{Id}}},$$

where $\underline{\underline{\sigma}}^i$ is the linear stress tensor, K^i is the bulk modulus, α^i is the thermal expansion coefficient and θ_0 is the reference temperature at which the bodies are stress free which we assume to be constant on $\Omega^s \cup \Omega^m$. From linear elasticity (Hooke's law) we have fourth order elasticity tensor $\underline{\underline{\underline{\underline{c}}}} = c_{ijkl}$, Cauchy's strain tensor $\underline{\underline{\varepsilon}} = \varepsilon_{ij}$ and holds

$$\underline{\underline{\sigma}}^i(\underline{u}^i) = \underline{\underline{\underline{\underline{c}}}} : \underline{\underline{\varepsilon}}^i(\underline{u}^i) = \lambda^i \mathrm{tr}(\underline{\underline{\varepsilon}}^i)\underline{\underline{\mathrm{Id}}} + 2\mu^i\underline{\underline{\varepsilon}}^i, \quad \underline{\underline{\varepsilon}}^i(\underline{u}^i) = \frac{1}{2}\left(\underline{\underline{\nabla}}(\underline{u}^i) + \underline{\underline{\nabla}}^\top(\underline{u}^i)\right).$$

Where Lamé parameters λ, μ are equal to

$$\lambda^i = \frac{E^i\nu^i}{(1+\nu^i)(1-2\nu^i)}, \quad \mu^i = \frac{E^i}{2(1+\nu^i)}, \quad K^i = \lambda^i + \frac{2}{d}\mu^i,$$

depending on the Young's modulus $E^i \in \mathbb{R}^+$ and Poisson ratio $\nu^i \in (0, 0.5)$.

Moreover we can modify term $-\underline{\mathrm{div}}(\underline{\underline{P}}^i(\underline{u}^i))$ in (2.1), so we get

$$\varrho^i \underline{\ddot{u}}^i - \underline{\mathrm{div}}(\underline{\underline{\sigma}}^i(\underline{u}^i)) + dK^i\alpha^i\underline{\nabla}\theta^i = \underline{f}^i, \quad \text{in } \Omega^i \times (0, T). \tag{2.5}$$

For further use denote

$$
\begin{aligned}
\underline{p}^i(\underline{u}^i) &= \underline{\underline{\sigma}}^i(\underline{u}^i)\underline{n}_0^i,\\
p_n^i(\underline{u}^i) &= \underline{\underline{\sigma}}^i(\underline{u}^i)\underline{n}_0^i \cdot \underline{n}_0^i,\\
\underline{p}_\tau^i(\underline{u}^i) &= \underline{p}^i(\underline{u}^i) - p_n^i(\underline{u}^i)\underline{n}_0^i,
\end{aligned}
$$

where p_n^i is normal pressure, p_τ^i is tangential pressure. For the definition of linearized contact conditions we assume a smooth mapping $\underline{R}_t(\underline{x}^s) : \Gamma_C^s \to \Gamma_C^m$ such that

- $\underline{R}_t(\Gamma_C^s) \subset \Gamma_C^m$
- the vector $\underline{\varphi}^s(\underline{x}^s, t) - \underline{\varphi}^m(\underline{R}_t(\underline{x}^s), t)$ is parallel with the actual outer normal $\underline{n}(\underline{x}^s, t)$ to Γ_C^s in $\underline{\varphi}^s(\underline{x}^s, t)$
- actual gap $g(\underline{x}^s, t) := [\underline{\varphi}^s(\underline{x}^s, t) - \underline{\varphi}^m(\underline{R}_t(\underline{x}^s), t)]\underline{n}(\underline{x}^s, t)$

Using the above defined mapping we define the jump of the vector function $\underline{w}(\underline{x}, t)$ across Γ_C^s and its normal and tangential part

$$
\begin{aligned}
[\underline{w}](\underline{x}^s, t) &:= \underline{w}^s(\underline{x}^s, t) - \underline{w}^m(\underline{R}_t(\underline{x}^s), t) \quad \forall \underline{x}^s \in \Gamma_C^s\\
[w]_n(\underline{x}^s, t) &:= [\underline{w}](\underline{x}^s, t) \cdot \underline{n}(\underline{x}^s, t)\\
[\underline{w}]_\tau(\underline{x}^s, t) &:= [\underline{w}](\underline{x}^s, t) - [w]_n(\underline{x}^s, t)\underline{n}(\underline{x}^s, t)
\end{aligned}
$$

There are two conditions on Γ_C^i. In the outward normal direction is prescribed the non-penetrability, i.e. $\forall \underline{x}^s \in \Gamma_C^s$

$$
\begin{aligned}
0 \geq g(\underline{x}^s, t) &= [\underline{\varphi}^s(\underline{x}^s, t) - \underline{\varphi}^m(\underline{R}_t(\underline{x}^s), t)]\underline{n}(\underline{x}^s, t)\\
&= [u]_n(\underline{x}^s, t) + [\underline{x}^s - \underline{R}_t(\underline{x}^s)]\underline{n}(\underline{x}^s, t) = ([u]_n - g_0)(\underline{x}^s, t)\\
0 \leq p_n(\underline{x}^s, t) &= \left(\underline{\underline{\sigma}}(\underline{u})\underline{n} \cdot \underline{n}\right)(\underline{x}^s, t)\\
0 &= (p_n g)(\underline{x}^s, t)
\end{aligned}
\tag{2.6}
$$

and friction condition

$$
\begin{aligned}
\|\underline{p}_\tau^i(\underline{u}^i)\| - \mathfrak{F}|p_n^i(\underline{u}^i)| &\leq 0\\
[\dot{\underline{u}}]_\tau + \beta^2 \underline{p}_\tau(\underline{u}) &= 0\\
[\dot{\underline{u}}]_\tau \left(\|\underline{p}_\tau^i(\underline{u}^i)\| - \mathfrak{F}|p_n^i(\underline{u}^i)|\right) &= 0
\end{aligned}
\tag{2.7}
$$

Problem to solve is given by the equilibrium condition (2.5), the boundary conditions (2.2), (2.3), the initial conditions (2.4) and the mechanical contact conditions (2.6) and (2.7).

2.2 Heat transfer

Similarly we are looking for $\theta^i(\underline{x}, t) : \Omega^i \times (0, T) \to \mathbb{R}$, which is strong solution of the heat conduction equation. We can get this equation from the first law of thermodynamics, and add prescribed boundary and initial conditions:

$$\begin{aligned} c^i\dot{\theta}^i - \mathrm{div}(\overbrace{\kappa^i \nabla\theta^i}^{-\underline{q}^i}) + \mathcal{H}^i(\underline{\dot{u}}) &= r^i, && \text{in } \Omega^i \times (0,T), && (2.8)\\ \theta^i &= \theta^i_D, && \text{on } \Gamma^i_{\theta D} \times (0,T), && (2.9)\\ \underline{q}^i \underline{n}^i_0 &= q^i_N, && \text{on } \Gamma^i_{\theta N} \times (0,T), && (2.10)\\ \underline{q}^i \underline{n}^i &= q^i_C, && \text{on } \Gamma^i_{\theta C} \times (0,T), && (2.11)\\ \theta^i &= \theta^i_0, && \text{in } \Omega^i \times \{0\}, && (2.12)\end{aligned}$$

where $\kappa^i > 0$ is the thermal conductivity, c^i is the specific heat capacity of the body Ω^i, $\underline{q}^i$ is the heat flux, r^i is the prescribed heat source and the heating term from the Joule effect is

$$\mathcal{H}^i = d\alpha^i K^i \theta_0 \mathrm{div}(\underline{\dot{u}}^i).$$

The condition for heat flux on contact boundary $q^i_C := \underline{q}^i \underline{n}^i$ can be written as

$$q^s_C = \gamma^s_C p_n(\theta^s - \theta_0), \quad q^m_C = \gamma^m_C p_n(\theta^m - \theta_0), \quad 0 = q^s_C + q^m_C + \underline{p}_\tau [\underline{\dot{u}}]_\tau, \qquad (2.13)$$

let's note, that material coefficient $\gamma^i_C(p_n)$, which affects heat transfer depending on presure is used as a linear model $\gamma^i_C(p_n) := \gamma^i_C p_n$. From previous equations follows

$$\begin{aligned} q^s_C &= \beta_C p_n [\theta] - \delta_C \mathfrak{F} p_n \|[\underline{\dot{u}}]_\tau\|,\\ q^m_C &= -\beta_C p_n [\theta] - (1-\delta_C)\mathfrak{F} p_n \|[\underline{\dot{u}}]_\tau\|,\\ \beta_C &= \frac{\gamma^s_C \gamma^m_C}{\gamma^s_C + \gamma^m_C}, \quad \delta_C = \frac{\gamma^s_C}{\gamma^s_C + \gamma^m_C}, \quad [\theta] = \theta^s - \theta^m,\end{aligned}$$

because

$$0 = q^s_C + q^m_C + \underline{p}_\tau [\underline{\dot{u}}]_\tau = p_n \left[\gamma^s_C(\theta^s - \theta_0) + \gamma^m_C(\theta^m - \theta_0)\right] + \underline{p}_\tau [\underline{\dot{u}}]_\tau.$$

From that follows

$$
\begin{aligned}
\theta_0 &= \frac{\gamma_C^s \theta^s + \gamma_C^m \theta^m}{\gamma_C^s + \gamma_C^m} + \frac{\underline{p}_\tau [\underline{\dot{u}}]_\tau}{p_n(\gamma_C^s + \gamma_C^m)}, \\
q_C^s &= \gamma_C^s p_n \left(\theta^s - \frac{\gamma_C^s \theta^s + \gamma_C^m \theta^m}{\gamma_C^s + \gamma_C^m} + \frac{\underline{p}_\tau [\underline{\dot{u}}]_\tau}{p_n(\gamma_C^s + \gamma_C^m)} \right) \\
&= p_n \left(\gamma_C^s \theta^s \left(1 - \frac{\gamma_C^s}{\gamma_C^s + \gamma_C^m} \right) - \frac{\gamma_C^s \gamma_C^m \theta^m}{\gamma_C^s + \gamma_C^m} \right) + \frac{\gamma_C^s}{\gamma_C^s + \gamma_C^m} \underline{p}_\tau [\underline{\dot{u}}]_\tau \\
&= p_n \beta_C \, [\theta] - \delta_C \|\underline{p}_\tau\| \|[\underline{\dot{u}}]_\tau\| = p_n \beta_C \, [\theta] - \delta_C \mathfrak{F} p_n \|[\underline{\dot{u}}]_\tau\|, \\
q_C^m &= \gamma_C^m p_n \left(\theta^m - \frac{\gamma_C^s \theta^s + \gamma_C^m \theta^m}{\gamma_C^s + \gamma_C^m} + \frac{\underline{p}_\tau [\underline{\dot{u}}]_\tau}{p_n(\gamma_C^s + \gamma_C^m)} \right) \\
&= p_n \left(\gamma_C^m \theta^m \left(1 - \frac{\gamma_C^m}{\gamma_C^s + \gamma_C^m} \right) - \frac{\gamma_C^s \gamma_C^m \theta^s}{\gamma_C^s + \gamma_C^m} \right) + \frac{\gamma_C^m}{\gamma_C^s + \gamma_C^m} \underline{p}_\tau [\underline{\dot{u}}]_\tau \\
&= -p_n \beta_C \, [\theta] - (1 - \delta_c) \mathfrak{F} p_n \|[\underline{\dot{u}}]_\tau\| = -q_C^s - \mathfrak{F} p_n \|[\underline{\dot{u}}]_\tau\|,
\end{aligned}
$$

which completes the way of derivation of the above formulas. We should stress that in the previous adjustments we used the relations

$$
\underline{p}_\tau [\underline{\dot{u}}]_\tau = -\|\underline{p}_\tau\| \|[\underline{\dot{u}}]_\tau\| \text{ and } \|\underline{p}_\tau\| \|[\underline{\dot{u}}]_\tau\| = \mathfrak{F} p_n \|[\underline{\dot{u}}]_\tau\|.
$$

Now the heat transfer problem is given by heat conduction equation (2.8), the boundary conditions (2.9)-(2.11), the initial condition (2.12) and the thermal flow conditions on contact interface (2.13).

3 Weak formulation

In this section we demonstrate how to get weak formulations from strong forms in previous section. First of all we need to define spaces of test functions for each body in given problem and sets of admissible displacements.

3.1 Spaces definition

In this part we define space of test functions, define

$$
\begin{aligned}
V^i &:= H^1(\Omega^i), & V &= V^s \times V^m,\\
V^i_D &:= \left\{\chi^i \in V^i \middle|\, T(\chi^i) = \theta^i_D \text{ on } \Gamma^i_{\theta D}\right\}, & V_D &= V^s_D \times V^m_D,\\
V^i_0 &:= \left\{\chi^i \in V^i \middle|\, T(\chi^i) = 0 \text{ on } \Gamma^i_{\theta D}\right\}, & V_0 &= V^s_0 \times V^m_0,\\
\underline{V}^i &:= \left(V^i\right)^d, & \underline{V} &= \underline{V}^s \times \underline{V}^m,\\
\underline{V}^i_D &:= \left\{\underline{v}^i \in \underline{V}^i \middle|\, T(\underline{v}^i) = \underline{u}^i_D \text{ on } \Gamma^i_{uD}\right\}, & \underline{V}_D &= \underline{V}^s_D \times \underline{V}^m_D,\\
\underline{V}^i_0 &:= \left\{\underline{v}^i \in \underline{V}^i \middle|\, T(\underline{v}^i) = \underline{0} \text{ on } \Gamma^i_{uD}\right\}, & \underline{V}_0 &= \underline{V}^s_0 \times \underline{V}^m_0,
\end{aligned}
$$

and convex sets

$$
\begin{aligned}
\underline{K}(t) &:= \left\{\underline{v} \in \underline{V}_D \middle|\, [\underline{v}]_n (\underline{x}^s, t) \le g_0(\underline{x}^s, t)\right\},\\
\underline{M}(\gamma) &:= \left\{\underline{\mu} \in \underline{V}' \middle|\, \mu_n \ge 0,\ \|\underline{\mu}_\tau\| \le \mathfrak{F}\gamma\right\},
\end{aligned}
$$

where $\underline{K}(t)$ is a convex set of functions satisfying the non-penetration condition (2.6), and $\underline{M}(\gamma)$ is convex set of Lagrange multipliers.

Theoretically, we are looking for solution from $\underline{V}_D$ for elasticity and from V_D for heat transfer. In this phase its too difficult, because of continous functions, but that will be discussed later.

3.2 Elasticity

Multiplying (2.5) by $\underline{w} \in \underline{V}_0$, and integrating over Ω^i we get

$$
\int_{\Omega^i} \varrho^i \ddot{\underline{u}}^i \underline{w}^i \, d\underline{x} - \int_{\Omega^i} \underline{\operatorname{div}}(\underline{\underline{\sigma}}^i(\underline{u}^i))\underline{w}^i \, d\underline{x} + \int_{\Omega^i} dK^i \alpha^i \underline{\nabla}\theta^i \underline{w}^i \, d\underline{x} = \int_{\Omega^i} \underline{f}^i \underline{w}^i \, d\underline{x}, \quad \forall \underline{w} \in \underline{V}_0,
$$

apply Green's theorem on term with $\underline{\text{div}}(\cdot)$

$$
\begin{aligned}
-\int_{\Omega^i} \underline{\text{div}}\left(\underline{\underline{\sigma}}^i\left(\underline{u}^i\right)\right)\underline{w}^i\,d\underline{x} &= \\
= -\int_{\partial\Omega^i} \underline{\underline{\sigma}}^i\left(\underline{u}^i\right)\underline{w}^i\underline{n}_0^i\left(\underline{x}^i\right)dS(\underline{x}) &+ \int_{\Omega^i} \underline{\underline{\sigma}}^i\left(\underline{u}^i\right) : \underline{\nabla}\left(\underline{w}^i\right)d\underline{x}, \quad \forall \underline{w} \in \underline{V}_0.
\end{aligned}
$$

Integral over boundary Ω^i divide into sum of integrals over Γ_D, Γ_N and Γ_C

$$
\begin{aligned}
\int_{\partial\Omega^i} \underline{\underline{\sigma}}^i\left(\underline{u}^i\right)\underline{w}^i\underline{n}_0^i\left(\underline{x}^i\right)dS(\underline{x}) &= \int_{\Gamma_D} \underline{\underline{\sigma}}^i\left(\underline{u}^i\right)\overbrace{\underline{w}^i}^{=0}\,\underline{n}_0^i\left(\underline{x}^i\right)dS(\underline{x}) + \\
&+ \int_{\Gamma_N} \overbrace{\underline{\underline{\sigma}}^i\left(\underline{u}^i\right)\underline{n}_0^i}^{\underline{p}_N^i}\,\underline{w}^i\left(\underline{x}^i\right)dS(\underline{x}) + \int_{\Gamma_C} \overbrace{\underline{\underline{\sigma}}^i\left(\underline{u}^i\right)\underline{n}_0^i}^{\underline{\lambda}}\,\underline{w}^i\left(\underline{x}^i\right)dS(\underline{x}),
\end{aligned}
$$

where $\underline{\lambda} \in \underline{M}$ approximate the contact stress. Because of contact boundary we have to apply jump between solids and apply (2.6), (2.7). Using above process we get weak formulation of elasticity problem

$$
\left.
\begin{aligned}
m_u(\underline{\ddot{u}}, \underline{w}) + a_u(\underline{u}, \underline{w}) + a_{u\theta}(\theta, \underline{w}) + b_u(\underline{w}, \underline{\lambda}) &= f_u(\underline{w}), & \forall \underline{w} \in \underline{V}_0 \\
b_{un}(\underline{u}, \underline{\mu} - \underline{\lambda}) + b_{u\tau}(\underline{\dot{u}}, \underline{\mu} - \underline{\lambda}) &\le \langle g, \mu_n - \lambda_n \rangle, & \forall \underline{\mu} \in \underline{M}(\lambda_n)
\end{aligned}
\right\} \mathcal{M}_u(\lambda_n)
$$

with the bilinear forms

$$
\begin{aligned}
m_u(\underline{\ddot{u}}, \underline{w}) &= \sum_{i \in \{s,m\}} \int_{\Omega^i} \varrho^i \underline{\ddot{u}}^i \underline{w}^i \,\mathrm{d}\underline{x}^i, \\
a_u(\underline{u}, \underline{w}) &= \sum_{i \in \{s,m\}} \int_{\Omega^i} \left(\underline{\underline{\underline{c}}} : \underline{\underline{\varepsilon}}^i(\underline{u}^i) \right) : \underline{\underline{\varepsilon}}^i(\underline{w}^i)\,\mathrm{d}\underline{x}^i, \\
a_{u\theta}(\theta, \underline{w}) &= \sum_{i \in \{s,m\}} \int_{\Omega^i} dK^i \alpha^i \underline{\nabla}\theta^i \underline{w}^i \,\mathrm{d}\underline{x}^i, \\
b_u(\underline{w}, \underline{\mu}) &= b_{un}(\underline{w}, \underline{\mu}) + b_{u\tau}(\underline{w}, \underline{\mu}), \\
b_{un}(\underline{w}, \underline{\mu}) &= \int_{\Gamma_C^s} [\underline{w}]_n \mu_n \,\mathrm{d}S(\underline{x}^s), \\
b_{u\tau}(\underline{w}, \underline{\mu}) &= \int_{\Gamma_C^s} [\underline{w}]_\tau \cdot \underline{\mu}_\tau \,\mathrm{d}S(\underline{x}^s),
\end{aligned}
$$

and the linear form

$$
f_u(\underline{w}) = \sum_{i \in \{s,m\}} \int_{\Gamma_{uN}^i} \underline{p}_N^i \underline{w}^i \,\mathrm{d}S(\underline{x}^i) + \sum_{i \in \{s,m\}} \int_{\Omega^i} \underline{f}^i \underline{w}^i \,\mathrm{d}\underline{x}^i.
$$

3.3 Heat transfer

Multiplying (2.8) by $\chi \in V_0$, integrating over Ω^i, and using the Green's theorem we have

$$\left.\begin{aligned} m_\theta(\dot{\theta},\chi) + a_\theta(\theta,\chi) + a_{\theta u}(\underline{\dot{u}},\chi) + \langle \lambda_\theta, [\chi]\rangle_{\Gamma_C^s} &= f_\theta(\chi), && \forall \chi \in V_0, \\ \langle \lambda_\theta, w\rangle_{\Gamma_C^s} + d_{\theta,\lambda_{un}}(\underline{\dot{u}}, w) &= b_{\theta,\lambda_{un}}(\theta, w), && \forall w \in W, \end{aligned}\right\} \quad \mathcal{M}_\theta(\lambda_{un})$$

with linear and bilinear forms

$$\begin{aligned} m_\theta(\dot{\theta},\chi) &= \sum_{i\in\{s,m\}} \int_{\Omega^i} c^i \dot{\theta}^i \chi^i \,\mathrm{d}\underline{x}^i, \\ a_\theta(\theta,\chi) &= \sum_{i\in\{s,m\}} \int_{\Omega^i} \kappa^i \underline{\nabla}\theta^i \cdot \underline{\nabla}\chi^i \,\mathrm{d}\underline{x}^i, \\ a_{\theta u}(\underline{\dot{u}},\chi) &= \sum_{i\in\{s,m\}} \int_{\Omega^i} d\theta_0 K^i \alpha^i \mathrm{div}(\underline{\dot{u}}^i)\chi^i \,\mathrm{d}\underline{x}^i, \\ f_\theta(\chi) &= \sum_{i\in\{s,m\}} \int_{\Gamma^i_{\theta N}} q^i_N \chi^i \,\mathrm{d}S(\underline{x}^i) + \sum_{i\in\{s,m\}} \int_{\Omega^i} r^i \chi^i \,\mathrm{d}\underline{x}^i, \\ b_{\theta,\lambda_{un}}(\vartheta, w) &= \int_{\Gamma_C^s} \lambda_{un}\beta_C[\vartheta] w \,\mathrm{d}S(\underline{x}^s), \\ d_{\theta,\lambda_{un}}(\underline{\dot{u}}, w) &= \int_{\Gamma_C^s} \delta_C \mathfrak{F} \lambda_{un} \|[\underline{\dot{u}}]_\tau\| w \,\mathrm{d}S(\underline{x}^s). \end{aligned}$$

Taking a closer look on $a_{\theta u}(\underline{\dot{u}},\chi)$ we see (because of $\underline{\nabla}\cdot(f\underline{g}) = f(\underline{\nabla}\cdot\underline{g}) + \underline{g}\cdot\underline{\nabla}(f)$ and the Green formula) that

$$\begin{aligned} a_{\theta u}(\underline{\dot{u}},\chi) = &\sum_{i\in\{s,m\}} d\theta_0 K^i \alpha^i \int_{\Omega^i} (\underline{\nabla}\cdot\underline{\dot{u}}^i)\chi^i \,\mathrm{d}\underline{x}^i = \\ = &\sum_{i\in\{s,m\}} d\theta_0 K^i \alpha^i \left(\int_{\partial\Omega^i} \chi^i \underline{\dot{u}}^i \cdot \underline{n}^i \,\mathrm{d}S(\underline{x}^i) - \int_{\Omega^i} \underline{\dot{u}}^i \cdot \underline{\nabla}(\chi^i) \,\mathrm{d}\underline{x}^i \right) = \\ = &\sum_{i\in\{s,m\}} d\theta_0 K^i \alpha^i \int_{\partial\Omega^i} \chi^i \underline{\dot{u}}^i \cdot \underline{n}^i \,\mathrm{d}S(\underline{x}^i) - \theta_0 a_{u\theta}(\chi, \underline{\dot{u}}) \end{aligned}$$

In accordance with REMARK 4.1 in [1], we can substitute $w = [\chi]$ into $(\mathcal{M}_\theta(\lambda_{un}))_2$ and denote

$$\hat{b}_{\theta,\lambda_{un}}(\vartheta,\chi) = b_{\theta,\lambda_{un}}(\vartheta,[\chi]), \quad \hat{d}_{\theta,\lambda_{un}}(\underline{\dot{u}},\chi) = d_{\theta,\lambda_{un}}(\underline{\dot{u}},[\chi]),$$

now we can substitute $\langle \lambda_\theta, [\chi]\rangle_{\Gamma_C^s}$ in $(\mathcal{M}_\theta(\lambda_{un}))$ and get

$$m_\theta(\dot{\theta},\chi) + a_\theta(\theta,\chi) + a_{\theta u}(\underline{\dot{u}},\chi) + \hat{b}_{\theta,\lambda_{un}}(\theta,\chi) - \hat{d}_{\theta,\lambda_{un}}(\underline{\dot{u}},\chi) = f_\theta(\chi), \quad \forall \chi \in V_0.$$

3.4 Fixed point formulation

We will simplify our problems by defining the auxiliary problem

$$\left.\begin{array}{rll} m_u(\underline{\ddot{u}},\underline{w}) + a_u(\underline{u},\underline{w}) + a_{u\theta}(\theta,\underline{w}) + b_u(\underline{w},\underline{\lambda}) & = f_u(\underline{w}), & \forall \underline{w} \in \underline{V}_0, \\ b_{un}(\underline{u},\underline{\mu}-\underline{\lambda}) + b_{u\tau}(\underline{\dot{u}},\underline{\mu}-\underline{\lambda}) & \leq \langle g, \mu_n - \lambda_n \rangle, & \forall \underline{\mu} \in \underline{M}(\gamma), \end{array}\right\} \mathcal{M}_u(\gamma)$$

for the given $\gamma \in H^1_+(\Omega^s)$, so the solution $(\underline{u},\underline{\lambda})$ of $(\mathcal{M}_u(\lambda_n))$ is the solution $(\underline{\hat{u}},\underline{\hat{\lambda}})$ of $(\mathcal{M}_u(\gamma))$ iff $\underline{\hat{\lambda}}$ is the fixed point of the mapping

$$\gamma \mapsto \lambda_n, \text{ where } \lambda_n \text{ is the solution of } (\mathcal{M}_u(\gamma)).$$

Analogously as in the elasticity problem we will use the fixed point approach and define the auxiliary problem

$$m_\theta(\dot{\theta},\chi) + a_\theta(\theta,\chi) + a_{\theta u}(\underline{\dot{u}},\chi) + \hat{b}_{\theta,\gamma}(\theta,\chi) - \hat{d}_{\theta,\gamma}(\underline{\dot{u}},\chi) = f_\theta(\chi), \quad \forall \chi \in V_0. \qquad \mathcal{M}_\theta(\gamma)$$

4 Discretization of spaces

In this part we will introduce discretized form of spaces in previous section, and using basis functions to get algebraic representation of known (bi)linear forms.

4.1 Discretized spaces definition

The spaces V, V_0, $\underline{V}$, $\underline{V}_0$, the affine sets V_D, $\underline{V}_D$ and the convex set $\underline{M}(\gamma)$ will be discretized by

$$
\begin{aligned}
V_h^i &= \left\{ v_h^i \in C(\overline{\Omega}^i) \middle| \; v_h^i \in \tilde{P}(el), \quad \forall el \in \mathcal{T}_h^i \right\}, & V_h &= V_h^s \times V_h^m, \\
V_{0,h}^i &= \left\{ v_h^i \in V_h^i \middle| \; v_h^i = 0 \text{ on } \Gamma_{\theta D}^i \right\}, & V_{0,h} &= V_{0,h}^s \times V_{0,h}^m, \\
V_{D,h}^i &= \left\{ v_h^i \in V_h^i \middle| \; v_h^i = \theta_D^i \text{ on } \Gamma_{\theta D}^i \right\}, & V_{D,h} &= V_{D,h}^s \times V_{D,h}^m,
\end{aligned}
$$

$$
\begin{aligned}
\underline{V}_h^i &= \left(V_h^i\right)^d, & \underline{V}_h &= \underline{V}_h^s \times \underline{V}_h^m, \\
\underline{V}_{0,h}^i &= \left\{ \underline{v}_h^i \in \underline{V}_h^i \middle| \; \underline{v}_h^i = \underline{0} \text{ on } \Gamma_{uD}^i \right\}, & \underline{V}_{0,h} &= \underline{V}_{0,h}^s \times \underline{V}_{0,h}^m, \\
\underline{V}_{D,h}^i &= \left\{ \underline{v}_h^i \in \underline{V}_h^i \middle| \; \underline{v}_h^i = \underline{u}_D^i \text{ on } \Gamma_{uD}^i \right\}, & \underline{V}_{D,h} &= \underline{V}_{D,h}^s \times \underline{V}_{D,h}^m,
\end{aligned}
$$

$$
\underline{\mathrm{M}}_h(\gamma) = \left\{ \underline{\mu}_h \in \underline{\mathrm{M}}_h(\gamma) \middle| \; \underline{\mu}_h = \sum_{k=1}^{n_{\Gamma_C^s \text{ nodes}}} \underline{\mu}_k \psi_k \right\}, \quad \underline{M}_h = \left(M_h\right)^d,
$$

where ψ_k are basis function of $\underline{\mathrm{M}}_h$. The discretization of spaces allows us to write each space as the linear combination of each base function, therefore denote using linear span

$$
\begin{aligned}
V_h^i &= \operatorname*{span}_{j=1,\dots,n_{\text{nodes}}^i} \{\varphi_j^i\}, & \underline{V}_h^i &= \operatorname*{span}_{\substack{j=1,\dots,n_{\text{nodes}}^i \\ k=1,\dots,d}} \{\underline{\varphi}_j^{i,k}\}, \\
V_{0,h}^i &= \operatorname*{span}_{\substack{j=1,\dots,n_{\text{nodes}}^i \\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}} \{\varphi_j^i\}, & \underline{V}_{0,h}^i &= \operatorname*{span}_{\substack{j=1,\dots,n_{\text{nodes}}^i \\ \underline{x}_j^i \notin \Gamma_{uD}^i \\ k=1,\dots,d}} \{\underline{\varphi}_j^{i,k}\}, \\
M_h &= \operatorname*{span}_{j=1,\dots,n_{\Gamma_C^s \text{ nodes}}} \{\psi_j\}, & \underline{M}_h &= \operatorname*{span}_{\substack{j=1,\dots,n_{\Gamma_C^s \text{ nodes}} \\ k=1,\dots,d}} \{\underline{\psi}_j^k\},
\end{aligned}
$$

and with the above notation we mean e.g. for the case of $d = 3$

$$\underline{\varphi}_j^{i,1}(\underline{x}) = \begin{bmatrix} \varphi_j^i(\underline{x}) \\ 0 \\ 0 \end{bmatrix}, \qquad \underline{\varphi}_j^{i,2}(\underline{x}) = \begin{bmatrix} 0 \\ \varphi_j^i(\underline{x}) \\ 0 \end{bmatrix}, \qquad \underline{\varphi}_j^{i,3}(\underline{x}) = \begin{bmatrix} 0 \\ 0 \\ \varphi_j^i(\underline{x}) \end{bmatrix},$$
$$\underline{\psi}_j^{1}(\underline{x}) = \begin{bmatrix} \psi_j(\underline{x}) \\ 0 \\ 0 \end{bmatrix}, \qquad \underline{\psi}_j^{2}(\underline{x}) = \begin{bmatrix} 0 \\ \psi_j(\underline{x}) \\ 0 \end{bmatrix}, \qquad \underline{\psi}_j^{3}(\underline{x}) = \begin{bmatrix} 0 \\ 0 \\ \psi_j(\underline{x}) \end{bmatrix},$$

where φ_j^i are the nodal basis functions.

4.1.1 Algebraic form of solution

The displacement $\underline{u}_h^i$ for $d \in \{2, 3\}$ and θ_h^i can be expresed using basis functions and summation over nodes as follows

$$\underline{u}_h^i = \sum_{\substack{j=1 \\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n_{\text{nodes}}^i} \sum_{k=1}^{d} [\![\underline{\mathrm{u}}^i]\!]_{k+d(j-1)} \underline{\varphi}_j^{i,k}, \quad \theta_h^i = \sum_{\substack{j=1 \\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n_{\text{nodes}}^i} [\![\underline{\theta}^i]\!]_j \varphi_j^i. \tag{4.1}$$

Similarly we need discretized form of Lagrange multipliers, which can be written as

$$\lambda = \begin{bmatrix} \lambda_n \\ \lambda_\tau \end{bmatrix}, \; \lambda_n \in \mathbb{R}^{n_{\Gamma_C^s \text{ nodes}}}, \; \lambda_\tau \in \mathbb{R}^{d \cdot n_{\Gamma_C^s \text{ nodes}}}, \text{ in 3D}: \; \lambda_\tau = \begin{bmatrix} \lambda_{\tau 1} \\ \lambda_{\tau 2} \end{bmatrix}, \tag{4.2}$$
$$\lambda_{n,h} = \sum_{j=1}^{n_{\Gamma_C^s \text{ nodes}}} [\![\lambda_n]\!]_j \psi_j, \quad \lambda_{\tau l,h} = \sum_{j=1}^{n_{\Gamma_C^s \text{ nodes}}} [\![\lambda_{\tau l}]\!]_j \psi_j, \quad l \in \begin{cases} \{1\}, & \text{in 2D,} \\ \{1, 2\}, & \text{in 3D,} \end{cases}$$
$$\underline{\lambda}_h = \lambda_{n,h} \cdot \underline{n} + \sum_{l=1}^{d} \lambda_{\tau l,h} \cdot \underline{\tau}_l,$$

where $\underline{n}$, $\underline{\tau}_l$ are orthogonal unit vectors and $n_{\Gamma_C^s \text{ nodes}}$ are nodes over contact interface on slave body.

4.2 Algebraic spaces and sets definition

In section 4.1 we introduced discretized spaces, but for computational resons, we also need spaces, where are only node-values. Let's introduce algebraic spaces related to discretized forms:

$$V_h \equiv \underline{\mathrm{V}}_\theta = \mathbb{R}^{n_{nodes}}, \qquad \underline{V}_h \equiv \underline{\mathrm{V}}_u = \mathbb{R}^{d \times n_{nodes}}.$$

Next, there are subspaces $\underline{\mathrm{V}}_\theta$, $\underline{\mathrm{V}}_u$, where nodes belonging to Dirichlet dof's are equal to zero. Dimensions of that spaces are

$$V_{0,h} \equiv \underline{\mathrm{V}}_{\theta 0} = \mathbb{R}^{n_{nodes}}, \qquad \underline{V}_{0,h} \equiv \underline{\mathrm{V}}_{u0} = \mathbb{R}^{d \times n_{nodes}}.$$

Finally, we introduce affine sets, where value for nodes on Dirichlet boundary is equal to value of the prescribed function

$$V_{D,h} \equiv \underline{\mathrm{V}}_{\theta D} = \mathbb{R}^{n_{nodes}}, \qquad \underline{V}_{D,h} \equiv \underline{\mathrm{V}}_{uD} = \mathbb{R}^{d \times n_{nodes}}.$$

4.3 Space-discretized form of elasticity problem

Now we show, how to write down algebraic representation of known linear forms, first of all relable equation ($\mathcal{M}_u(\gamma)$), so we get weak discretized formulation of elasticity problem

$$\begin{aligned} m_u(\underline{\ddot{u}}_h, \underline{w}_h) + a_u(\underline{u}_h, \underline{w}_h) + a_{u\theta}(\theta_h, \underline{w}_h) + b_u(\underline{w}_h, \underline{\lambda}_h) = f_u(\underline{w}_h), \qquad & \forall \underline{w}_h \in \underline{V}_{0,h}, \\ b_n(\underline{u}_h, \underline{\mu}_h - \underline{\lambda}_h) + b_\tau(\underline{\dot{u}}_h, \underline{\mu}_h - \underline{\lambda}_h) \leq \langle g, \mu_{n,h} - \lambda_{n,h} \rangle, \qquad & \forall \underline{\mu}_h \in \underline{M}_h(\gamma), \end{aligned}$$

after that using (4.1) we can write bilinear form in discretized formulation

$$\begin{aligned} m_u(\underline{\ddot{u}}_h, \underline{w}_h) &= \sum_{i \in \{m,s\}} m_u(\underline{\ddot{u}}_h^i, \underline{w}_h^i) = \\ &= \sum_{i \in \{m,s\}} m_u \left(\sum_{\substack{j=1 \\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n_{\text{nodes}}^i} \sum_{k=1}^{d} [\underline{\ddot{\mathrm{u}}}^i]_{k+d(j-1)} \underline{\varphi}_j^{i,k}, \sum_{\substack{j=1 \\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n_{\text{nodes}}^i} \sum_{k=1}^{d} [\underline{\mathrm{w}}^i]_{k+d(j-1)} \underline{\varphi}_j^{i,k} \right) = \\ &= [\underline{\mathrm{w}}^{s\top} \, \underline{\mathrm{w}}^{m\top}] \begin{bmatrix} \underline{\underline{\mathrm{M}}}_u^s & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{\mathrm{M}}}_u^m \end{bmatrix} \begin{bmatrix} \underline{\ddot{\mathrm{u}}}^s \\ \underline{\ddot{\mathrm{u}}}^m \end{bmatrix} = \underline{\mathrm{w}}^\top \underline{\underline{\mathrm{M}}}_u \underline{\ddot{\mathrm{u}}}, \end{aligned}$$

where $\underline{\underline{\mathrm{M}}}_u$ is the mass matrix of elasticity problem and the number $[\![\underline{\underline{\mathrm{M}}}_u^i]\!]_{o,p}$ in the matrix $\underline{\underline{\mathrm{M}}}_u^i$ on the position (o,p) is

$$[\![\underline{\underline{\mathrm{M}}}_u]\!]_{o,p}^i = \sum_{\substack{j_1,j_2=1 \\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n_{\text{nodes}}^i} \sum_{k_1,k_2=1}^{d} \delta_{o,k_1+d(j_1-1)} \delta_{p,k_2+d(j_2-1)} \int_{\Omega^i} \varrho^i \underline{\varphi}_{j_1}^{i,k_1} \underline{\varphi}_{j_2}^{i,k_2} \, \mathrm{d}\underline{x}^i,$$

we note here that $\underline{\varphi}_{j_1}^{i,k_1}\underline{\varphi}_{j_2}^{i,k_2}$ is the scalar product in $\mathbb{R}^d$. The bilinear form a_u can be written then as

$$\begin{aligned} a_u(\underline{u}_h,\underline{w}_h) &= \sum_{i\in\{m,s\}} a_u(\underline{u}_h^i,\underline{w}_h^i) = \\ &= \sum_{i\in\{m,s\}} a_u\left(\sum_{\substack{j=1\\ \underline{x}_j^i\notin\Gamma_{\theta D}^i}}^{n_{\text{nodes}}^i}\sum_{k=1}^{d}[\underline{\mathrm{u}}^i]_{k+d(j-1)}\underline{\varphi}_j^{i,k},\ \sum_{\substack{j=1\\ \underline{x}_j^i\notin\Gamma_{\theta D}^i}}^{n_{\text{nodes}}^i}\sum_{k=1}^{d}[\underline{\mathrm{w}}^i]_{k+d(j-1)}\underline{\varphi}_j^{i,k}\right) = \\ &= [\underline{\mathrm{w}}^{s\top}\ \underline{\mathrm{w}}^{m\top}]\begin{bmatrix}\underline{\underline{\mathrm{K}}}^s & \underline{\underline{0}}\\ \underline{\underline{0}} & \underline{\underline{\mathrm{K}}}^m\end{bmatrix}\begin{bmatrix}\underline{\mathrm{u}}^s\\ \underline{\mathrm{u}}^m\end{bmatrix} = \underline{\mathrm{w}}^\top\underline{\underline{\mathrm{K}}}\,\underline{\mathrm{u}}, \end{aligned}$$

where $\underline{\underline{\mathrm{K}}}_u$ is the stiffness matrix of elasticity problem, and holds

$$\begin{aligned} [\![\underline{\underline{\mathrm{K}}}_u^i]\!]_{o,p} = &\sum_{\substack{j_1,j_2=1\\ \underline{x}_j^i\notin\Gamma_{\theta D}^i}}^{n_{\text{nodes}}^i}\sum_{k_1,k_2=1}^{d}\delta_{o,k_1+d(j_1-1)}\delta_{p,k_2+d(j_2-1)} \\ &\int_{\Omega^i}\underline{\mathrm{SVoi}}^\top\left(\underline{\underline{\nabla}}_{\text{sym}}\left(\underline{\varphi}_{j_1}^{i,k_1}\right)\right)\underline{\underline{\mathrm{C}}}\,\underline{\mathrm{SVoi}}\left(\underline{\underline{\nabla}}_{\text{sym}}\left(\underline{\varphi}_{j_2}^{i,k_2}\right)\right)\mathrm{d}\underline{x}^i. \end{aligned}$$

The linear form $f_u(w)$ can be written as

$$f_u(\underline{w}_h) = \sum_{i\in\{m,s\}} f_u\left(\sum_{\substack{j=1\\ \underline{x}_j^i\notin\Gamma_{\theta D}^i}}^{n_{\text{nodes}}^i}\sum_{k=1}^{d}[\![\underline{\mathrm{w}}^i]\!]_{k+d(j-1)}\underline{\varphi}_j^{i,k}\right) = [\underline{\mathrm{w}}^{s\top}\ \underline{\mathrm{w}}^{m\top}]\begin{bmatrix}\underline{\mathrm{f}}_u^s\\ \underline{\mathrm{f}}_u^m\end{bmatrix} = \underline{\mathrm{w}}^\top\underline{\mathrm{f}}_u,$$

where $\underline{\mathrm{f}}_u$ is vector of the load forces, which contains Neumann boundary conditions and is equal to

$$[\![\underline{\mathrm{f}}_u^i]\!]_o = \sum_{\substack{j=1\\ \underline{x}_j^i\notin\Gamma_{\theta D}^i}}^{n_{\text{nodes}}}\sum_{k=1}^{d}\delta_{o,k+d(j-1)}\left(\int_{\Gamma_N^i}\underline{p}_N^i\underline{\varphi}_j^{i,k}\,\mathrm{d}\underline{x}^i + \int_{\Omega^i}\underline{\tilde{f}}^i\underline{\varphi}_j^{i,k}\,\mathrm{d}\underline{x}^i\right).$$

The bilinear form $a_{u\theta}$ can be written then as

$$
\begin{aligned}
a_{u\theta}(\theta_h, \underline{w}_h) &= \sum_{i\in\{m,s\}} a_{u\theta}(\theta_h^i, \underline{w}_h^i) = \\
&= \sum_{i\in\{m,s\}} a_{u\theta}\left(\sum_{\substack{j=1\\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n^i_{\text{nodes}}} [\![\underline{\theta}^i]\!]_j \varphi_j^i, \sum_{\substack{j=1\\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n^i_{\text{nodes}}} \sum_{k=1}^{d} [\![\underline{w}^i]\!]_{k+d(j-1)} \underline{\varphi}_j^{i,k}\right) = \\
&= [\underline{w}^{s\top}\, \underline{w}^{m\top}] \begin{bmatrix} \underline{\underline{K}}_{u\theta}^s & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{K}}_{u\theta}^m \end{bmatrix} \begin{bmatrix} \underline{\theta}^s \\ \underline{\theta}^m \end{bmatrix} = \underline{w}^\top \underline{\underline{K}}_{u\theta} \underline{\theta},
\end{aligned}
$$

where

$$
\begin{aligned}
[\![\underline{\underline{K}}_{u\theta}^i]\!]_{o,p} &= d \sum_{\substack{j_1,j_2=1\\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n^i_{\text{nodes}}} \sum_{k_1=1}^{d} \delta_{o,k_1+d(j_1-1)} \delta_{p,k_2} K^i \alpha^i \int_{\Omega^i} \underline{\varphi}_{j_1}^{i,k_1} \cdot \underline{\nabla} \varphi_{j_2}^i \,\mathrm{d}\underline{x}^i = \\
&= d \sum_{\substack{j_1,j_2=1\\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n^i_{\text{nodes}}} \sum_{k_2=1}^{d} \delta_{o,k_1+d(j_1-1)} K^i \alpha^i \int_{\Omega^i} \underline{\varphi}_{j_1}^{i,k_1} \cdot \underline{\nabla} \varphi_{p}^i \,\mathrm{d}\underline{x}^i.
\end{aligned}
$$

And

$$
\begin{aligned}
b_{un}(\underline{w}_h, \underline{\mu}_h) &= \int_{\Gamma_C^s} [\underline{w}_h]_n \mu_{n,h} \,\mathrm{d}S(\underline{x}^s) = \int_{\Gamma_C^s} (\underline{w}_h^s \underline{n} - \underline{w}_h^m \underline{n}) \mu_{n,h} \,\mathrm{d}S(\underline{x}^s) = \\
&= b_{un}\left(\sum_{\substack{j=1\\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n^i_{\text{nodes}}} \sum_{k=1}^{d} [\![\underline{w}^i]\!]_{k+d(j-1)} \underline{\varphi}_j^{i,k}, \sum_{j=1}^{n_{\Gamma_C^s}\ \text{nodes}} [\![\underline{\mu}_n]\!]_j \psi_j\right) = \\
&= [\underline{w}^{s\top}\, \underline{w}^{m\top}] \begin{bmatrix} \underline{\underline{B}}_n^s & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{B}}_n^m \end{bmatrix} \begin{bmatrix} \underline{\mu}_n \\ \underline{\mu}_\tau \end{bmatrix} = \underline{w}^\top \underline{\underline{B}}_n \underline{\mu}_n,
\end{aligned}
$$

where

$$[\![\underline{\underline{\mathrm{B}}}_n^s]\!]_{o,p} = \sum_{\substack{j=1\\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n_{\mathrm{nodes}}^s} \sum_{k=1}^{d} \sum_{o=1}^{n_{\Gamma_C^s} \text{ nodes}} \delta_{p,k+d(j-1)} \int_{\Gamma_C^s} \underline{\varphi}_j^{s,k} \cdot \underline{n} \psi_o \,\mathrm{d}S(\underline{x}^s),$$

$$[\![\underline{\underline{\mathrm{B}}}_n^m]\!]_{o,p} = - \sum_{\substack{j=1\\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n_{\mathrm{nodes}}^m} \sum_{k=1}^{d} \sum_{o=1}^{n_{\Gamma_C^s} \text{ nodes}} \delta_{p,k+d(j-1)} \int_{\Gamma_C^s} \underline{\varphi}_j^{m,k} \cdot \underline{n} \psi_o \,\mathrm{d}S(\underline{x}^s).$$

The discretization of $b_{u\tau}$ will be analogous to b_{un}, i.e.

$$b_{u\tau}(\underline{w}_h, \underline{\mu}_h) = \sum_{l=1}^{d} \underline{\mu}_{\tau l}^\top \underline{\underline{\mathrm{B}}}_{\tau l} \underline{\mathrm{w}},$$

one can also easy see that

$$\underline{\underline{\mathrm{B}}} = \begin{bmatrix} \underline{\underline{\mathrm{B}}}_n \\ \underline{\underline{\mathrm{B}}}_{\tau 1} \\ (\underline{\underline{\mathrm{B}}}_{\tau 2}) \end{bmatrix}.$$

The space-discretized version of $(\mathcal{M}_u(\gamma))$ is to find $\underline{\mathrm{u}} \in \underline{\mathrm{V}}_D$ such that

$$\left.\begin{aligned} \underline{\mathrm{w}}^\top \left(\underline{\underline{\mathrm{M}}}_u \ddot{\underline{\mathrm{u}}} + \underline{\underline{\mathrm{K}}}_u \underline{\mathrm{u}} + \underline{\underline{\mathrm{B}}}^\top \underline{\lambda} + \underline{\underline{\mathrm{K}}}_{u\theta} \underline{\theta}\right) &= \underline{\mathrm{w}}^\top \underline{\mathrm{f}}, & \forall \underline{\mathrm{v}} : \ \underline{\mathrm{V}}_0. \\ \left(\underline{\mu} - \underline{\lambda}\right)_n^\top \underline{\underline{\mathrm{B}}}_n \underline{\mathrm{u}} + \left(\underline{\mu} - \underline{\lambda}\right)_\tau^\top \underline{\underline{\mathrm{B}}}_\tau \dot{\underline{\mathrm{u}}} &\le \left(\underline{\mu} - \underline{\lambda}\right)_n^\top \underline{\mathrm{g}}, & \forall \underline{\mu} \in \underline{\mathrm{M}}(\gamma). \end{aligned}\right\} \qquad (\underline{\mathrm{M}}(\gamma))$$

4.4 Space-discretized form of heat transfer problem

In a similar way relable $(\mathcal{M}_\theta(\gamma))$, which implies weak discretized formulation of heat transfer problem

$$m_\theta(\dot{\theta}_h, \chi_h) + a_\theta(\theta_h, \chi_h) + a_{\theta u}(\dot{\underline{u}}_h, \chi_h) + \hat{b}_{\theta,\gamma}(\theta_h, \chi_h) - \hat{d}_{\theta,\gamma}(\dot{\underline{u}}_h, \chi_h) = f_\theta(\chi_h), \quad \forall \chi_h \in V_{0,h}.$$

The mass term can be written as

$$m_\theta(\dot{\theta}_h, \chi_h) = \sum_{i\in\{m,s\}} m_\theta(\dot{\theta}_h^i, \chi_h^i) =$$
$$= \sum_{i\in\{m,s\}} m_\theta \left(\sum_{\substack{j=1 \\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n_{\text{nodes}}^i} [\![\dot{\underline{\theta}}^i]\!]_j \varphi_j^i, \sum_{\substack{j=1 \\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n_{\text{nodes}}^i} [\![\underline{\chi}^i]\!]_j \varphi_j^i \right) =$$
$$= [\underline{\chi}^{s\top}, \underline{\chi}^{m\top}] \begin{bmatrix} \underline{\underline{M}}_\theta^s & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{M}}_\theta^m \end{bmatrix} \begin{bmatrix} \dot{\underline{\theta}}^s \\ \dot{\underline{\theta}}^m \end{bmatrix} = \underline{\chi}^\top \underline{\underline{M}}_\theta \dot{\underline{\theta}},$$

where the number $\underline{\underline{M}}_{\theta[o,p]}^i$ in the matrix $\underline{\underline{M}}_\theta^i$ on the position $[o,p]$ is

$$\underline{\underline{M}}_{\theta[o,p]}^i = \sum_{\substack{j_1,j_2=1 \\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n_{\text{nodes}}^i} \delta_{o,j_1}\delta_{p,j_2} \int_{\Omega^i} c^i \varphi_{j_1}^i \varphi_{j_2}^i \, \mathrm{d}\underline{x}^i = \int_{\Omega^i} c^i \varphi_o^i \varphi_p^i \, \mathrm{d}\underline{x}^i.$$

The stiffness matrix is assembled from term a_θ as follows

$$a_\theta(\theta_h, \chi_h) = \sum_{i\in\{m,s\}} a_\theta(\theta_h^i, \chi_h^i) = \sum_{i\in\{m,s\}} a_\theta \left(\sum_{\substack{j=1 \\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n_{\text{nodes}}^i} [\![\underline{\theta}^i]\!]_j \varphi_j^i, \sum_{\substack{j=1 \\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n_{\text{nodes}}^i} [\![\underline{\chi}^i]\!]_j \varphi_j^i \right) =$$
$$= [\underline{\chi}^{s\top}\ \underline{\chi}^{m\top}] \begin{bmatrix} \underline{\underline{K}}_\theta^s & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{K}}_\theta^m \end{bmatrix} \begin{bmatrix} \underline{\theta}^s \\ \underline{\theta}^m \end{bmatrix} = \underline{\chi}^\top \underline{\underline{K}}_\theta \underline{\theta},$$

where the number $\underline{\underline{K}}_{\theta[o,p]}^i$ in the matrix $\underline{\underline{K}}_\theta^i$ on the position $[o,p]$ is

$$\underline{\underline{K}}_{\theta[o,p]}^i = \sum_{\substack{j_1,j_2=1 \\ \underline{x}_j^i \notin \Gamma_{\theta D}^i}}^{n_{\text{nodes}}^i} \delta_{o,j_1}\delta_{p,j_2} \int_{\Omega^i} \kappa^i \underline{\nabla}\varphi_{j_1}^i \cdot \underline{\nabla}\varphi_{j_2}^i \, \mathrm{d}\underline{x}^i = \int_{\Omega^i} \kappa^i \underline{\nabla}\varphi_o^i \cdot \underline{\nabla}\varphi_p^i \, \mathrm{d}\underline{x}^i.$$

The term

$$a_{\theta u}(\dot{\underline{u}}_h, \chi_h) = \sum_{i\in\{m,s\}} a_{\theta u}(\dot{\underline{u}}_h^i, \chi_h^i) = [\underline{\chi}^{s\top}, \underline{\chi}^{m\top}] \begin{bmatrix} \underline{\underline{K}}_{\theta u}^s & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{K}}_{\theta u}^m \end{bmatrix} \begin{bmatrix} \dot{\underline{u}}^s \\ \dot{\underline{u}}^m \end{bmatrix} = \underline{\chi}^\top \underline{\underline{K}}_{\theta u} \dot{\underline{u}},$$

where

$$[\![\underline{\underline{\mathrm{K}}}^i_{\theta u}]\!]_{o,p} = d\theta_0 \sum_{\substack{j_1,j_2=1 \\ \underline{x}^i_j \notin \Gamma^i_{\theta D}}}^{n^i_{\text{nodes}}} \sum_{k_2=1}^{d} \delta_{o,j_1}\delta_{p,k_2+d(j_2-1)} K^i \alpha^i \int_{\Omega^i} \underline{\nabla}\varphi^i_{j_1} \cdot \underline{\varphi}^{i,k_2}_{j_2} \, \mathrm{d}\underline{x}^i.$$

The linear form $f_u(w)$ can be written as

$$f_\theta(\chi_h) = \sum_{i\in\{m,s\}} f_u \left(\sum_{\substack{j=1 \\ \underline{x}^i_j \notin \Gamma^i_{\theta D}}}^{n^i_{\text{nodes}}} [\![\underline{\chi}^i]\!]_j \varphi^i_j \right) = [\underline{\chi}^{s\top} \; \underline{\chi}^{m\top}] \begin{bmatrix} \underline{\mathrm{f}}^s_\theta \\ \underline{\mathrm{f}}^m_\theta \end{bmatrix} = \underline{\chi}^\top \underline{\mathrm{f}}_\theta,$$

where

$$[\![\underline{\mathrm{f}}^i_\theta]\!]_o = \sum_{\substack{j=1 \\ \underline{x}^i_j \notin \Gamma^i_{\theta D}}}^{n_{\text{nodes}}} \left(\int_{\Gamma^i_N} \underline{q}^i_N \underline{\varphi}^{i,k}_j \, \mathrm{d}\underline{x}^i + \int_{\Omega^i} \underline{r}^i \underline{\varphi}^{i,k}_j \, \mathrm{d}\underline{x}^i \right).$$

The discretization of the term $\hat{b}_{\theta,\gamma}(\theta_h, \chi_h)$ is

$$\begin{aligned} \hat{b}_{\theta,\gamma}(\theta_h, \chi_h) &= \beta_C \int_{\Gamma^s_C} \gamma[\theta_h][\chi_h] \, \mathrm{d}S(\underline{x}) = \\ &= [\underline{\chi}^{s\top}, \underline{\chi}^{m\top}] \begin{bmatrix} \underline{\underline{\mathrm{L}}}^{ss}_{\theta,\gamma} & \underline{\underline{\mathrm{L}}}^{sm}_{\theta,\gamma} \\ \text{sym.} & \underline{\underline{\mathrm{L}}}^{mm}_{\theta,\gamma} \end{bmatrix} \begin{bmatrix} \underline{\theta}^s \\ \underline{\theta}^m \end{bmatrix} = \underline{\chi}^\top \underline{\underline{\mathrm{L}}}_{\theta,\gamma} \underline{\theta}, \end{aligned}$$

where

$$\begin{aligned} [\![\underline{\underline{\mathrm{L}}}^{ss}_{\theta,\gamma}]\!]_{ij} &= \beta_C \int_{\Gamma^s_C} \gamma \varphi^s_i(\underline{x}) \varphi^s_j(\underline{x}) \, \mathrm{d}S(\underline{x}) = \beta_C \sum_{k=1}^{n_{\Gamma_C \text{ nodes}}} \gamma_k \int_{\Gamma^s_C} \varphi^s_k \varphi^s_i \varphi^s_j \, \mathrm{d}S(\underline{x}), \\ [\![\underline{\underline{\mathrm{L}}}^{mm}_{\theta,\gamma}]\!]_{ij} &= \beta_C \int_{\Gamma^s_C} \gamma \varphi^m_i(\underline{R}_t(\underline{x})) \varphi^m_j(\underline{R}_t(\underline{x})) \, \mathrm{d}S(\underline{x}) = \beta_C \sum_{k=1}^{n_{\Gamma_C \text{ nodes}}} \gamma_k \int_{\Gamma^s_C} \varphi^s_k \varphi^m_i \varphi^m_j \, \mathrm{d}S(\underline{x}), \\ [\![\underline{\underline{\mathrm{L}}}^{sm}_{\theta,\gamma}]\!]_{ij} &= -\beta_C \int_{\Gamma^s_C} \gamma \varphi^s_i(\underline{x}) \varphi^m_j(\underline{R}_t(\underline{x})) \, \mathrm{d}S(\underline{x}) = -\beta_C \sum_{k=1}^{n_{\Gamma_C \text{ nodes}}} \gamma_k \int_{\Gamma^s_C} \varphi^s_k \varphi^s_i \varphi^m_j \, \mathrm{d}S(\underline{x}). \end{aligned}$$

Finally, discretization of bilinear form $\hat{d}_{\theta,\gamma}(\underline{\dot{u}}, \chi)$ is

$$\begin{aligned} \hat{d}_{\theta,\gamma}(\underline{\dot{u}}_h, \chi_h) &= \int_{\Gamma^s_C} \delta_C \mathfrak{F} \gamma \| [\underline{\dot{u}}]_{\tau,h} \| [\chi_h] \, \mathrm{d}S(\underline{x}^s) = \\ &= [\underline{\chi}^{s\top}, \underline{\chi}^{m\top}] \begin{bmatrix} \underline{\mathrm{d}}^s_{\theta,\gamma}(\underline{\dot{\mathrm{u}}}) \\ \underline{\mathrm{d}}^m_{\theta,\gamma}(\underline{\dot{\mathrm{u}}}) \end{bmatrix} = \underline{\chi}^\top \underline{\mathrm{d}}_{\theta,\gamma}(\underline{\dot{\mathrm{u}}}), \end{aligned}$$

where

$$[\![\underline{\mathrm{d}}^s_{\theta,\gamma}(\dot{\underline{\mathrm{u}}})]\!]_i = \delta_C \mathfrak{F} \int_{\Gamma^s_C} \gamma \|[\dot{\underline{u}}]_{\tau,h}\| \varphi^s_i(\underline{x}^s)\,\mathrm{d}S(\underline{x}^s)$$

$$[\![\underline{\mathrm{d}}^m_{\theta,\gamma}(\dot{\underline{\mathrm{u}}})]\!]_i = \delta_C \mathfrak{F} \int_{\Gamma^s_C} \gamma \|[\dot{\underline{u}}]_{\tau,h}\| \varphi^m_i(\underline{R}_t(\underline{x}^s))\,\mathrm{d}S(\underline{x}^s).$$

The space-discretized version of $(\mathcal{M}_\theta(\gamma))$ is then

$$\underline{\underline{\mathrm{M}}}_\theta \dot{\underline{\theta}} + \underline{\underline{\mathrm{K}}}_\theta \underline{\theta} + \underline{\underline{\mathrm{K}}}_{\theta u} \dot{\underline{\mathrm{u}}} + \underline{\underline{\mathrm{L}}}_{\theta,\gamma} \underline{\theta} = \underline{\mathrm{f}}_\theta + \underline{\mathrm{d}}_{\theta u,\gamma}(\dot{\underline{\mathrm{u}}}). \tag{4.3}$$

5 Time discretization

Now we have weak formulation in discretized space, but we also need to discretized time to timesteps. Due to the fact that in elasticity problem we have second time derivative of displacement, we can use time discretization employ Newmark integration scheme, see [7]. In heat transfer part of problem we use backward formula approximation, for discretization scheme.

5.1 Newmark integration scheme

Newmark integration scheme is method for solving differential equations of the second order, which uses known values in t^k to get solution in time interval t^{k+1}. With simple modification, we can use this method to get time discretized form of displacement problem.

We split the time interval into n_t times $t^k = k\tau$, $\tau = T/n_t$ and denote

$$\underline{\mathrm{u}}^k \approx \underline{\mathrm{u}}(\cdot, t^k).$$

Standard Newmark integration scheme is defined, according to [8], as follows

$$\underline{\mathrm{u}}^{k+1} = \underline{\mathrm{u}}^k + \tau\underline{\dot{\mathrm{u}}}^k + \frac{\tau^2}{2}\left[(1-2\beta)\,\underline{\ddot{\mathrm{u}}}^k + 2\beta\underline{\ddot{\mathrm{u}}}^{k+1}\right], \tag{5.1}$$

$$\underline{\dot{\mathrm{u}}}^{k+1} = \underline{\dot{\mathrm{u}}}^k + \tau\left[(1-\gamma)\,\underline{\ddot{\mathrm{u}}}^k + \gamma\underline{\ddot{\mathrm{u}}}^{k+1}\right], \tag{5.2}$$

to get acceptable order of convergence and to stabilize this method set constants $\frac{1}{2} = \gamma = 2\beta$, this choice provides the energy conserving of algorithm. Now from (5.1) we can express

$$\underline{\ddot{\mathrm{u}}}^{k+1} = \frac{4}{\tau^2}(\underline{\mathrm{u}}^{k+1} - \underline{\mathrm{u}}^k) - \frac{4}{\tau}\underline{\dot{\mathrm{u}}}^k - \underline{\ddot{\mathrm{u}}}^k,$$

which we immediately insert into (5.2) and obtain

$$\underline{\dot{\mathrm{u}}}^{k+1} = \frac{2}{\tau}(\underline{\mathrm{u}}^{k+1} - \underline{\mathrm{u}}^k) - \underline{\dot{\mathrm{u}}}^k.$$

Substituting formulas for $\underline{\ddot{\mathrm{u}}}^{k+1}$, $\underline{\dot{\mathrm{u}}}^{k+1}$ into the equilibrium equality ($\underline{\mathrm{M}}(\gamma)$) one will obtain for t^{k+1} the problem of finding $\left(\underline{\mathrm{u}}^{k+1}, \underline{\lambda}^{k+1}\right) \in \underline{\mathrm{V}}_{uD} \times \underline{\mathrm{M}}(\gamma)$ such that

$$
\begin{aligned}
&\forall \underline{\mathrm{w}} \in \underline{\mathrm{V}}_{u0},\ \forall \underline{\mu} \in \underline{\mathrm{M}}(\gamma): \\
&\underline{\mathrm{w}}^\top \left[\left(\underline{\underline{\mathrm{K}}}_u + \frac{4}{\tau^2} \underline{\underline{\mathrm{M}}}_u \right) \underline{\mathrm{u}}^{k+1} + \left(\underline{\underline{\mathrm{B}}}^{k+1} \right)^\top \underline{\lambda}^{k+1} + \underline{\underline{\mathrm{K}}}_{u\theta} \underline{\theta}^{k+1} \right] = \underline{\mathrm{w}}^\top \underline{\mathrm{f}}_u^{k+1}(\underline{\mathrm{u}}^k, \underline{\dot{\mathrm{u}}}^k, \underline{\ddot{\mathrm{u}}}^k), \\
&\left(\underline{\mu} - \underline{\lambda}^{k+1} \right)^\top \begin{bmatrix} \underline{\underline{\mathrm{B}}}_n^{k+1} \\ \frac{2}{\tau} \underline{\underline{\mathrm{B}}}_\tau^{k+1} \end{bmatrix} \underline{\mathrm{u}}^{k+1} \leq \left(\underline{\mu} - \underline{\lambda}^{k+1} \right)^\top \begin{bmatrix} \underline{\mathrm{g}}^{k+1} \\ \underline{\underline{\mathrm{B}}}_\tau^{k+1} \left(\frac{2}{\tau} \underline{\mathrm{u}}^k + \underline{\dot{\mathrm{u}}}^k \right) \end{bmatrix},
\end{aligned}
\tag{5.3}
$$

where

$$
\underline{\mathrm{f}}_u^{k+1}(\underline{\mathrm{u}}^k, \underline{\dot{\mathrm{u}}}^k, \underline{\ddot{\mathrm{u}}}^k) = \underline{\tilde{\mathrm{f}}}_u^{k+1} + \underline{\underline{\mathrm{M}}}_u \left(\frac{4}{\tau^2} \underline{\mathrm{u}}^k + \frac{4}{\tau} \underline{\dot{\mathrm{u}}}^k + \underline{\ddot{\mathrm{u}}}^k \right).
$$

Let us choose $\underline{\mu}^\top \in \{[\underline{\mu}_n^\top,\ \underline{\lambda}_\tau^{k+1\top}],\ [\underline{\lambda}_n^{k+1\top},\ \frac{\tau}{2} \underline{\mu}_\tau^\top]\}$ and decouple inequality into two inequalities

$$
\begin{aligned}
\left(\underline{\mu} - \underline{\lambda}^{k+1} \right)_n^\top \underline{\underline{\mathrm{B}}}_n^{k+1} \underline{\mathrm{u}}^{k+1} &\leq \left(\underline{\mu} - \underline{\lambda}^{k+1} \right)_n^\top \underline{\mathrm{g}}^{k+1}, && \forall \underline{\mu} \in \underline{\mathrm{M}}(\gamma), \\
\frac{2}{\tau} \left(\underline{\mu} - \underline{\lambda}^{k+1} \right)_\tau^\top \underline{\underline{\mathrm{B}}}_\tau^{k+1} \underline{\mathrm{u}}^{k+1} &\leq \frac{2}{\tau} \left(\underline{\mu} - \underline{\lambda}^{k+1} \right)_\tau^\top \underline{\underline{\mathrm{B}}}_\tau^{k+1} \left(\underline{\mathrm{u}}^k + \frac{\tau}{2} \underline{\dot{\mathrm{u}}}^k \right), && \forall \underline{\mu} \in \underline{\mathrm{M}}(\gamma),
\end{aligned}
$$

and summed again together as

$$
\left(\underline{\mu} - \underline{\lambda}^{k+1} \right)^\top \begin{bmatrix} \underline{\underline{\mathrm{B}}}_n^{k+1} \\ \underline{\underline{\mathrm{B}}}_\tau^{k+1} \end{bmatrix} \underline{\mathrm{u}}^{k+1} \leq \left(\underline{\mu} - \underline{\lambda}^{k+1} \right)^\top \begin{bmatrix} \underline{\mathrm{g}}^{k+1} \\ \underline{\underline{\mathrm{B}}}_\tau^{k+1} \left(\underline{\mathrm{u}}^k + \frac{\tau}{2} \underline{\dot{\mathrm{u}}}^k \right) \end{bmatrix}, \quad \forall \underline{\mu} \in \underline{\mathrm{M}}(\gamma). \tag{5.4}
$$

Equations (5.3) and (5.4) are discretized equations of elasticity part of our problem, note that in (5.3) still occurs unknown term $\underline{\theta}^{k+1}$.

5.2 Backward formula scheme

As in previous section, we need to discretized time into timesteps also in heat tranfer part of problem. We split the time interval into n_t times $t^k = k\tau$, $\tau = T/n_t$ and denote

$$
\underline{\theta}^k \approx \underline{\theta}(\cdot, t^k),
$$

Newmark integration scheme can not be used, because the heat constitutive equation lack's second time derivative. We simply use backward formula from Backward Euler method

$$
\dot{\theta}^{k+1} = \frac{1}{\tau} \left(\theta^{k+1} - \theta^k \right), \tag{5.5}
$$

and from the Newmark time discretization for displacement we have

$$
\underline{\dot{\mathrm{u}}}^{k+1} = \frac{2}{\tau} (\underline{\mathrm{u}}^{k+1} - \underline{\mathrm{u}}^k) - \underline{\dot{\mathrm{u}}}^k.
$$

Now substitue $\dot{\underline{\theta}}$ and $\underline{u}^{k+1}$ into (4.3) and rearangement this equation into

$$\left(\frac{1}{\tau}\underline{\underline{M}}_\theta + \underline{\underline{K}}_\theta + \underline{\underline{L}}_{\theta,\gamma}\right)\underline{\theta}^{k+1} + \frac{2}{\tau}\underline{\underline{K}}_{\theta u}\underline{u}^{k+1} = \underline{f}_\theta^{k+1}(\underline{u}^k, \dot{\underline{u}}^k, \underline{\theta}^k) + \underline{d}_{\theta u,\gamma}(\dot{\underline{u}}^{k+1}), \tag{5.6}$$

where

$$\underline{f}_\theta^{k+1}(\underline{u}^k, \dot{\underline{u}}^k, \underline{\theta}^k) = \tilde{\underline{f}}_\theta^{k+1} + \frac{1}{\tau}\underline{\underline{M}}_\theta\underline{\theta}^k + \underline{\underline{K}}_{\theta u}\left(\frac{2}{\tau}\underline{u}^k - \dot{\underline{u}}^k\right).$$

6 Numerical Realization

In this chapter we complete derivation of main problem. Before that we outline, how assembling of vectors is realized

6.1 Assembly of matrices

In previous text we introduced discretized formulation of problems and assembly of matrices using basis functions. Although in numerical example we solve problem using the eight node hexahedron (brick) elements with appropriate shape functions, in this chapter we explain discretization of domains in the 2D with triangulation. Note, that there are several types of discretization [10, 11].

Each body Ω^i is divided into triangular elements, who are non-overlapping. Note that in our problem we use domains, that can be discretisable without any error. These elements could be written as $\mathcal{T}_h^i := \{\mathcal{T}_k^i\}_{k=1}^{n_{\text{el.}}}$, where $n_{\text{el.}}$ is number of elements in body. The set of all nodes is $\widetilde{P}^{i,h}(\text{el.}) := \{x_i\}_{i=1}^{n_{\text{nodes}}}$, where n_{nodes} is number of all nodes. Each node in element is numbered counter clockwise, its called local numbering. Similarly, the boundary parts are replaced by their discretized form $\Gamma_{u\cdot}^{i,h}, \Gamma_{\theta\cdot}^{i,h}, \Gamma_C^h$.

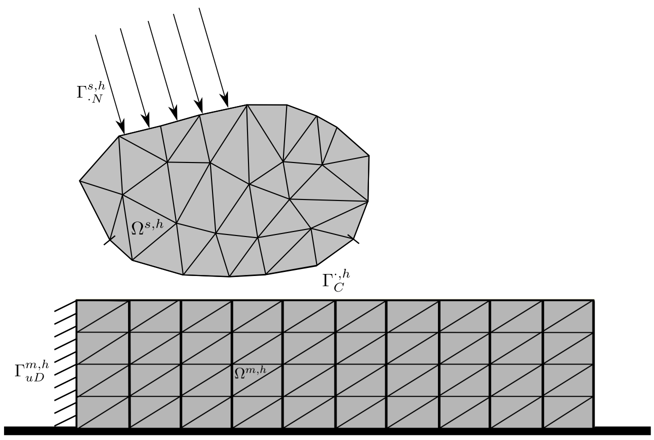


Figure 6.1: Possible discretization of the domains Ω^i

We use finite element discretization from section 4.2, be cautious to see different notations between algebraic vector $\underline{\mathrm{V}}$ and analytics vector $\underline{V}$. The solution $\underline{u}(t) \in \underline{V}_D$ is approximated by $\underline{\mathrm{u}}^h(t) \in \underline{\mathrm{V}}_{uD}$, respectively $\underline{\theta}^h(t) \in \underline{\mathrm{V}}_{\theta D}$. The set $\underline{\mathrm{V}}_{uD}$ consists of vector functions $\underline{\tilde{u}}(\underline{\mathrm{x}}, t) = (\tilde{u}(\underline{x}, t), \tilde{v}(\underline{x}, t))$ whose components are piecewise linear over each element and $\tilde{u}, \tilde{v} : \mathbb{R}^2 \times T \to \mathbb{R}$. Similarly $\underline{\theta}^h(t)$ consists of $\tilde{\theta}(\underline{\mathrm{x}}, t) = (\tilde{u}(\underline{x}, t), \tilde{\chi}(\underline{x}, t))$. Choose basis function with respect to definitons in 4.1, $\varphi_i(\underline{x}_j) = \delta_{ij}$, where δ_{ij} is Kronecker's delta.

Now, we approximate solution and test functions

$$\underline{\tilde{u}}(\underline{x},t) = \sum_{i=1}^{n_{\text{nodes}}} \underline{u}_i(t)\underline{\varphi}_i(x), \qquad \underline{\tilde{v}}(\underline{x},t) = \sum_{i=1}^{n_{\text{nodes}}} \underline{v}_i(t)\underline{\varphi}_i(x),$$
$$\tilde{\theta}(\underline{x},t) = \sum_{i=1}^{n_{\text{nodes}}} \theta_i(t)\varphi_i(x), \qquad \tilde{\chi}(\underline{x},t) = \sum_{i=1}^{n_{\text{nodes}}} \chi_i(t)\varphi_i(x).$$

where $\underline{u}_i$, θ_i are unknowns of the functions $\underline{\tilde{u}}$, $\tilde{\theta}$ at the nodes. Because of easy implementation, the assembling of matricis and vectors is performed over individual elements. For more see [9].

6.2 Domain decomposition

Because of possible computationally comprehensive problems, we introduce so-called gluing condition. It helps in finding solution faster, due to smaller subdomains. After decomposition of main domains, we need to prescribed neumann condition on new boundary nodes. It is ensured by new matrix $\underline{\underline{B}}_{\text{gl.}} \in \mathbb{R}^{n_{\text{nodes}} \times n_{\text{nodes}}}$, with $[\cdots\ 1\ -1\ \cdots]$ on relevant positions of boundary nodes and zeros elsewhere.

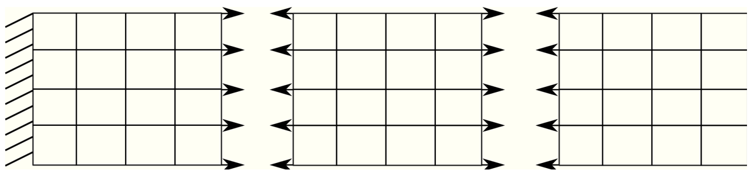


Figure 6.2: Domain decomposition

6.3 Decoupling

From previous section we have (5.6) as the main equation for heat flux and two equations for displacement (5.3), (5.4), but there are still "displacement terms" in heat transfer equation and vice versa. Some might say, that for decoupling, we could express $\underline{\theta}^{k+1}$, substitued into (5.3), then $\underline{u}^{k+1}$ put into (5.4) and find solutions, but this can not be done, because in that case full non-symetric matricies appears.

Instead, we find solution in each timestep using successive iterations. In this case we want to find solution $(\underline{u}^{k+1}, \underline{\theta}^{k+1}, \underline{\lambda}^{k+1})$ in the time t^{k+1} as the limit of $(\underline{u}^{k+1,p}, \underline{\theta}^{k+1,p}, \underline{\lambda}^{k+1,p})$, using starting values from previous timestep

$$(\underline{u}^{k+1,0}, \underline{\theta}^{k+1,0}, \underline{\lambda}^{k+1,0}) = (\underline{u}^{k}, \underline{\theta}^{k}, \underline{\lambda}^{k}).$$

When we know $(\underline{u}^{k+1,p}, \underline{\theta}^{k+1,p}, \underline{\lambda}^{k+1,p})$, we want to compute $(\underline{u}^{k+1,p+1}, \underline{\theta}^{k+1,p+1}, \underline{\lambda}^{k+1,p+1})$. Here we want to stress the fact, that

$$\gamma^{k+1,p} \equiv (\underline{\lambda}^{k+1,p})_n,$$

so in the successive iteration scheme we don't need to compute $\gamma^{k+1,p}$, only extract it from $\underline{\lambda}^{k+1,p}$. We decided to solve the heat problem (5.6) at first. So assembling $\underline{\underline{L}}_{\theta,\underline{\gamma}^{k+1,p}}$, $\underline{d}_{\theta u,\underline{\gamma}^{k+1,p}}$ and substituing into (5.6) we obtain

$$\underline{\chi}^\top \left(\frac{1}{\tau} \underline{\underline{M}}_\theta + \underline{\underline{K}}_\theta + \underline{\underline{L}}_{\theta,\underline{\gamma}^{k+1,p}} \right) \underline{\theta}^{k+1,p+1} = \underline{\chi}^\top \tilde{\underline{f}}_\theta^{k+1,p}, \quad \forall \underline{\chi} \in \underline{V}_{\theta 0}, \tag{6.1}$$

$$\tilde{\underline{f}}_\theta^{k+1,p} = \underline{f}_\theta^{k+1}(\underline{u}^k, \dot{\underline{u}}^k, \underline{\theta}^k) + \underline{d}_{\theta u,\gamma^{k+1,p}}(\underline{u}^{k+1,p}) - \frac{2}{\tau} \underline{\underline{K}}_{\theta u} \underline{u}^{k+1,p}.$$

Because of the computational reasons we approximate $\underline{\underline{L}}_{\theta,\underline{\gamma}^{k+1,p}} \underline{\theta}^{k+1,p+1}$ by $\underline{\underline{L}}_{\theta,\underline{\gamma}^{k+1,p}} \underline{\theta}^{k+1,p}$ so instead of (6.1) we use

$$\underline{\chi}^\top \left(\frac{1}{\tau} \underline{\underline{M}}_\theta + \underline{\underline{K}}_\theta \right) \underline{\theta}^{k+1,p+1} = \underline{\chi}^\top \underline{f}_\theta^{k+1,p} := \underline{\chi}^\top \left(\tilde{\underline{f}}_\theta^{k+1,p} - \underline{\underline{L}}_{\theta,\underline{\gamma}^{k+1,p}} \underline{\theta}^{k+1,p} \right), \quad \forall \underline{\chi} \in \underline{V}_{\theta 0}. \tag{6.2}$$

and obtain $\underline{\theta}^{k+1,p+1}$. Because the possitive definite matrix $\frac{1}{\tau}\underline{\underline{M}}_\theta + \underline{\underline{K}}_\theta$ is sparse, block diagonal and does not change during the computation it can be factorized and the action of it's inverse can be effectively computed.

After that we will solve the contact problem with known fixed temperature distribution $\underline{\theta}^{k+1,p+1}$, which will affect the righthand side of (5.3) and obtain the system

$$\underline{w}^\top \left(\frac{4}{\tau^2} \underline{\underline{M}}_u + \underline{\underline{K}}_u \right) \underline{u}^{k+1,p+1} + \underline{w}^\top \left(\underline{\underline{B}}^{k+1} \right)^\top \underline{\lambda}^{k+1,p+1} = \underline{w}^\top \overbrace{\left(\underline{f}_u^{k+1}(\underline{u}^k, \dot{\underline{u}}^k, \ddot{\underline{u}}^k) - \underline{\underline{K}}_{u\theta} \underline{\theta}^{k+1,p+1} \right)}^{\underline{f}_u^{k+1,p}}, \tag{6.3}$$

$$\left(\underline{\mu} - \underline{\lambda}^{k+1,p+1} \right)^\top \underline{\underline{B}}^{k+1} \underline{u}^{k+1,p+1} \le \left(\underline{\mu} - \underline{\lambda}^{k+1,p+1} \right)^\top \left[\begin{array}{c} \underline{g}^{k+1} \\ \underline{\underline{B}}_\tau^{k+1} \left(\underline{u}^k + \frac{\tau}{2} \dot{\underline{u}}^k \right) \end{array} \right], \quad \forall \underline{\mu} \in \underline{M}(\gamma). \tag{6.4}$$

Also here the matrix $\frac{4}{\tau^2}\underline{\underline{M}}_u + \underline{\underline{K}}_u$ is sparse, block diagonal and constant during the computation.

6.4 Treating Dirichlet boundary conditions using additional Lagrange multipliers

In this part, we want to demonstrate in generaly terms how to solve our problems. We use equivalency of solution of boundary value problem with minimization quadratic functional. Introducing indicator function and extending sets over we minimize, we can rewrite problem with homogeneous conditions into nonhomogeneous contact problem.

6.4.1 Deriving of the general procedure

So far we didn't mention the nonhomogeneous Dirichlet boundary conditions. Taking closer look at (6.2), after rearanging the Dirichlet dof's at the end of the vector we have the abstract problem

$$\begin{bmatrix} \underline{v}_F^\top & \underline{0}_D^\top \end{bmatrix} \begin{bmatrix} \underline{\underline{A}}_{FF} & \underline{\underline{A}}_{FD} \\ \underline{\underline{A}}_{FD}^\top & \underline{\underline{A}}_{DD} \end{bmatrix} \begin{bmatrix} \underline{u}_F \\ \underline{u}_D \end{bmatrix} = \underline{v}^\top \underline{\underline{A}}\,\underline{u} = \underline{v}^\top \underline{b} = \begin{bmatrix} \underline{v}_F^\top & \underline{0}_D^\top \end{bmatrix} \begin{bmatrix} \underline{b}_F \\ \underline{b}_D \end{bmatrix}, \quad \forall \underline{v}_F \in \mathbb{R}^{n_F},$$

for prescribed $\underline{u}_D \in \mathbb{R}^{n_D}$, where n_D is the count of Dirichlet dof's and n_F is the count of the rest dof's. This problem is equivalent to the unconstrain minimization problem

$$\arg\min_{\underline{v}_F} \frac{1}{2}\underline{v}_F^\top \underline{\underline{A}}_{FF}\underline{v}_F - \underline{v}_F^\top \left(\underline{b}_F - \underline{\underline{A}}_{FD}\underline{u}_D\right). \tag{6.5}$$

The minimization in (6.5) is over $\underline{v}_F \in \mathbb{R}^{n_F}$, but can be equivalently written as the equality constrained minimization over $\underline{v} \in \mathbb{R}^{n_{\text{nodes}}}$

$$\arg\min_{\underline{v}\colon \underline{v}_D = \underline{u}_D} \frac{1}{2}\underline{v}^\top \underline{\underline{A}}\,\underline{v} - \underline{v}^\top \underline{b}.$$

Indeed, after expanding the multiplication of block matrices we have

$$\arg\min_{\underline{v}\colon \underline{v}_D = \underline{u}_D} \frac{1}{2}\underline{v}_F^\top \underline{\underline{A}}_{FF}\underline{v}_F + \underline{v}_F^\top \underline{\underline{A}}_{FD}\underline{v}_D + \overbrace{\frac{1}{2}\underline{v}_D^\top \underline{\underline{A}}_{DD}\underline{v}_D}^{\text{const.}} - \underline{v}_F \underline{b}_F - \overbrace{\underline{v}_D \underline{b}_D}^{\text{const.}},$$

which is the same as (6.5) except the constants which will not influence the resulting $\underline{v}_F$ where the minimum occures. Now, we want to get rid of the equality constraint. That can be enforced by adding indicator function $\sup_{\underline{\mu}_D} \underline{\mu}\,(\underline{v}_D - \underline{u}_D)$, which is equals to 0 if $\underline{v}_D = \underline{u}_D$ and equals to ∞ if $\underline{v}_D \neq \underline{u}_D$, where $\underline{\mu} \in \mathbb{R}^{n_D}$:

$$\arg\min_{\underline{v}} \sup_{\underline{\mu}_D} \frac{1}{2}\underline{v}^\top \underline{\underline{A}}\,\underline{v} - \underline{v}^\top \underline{b} + \underline{\mu}_D^\top \left(\underline{\underline{B}}_D \underline{v} - \underline{u}_D\right).$$

Here we introduce the mapping matrix $\underline{\underline{\mathrm{B}}}_D$, that maps from $\underline{\mathrm{v}}$ to $\underline{\mathrm{v}}_D$. By the duality approach we can swap the minimum and supremum

$$\arg\max_{\underline{\mu}_D} \inf_{\underline{\mathrm{v}}} \frac{1}{2}\underline{\mathrm{v}}^\top \underline{\underline{\mathrm{A}}}\,\underline{\mathrm{v}} - \underline{\mathrm{v}}^\top \underline{\mathrm{b}} + \underline{\mu}_D^\top \left(\underline{\underline{\mathrm{B}}}_D \underline{\mathrm{v}} - \underline{\mathrm{u}}_D\right).$$

The infimum is achieved in $\underline{\mathrm{v}} = \underline{\underline{\mathrm{A}}}^{-1}\left(\underline{\mathrm{b}} - \underline{\underline{\mathrm{B}}}_D \underline{\mu}_D\right)$ and substituting this into the above saddle point problem we get

$$\arg\max_{\underline{\mu}_D} \frac{1}{2}\left(\underline{\mathrm{b}} - \underline{\underline{\mathrm{B}}}_D \underline{\mu}_D\right)^\top \underline{\underline{\mathrm{A}}}^{-1}\left(\underline{\mathrm{b}} - \underline{\underline{\mathrm{B}}}_D \underline{\mu}_D\right) - \left(\underline{\mathrm{b}} - \underline{\underline{\mathrm{B}}}_D \underline{\mu}_D\right)^\top \underline{\underline{\mathrm{A}}}^{-1}\left(\underline{\mathrm{b}} - \underline{\underline{\mathrm{B}}}_D \underline{\mu}_D\right) - \underline{\mu}_D^\top \underline{\mathrm{u}}_D.$$

That can be written as the unconstraint minimization problem

$$\arg\min_{\underline{\mu}_D} \frac{1}{2}\underline{\mu}_D^\top \underline{\underline{\mathrm{B}}}_D \underline{\underline{\mathrm{A}}}^{-1} \underline{\underline{\mathrm{B}}}_D^\top \underline{\mu}_D - \underline{\mu}_D^\top \left(\underline{\underline{\mathrm{B}}}_D \underline{\underline{\mathrm{A}}}^{-1} \underline{\mathrm{b}} - \underline{\mathrm{u}}_D\right), \tag{6.6}$$

which is equivalent to

$$\left(\underline{\underline{\mathrm{B}}}_D \underline{\underline{\mathrm{A}}}^{-1} \underline{\underline{\mathrm{B}}}_D^\top\right) \underline{\lambda}_D = \underline{\underline{\mathrm{B}}}_D \underline{\underline{\mathrm{A}}}^{-1} \underline{\mathrm{b}} - \underline{\mathrm{u}}_D.$$

Similarly we demonstrate, how to modify (6.3) and (6.4) into minimize problem, where the nonhomogeneous Dirichlet boundary conditions are taken into account. We use analogous procedures, therefore explanation is simplified. From (6.3) we get

$$\underline{\mathrm{v}}^\top \underline{\underline{\mathrm{A}}}\,\underline{\mathrm{u}} + \underline{\mathrm{v}}^\top \underline{\underline{\mathrm{B}}}_C^\top \underline{\lambda}_C = \underline{\mathrm{v}}^\top \underline{\mathrm{b}},$$

$$\begin{bmatrix} \underline{\mathrm{v}}_F^\top & \underline{0}_D^\top \end{bmatrix} \begin{bmatrix} \underline{\underline{\mathrm{A}}}_{FF} & \underline{\underline{\mathrm{A}}}_{FD} & \underline{\underline{\mathrm{B}}}_{C_F}^\top \\ \underline{\underline{\mathrm{A}}}_{FD}^\top & \underline{\underline{\mathrm{A}}}_{DD} & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{\mathrm{u}}_F \\ \underline{\mathrm{u}}_D \\ \underline{\lambda}_C \end{bmatrix} = \begin{bmatrix} \underline{\mathrm{v}}_F^\top & \underline{0}_D^\top \end{bmatrix} \begin{bmatrix} \underline{\mathrm{b}}_F \\ \underline{\mathrm{b}}_D \end{bmatrix},$$

and from (6.4) we have

$$\left(\underline{\mu}_C - \underline{\lambda}_C\right)^\top \begin{bmatrix} \underline{\underline{\mathrm{B}}}_{C_F} & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{\mathrm{u}}_F \\ \underline{\mathrm{u}}_D \end{bmatrix} = \left(\underline{\mu}_C - \underline{\lambda}_C\right)^\top \underline{\underline{\mathrm{B}}}_C\,\underline{\mathrm{u}} \le \left(\underline{\mu}_C - \underline{\lambda}_C\right)^\top \underline{\tilde{\mathrm{g}}}.$$

This is equivalent to to the saddle point problem, for all $\underline{\mu}_C \in \tilde{M}$

$$\arg\min_{\underline{\mathrm{v}}_F} \sup_{\underline{\mu}_C \in \tilde{M}} \frac{1}{2}\underline{\mathrm{v}}_F^\top \underline{\underline{\mathrm{A}}}_{FF} \underline{\mathrm{v}}_F - \underline{\mathrm{v}}_F^\top \left(\underline{\mathrm{b}}_F - \underline{\underline{\mathrm{A}}}_{FD} \underline{\mathrm{u}}_D\right) + \underline{\mu}_C^\top \left(\underline{\underline{\mathrm{B}}}_{C_F} \underline{\mathrm{v}}_F - \underline{\tilde{\mathrm{g}}}\right).$$

As in the previous case, we want to rewrite this problem to minimalization over $\underline{\mathrm{v}} \in \mathbb{R}^{n_{\mathrm{nodes}}}$, i.e.

$$\underset{\underline{v}:\,\underline{v}_D=\underline{u}_D}{\arg\min}\ \sup_{\underline{\mu}_C\in\tilde{M}}\ \frac{1}{2}\underline{v}^\top\underline{\underline{A}}\underline{v}-\underline{v}^\top\underline{b}+\underline{\mu}_C^\top\left(\underline{\underline{B}}_C\underline{v}-\underline{\tilde{g}}\right),$$

where $\underline{\underline{B}}_C=\left[\underline{\underline{B}}_{C_F}\,\underline{\underline{0}}\right]$. Applying indicator function we get

$$\underset{\underline{v}}{\arg\min}\ \sup_{\substack{\underline{\mu}_C\in\widetilde{M}\\ \underline{\mu}_D}}\ \frac{1}{2}\underline{v}^\top\underline{\underline{A}}\,\underline{v}-\underline{v}^\top\underline{b}+\underline{\mu}_C^\top\left(\underline{\underline{B}}_C\underline{v}-\underline{\tilde{g}}\right)+\underline{\mu}_D^\top\left(\underline{\underline{B}}_D\underline{v}-\underline{u}_D\right),$$

$$\underset{\underline{v}}{\arg\min}\ \sup_{\underline{\mu}\in M}\ \frac{1}{2}\underline{v}^\top\underline{\underline{A}}\,\underline{v}-\underline{v}^\top\underline{b}+\underline{\mu}^\top\left(\underline{\underline{B}}\,\underline{v}-\underline{g}\right),$$

where $\underline{\mu}=\left[\underline{\mu}_C\,\underline{\mu}_D\right]\in M=\tilde{M}\times\mathbb{R}^{n_D}$, $\underline{\underline{B}}=\left[\underline{\underline{B}}_C\,\underline{\underline{B}}_D\right]$ and $\underline{g}^\top=\left[\underline{\tilde{g}}^\top\,\underline{u}_D^\top\right]$. After swapping the minimum and supremum and finding minimum over $\underline{v}\in\mathbb{R}^{n_{\text{nodes}}}$, we obtain

$$\underset{\underline{\mu}\in M}{\arg\min}\ \frac{1}{2}\underline{\mu}^\top\underline{\underline{B}}\,\underline{\underline{A}}^{-1}\underline{\underline{B}}^\top\underline{\mu}-\underline{\mu}^\top\left(\underline{\underline{B}}\,\underline{\underline{A}}^{-1}\underline{b}-\underline{g}\right),\tag{6.7}$$

is the minimization of similar quadratic function that in (6.6) but yet over the convex set M.

6.4.2 Notation

Using this procedure we obtain from (6.6) equation of nonhomogeneous boundary value problem for heat transfer as follows

$$\left(\underline{\underline{B}}_D\left(\frac{1}{\tau}\underline{\underline{M}}_\theta+\underline{\underline{K}}_\theta\right)^{-1}\underline{\underline{B}}_D^\top\right)\underline{\lambda}_D=\underline{\underline{B}}_D\left(\frac{1}{\tau}\underline{\underline{M}}_\theta+\underline{\underline{K}}_\theta\right)^{-1}\underline{f}_\theta^{k+1,p}-\underline{u}_D,\tag{6.8}$$

$$\underline{\theta}^{k+1,p+1}=\left(\frac{1}{\tau}\underline{\underline{M}}_\theta+\underline{\underline{K}}_\theta\right)^{-1}\left(\underline{f}_\theta^{k+1,p}-\underline{\underline{B}}_D\underline{\lambda}_D\right).$$

Accoridng to choice of algorithm, this problem can be solved also in the unconstrained minimization form (6.6). For elasticity from (6.7) we get constrained minimization problem

$$\underline{\lambda}^{k+1,p+1}=\underset{\underline{\mu}\in M(\gamma)}{\arg\min}\ \frac{1}{2}\underline{\mu}^\top\underline{\underline{B}}\left(\frac{4}{\tau^2}\underline{\underline{M}}_u+\underline{\underline{K}}_u\right)^{-1}\underline{\underline{B}}^\top\underline{\mu}-\underline{\mu}^\top\left(\underline{\underline{B}}\left(\frac{4}{\tau^2}\underline{\underline{M}}_u+\underline{\underline{K}}_u\right)^{-1}\underline{f}_u^{k+1,p}-\underline{g}\right),\tag{6.9}$$

$$\underline{u}^{k+1,p+1}=\left(\frac{4}{\tau^2}\underline{\underline{M}}_u+\underline{\underline{K}}_u\right)^{-1}\left(\underline{f}_u^{k+1,p}-\underline{\underline{B}}\,\underline{\lambda}^{k+1,p+1}\right).$$

Finally, we obtained two equations, that can be solved with suitable algorithm.

6.5 Scheme of algorithm

Now we outline process of finding solution with the MatSol library, which is developed by the team of IT4Innovations at VŠB-TU Ostrava. Complete selection of data to solve can by found in next chapter.

Because of small problem, we can afford solve system (6.9) with some direct method. Modified Proportioning with Gradient Projection (MPGP) algorithm for finding solution of the constrained minimization problem (7.1) is used.

Firstly, all unchanging matrices and vectors are assembled. After that we enter timestep loop and set initial values of variebles (following section 6.3) and assemble $\underline{f}_u^{k+1}(\underline{u}^k, \dot{\underline{u}}^k, \ddot{\underline{u}}^k)$ from (5.3) and $\underline{f}_\theta^{k+1}(\underline{u}^k, \dot{\underline{u}}^k, \underline{\theta}^k)$ from (5.6). Matrices $\underline{\underline{B}}$, $\underline{\underline{L}}_{\theta,\gamma}$ and vector $\underline{d}_{\theta,\gamma}$ are also assembled before entering th successive iteration loop. Now we update (if needed) load vectors, for simplification we add $\underline{\underline{L}}_{\theta,\gamma}$ and $\underline{d}_{\theta,\gamma}$ as mentioned in (6.1), (6.2). By now, we can solve heat transfer part of problem (6.8), and obtain distribution of heat in bodies.

We add obtained solution into right side vector of elasticity problem and solve system (6.9) using MPRGP algorithm. This si repeated in successive iteration loop in prescribed number of iterations and whole process in every timestep.

You can se one timestep loop in following scheme of algorithm.

Algorithm 1 One loop of timestep

Choose $\underline{u}^{k+1,0} = \underline{u}^k$; $\underline{\theta}^{k+1,0} = \underline{\theta}^k$; $\underline{\lambda}^{k+1,0} = \underline{\lambda}^k$; $\gamma^{k+1,0} = \left(\underline{\lambda}^{k+1,0}\right)_n$
for every timestep
    assemble $\underline{f}_u^{k+1,p}$; $\underline{f}_\theta^{k+1,p}$; $\underline{\underline{B}}^p$; $\underline{\underline{L}}_{\theta,\gamma}^p$; $\underline{d}_{\theta,\gamma}^p$;
    while stopping criterion is not satisfied **do**
        update $\underline{f}_\theta^{k+1,p}$;
        solve (6.8);
        return $\underline{\theta}^{k+1,p+1}$;
        update $\underline{f}_u^{k+1,p}$;
        solve (6.9);
        return $\underline{u}^{k+1,p+1}$;
        $it = it + 1$;
    end while
end for

7 Numerical example

In this section, we show numerical example in the three dimensional situation. We use one geometry and looking into different qualities. Firstly heat flux between bodies and then formation of heat due to fricition between bodies. Our numerical realization uses MatSol library, which is developed by team from IT4Innovations at the Department of Applied Mathematics at VŠB-TU Ostrava.

7.1 Initial settings

We consider two bodies in 3D, in following configuration, master body is cuboid of size and initial position $\Omega^m := [-0.15, 0.15] \times [-0.15, 0.85] \times [-0.2, 0]$ and slave body is cube $\Omega^s := [-0.1, 0.1] \times [-0.1, 0.1] \times [1 \cdot 10^{-4}, 0.2]$. In fig. 7.1 you can see cross-section of bodies and denotation of boundarise and forces.

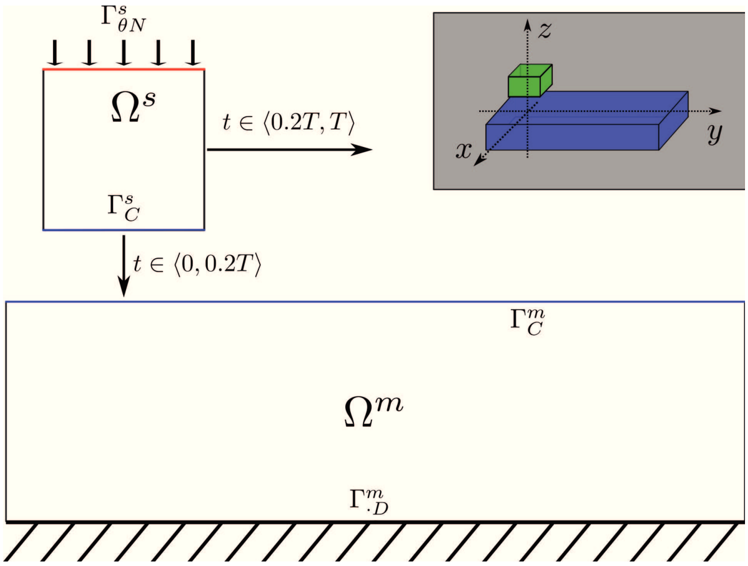


Figure 7.1: Front view of example problem

Lower body is clamped at its lower face and prescribed temperature $20°C$. Upper body have prescribed dirichlet conditions for elasticity on upper face, to stay parallell to the ground. Possible contact boundaries are colored blue. Unwritten boundaries consider as Neumann boundary with zero conditon in both options (elasticity or heat transfer).

In this dynamic problem with this boundary conditions we will be interested in 100 timesteps of size $\tau = 0.01$, i.e. $T = 1$ $[s]$, note that motion can continue, but we are interested in first second of moving. Motion of upper body is as following firstly upper body approaches lower body until they are in contact, in the rest of time it shifts on the master body in direction of axis y. In following table, you can see values of material parameters (in SI base units):

	Master body (Ω^m)	Slave body (Ω^s)	Units
Density (ρ)	7874	7859	$kg \cdot m^{-3}$
Poisson ratio (ν)	0.3	0.29	Pa
Thermal expansion (α)	$1.15 \cdot 10^{-13}$	$1.2 \cdot 10^{-13}$	-
Heat capacity (c_θ)	450	466	K^{-1}
Thermal conductivity (κ)	70	40	$J \cdot kg^{-1} \cdot K^{-1}$
Reference temperature (θ_0)	20	20	$W \cdot m^{-1} K^{-1}$

Table 7.1: Material properties

Furthermore, we use the parameters $\gamma_C^m = 2.5 \cdot 10^{-3}$, $\gamma_C^s = 1$, which lead to $\beta_C = 1.25 \cdot 10^{-3}$ and $\delta_C = 0.5$.

7.2 Solutions

In this part we show obtained solutions of following problems. Due to small geometry of examples, we solve them without domain decomposition, although implementation in algorithm is done.

7.2.1 Heat transfer between bodies

In this example, we want to demonstrate transfer of heat between bodies. Its ensured by prescribtion of heat flux on upper face of upper body. We set space discretization as follows: upper body $6 \times 6 \times 6$, lower body $8 \times 30 \times 6$.

For better visualisation we neglect formation of heat from friction on contact part. In following figure 7.2, you can see propagation of heat in every 0.09s. Note, that upper body acquire values out of colorbar range, its for better view of transfer conditions.

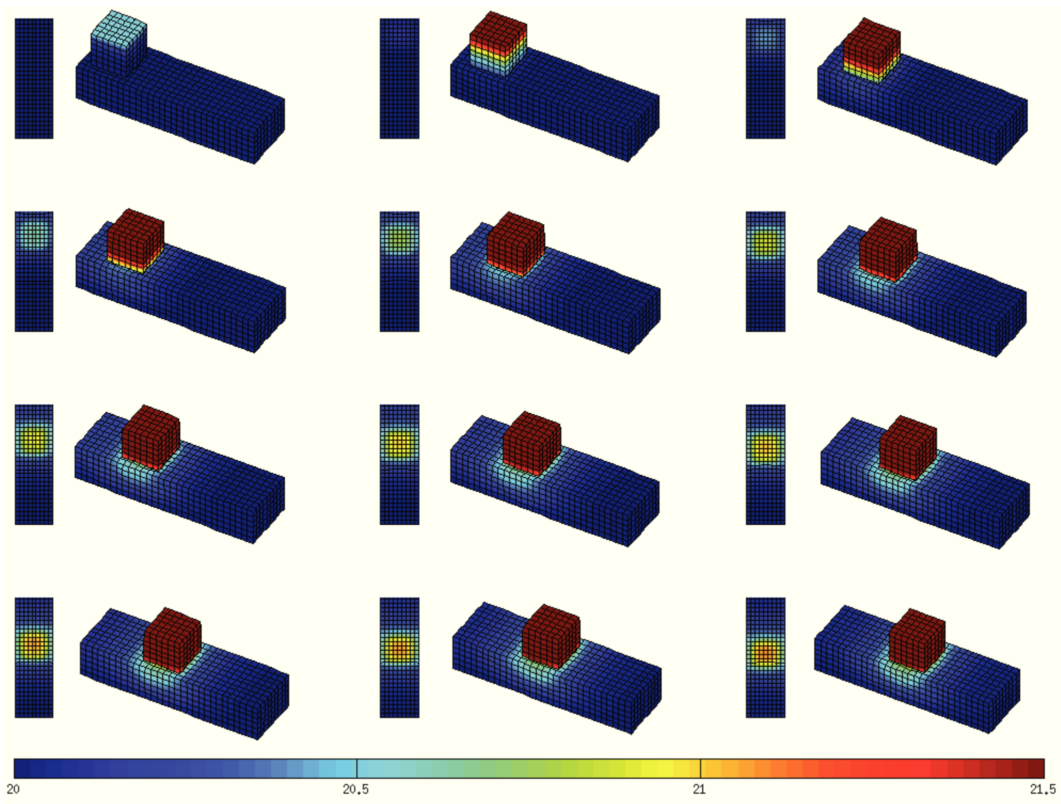


Figure 7.2: Heat transfer: Temperature distribution at each $\frac{1}{12}$ of 1 second

7.2.2 Formation of heat between bodies

In this example, we show formation of heat due to Coulomb friction. We are interested in formation of heat in first second of move. The static friction coefficient is set $\mathfrak{F} = 0.9$ and discretization is same as in previous example. In every timestep was three or less successive iterations, worst number of MPRGP outside iterrations is 17, but usually number of itterations was below 80. Heat transfer between bodies is neglected, to see more details in figure 7.3.

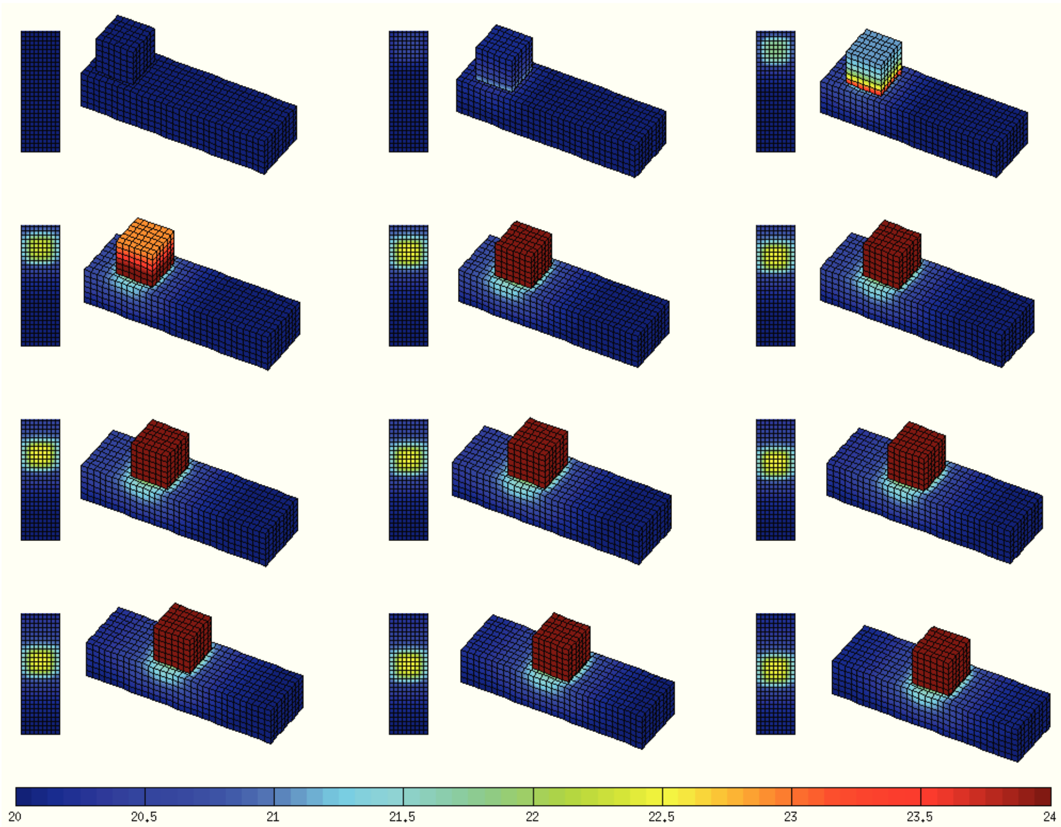


Figure 7.3: Formation of the heat: Temperature distribution at each $\frac{1}{12}$ of 1 second

Conclusion

Main goal of this thesis was to transform the mathematical model of thermodynamical elasticity contact problem to numerically convenient computational procedure. We introduced strong formulation as a balance equations with initial and boundary conditions. After that we derived weak formulation which was discretized firstly in space and after that in time using Newmark discretization scheme for elasticity and backward formula for heat. Finally we used equivalency of discretized weak formulation with minimization of the energy funcitonal. At that point was the nonhomogeneous Dirichlet conditions enforced by additional Lagrange multipliers without changing the assembled matrices. Great advantage of this approach is that each equation contains easily distinguishable terms, for heat transfer between body, formation of the heat, etc. That terms, ans so also the physical phenomenons, can be easily switched on or off according to what feature we want to focus on. Although we apply a few aproximations e.g. fix-point or substitution from previous timestep, in numerical experiment, we obtain acceptable data. We were able to simulate creation of heat only from the friction between the bodies in contact.

In first chapter, we introduced reference temperature θ_0, it is necessary to ensure, that this coefficient is non-zero. It appears in terms in sense as subtracted value, but also as a multiplicator and in case $\theta_0 = 0$ can vanish some term in whole formula. We are not sure about the physical model in that point and the only meaningful conclusion here was to express the temperature numericaly in Kelvins and not Celsius degrees.

From numerical perspective, there is a huge space for optimalization of the formulation and algorithms. As a continuation of this work it would be appropriate to implement the parallelization of the whole computation schema. It would open the space for finding solutions of bigger, better discretized problems, especially for real world benchmarks.

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