**A Proposal for A Two-Parameter Spectral Turbulence Closure**

**Branko Kosović**

Lawrence Livermore National Laboratory

The simplest spectral turbulence model is based on a spectral eddy viscosity. The first spectral eddy viscosity model was developed by Heisenberg (1948). Spectral eddy viscosity, $\nu_e(k|k_c,t)$ is defined as:

$$\nu_e(k|k_c,t) = -\frac{T_e(k|k_c,t)}{k^2 E(k,t)}$$

Spectral eddy viscosity is a function of the wave number $k$, cut-off wave number $k_c$, and time $t$. Spectral energy transfer is denoted with $T_e(k|k_c,t)$ and $E(k,t)$ is spectral energy density. Here, it is implicitly assumed that the spectral eddy viscosity depends only on the spectral energy density and its transfer.

André and Lesieur (1977) have recognized the importance of helicity in modulating energy transfer from large to small scales in high Reynolds number isotropic turbulence. However, only recently has the role of helicity in energy transfer been studied in greater detail e.g. Holm and Kerr (2002), Chen *et al.* (2003).

In order to correctly predict energy transfer in large eddy simulation of turbulent flows, the effect of helicity on the spectral energy transfer must be accounted for in subgrid turbulence models. We therefore develop a two-parameter spectral model that accounts for the helicity effects in homogeneous incompressible turbulent flow. In such a flow, Fourier transforms of velocity and vorticity span locally two-dimensional space perpendicular to a wave vector $\mathbf{k}$. Velocity and vorticity vectors in Fourier space form a complete basis that can be used to represent the Fourier transform of the divergence-free projection of the nonlinear term and its components. We use the velocity-vorticity basis to develop a two-parameter spectral model. By using the evolution equations for the two invariants of three-dimensional Euler equations, kinetic energy and helicity, and assuming that the flow is isotropic and has reached statistical equilibrium, we derive two equations for two unknown closure parameters. We finally arrive at a system of two algebraic equations with two unknowns represented by two model parameters.

The incompressible Navier-Stokes equations in Fourier space are

$$\left[\frac{d}{dt} + \nu k^2\right] \hat{u}_i(\mathbf{k},t) = -\hat{\zeta}_i^{<}(\mathbf{k},t) - \hat{\zeta}_i^{>}(\mathbf{k},t)$$

Here, $\hat{\zeta}_i^{<}(\mathbf{k}) + \hat{\zeta}_i^{>}(\mathbf{k})$ represents the divergence-free projection of the Fourier transform of the nonlinear term. Both resolved, $\hat{\zeta}_i^{<}$, and subgrid, $\hat{\zeta}_i^{>}(\mathbf{k})$, components are divergence free and therefore perpendicular to the local wave vector $\mathbf{k}$.

The Fourier transform of the vorticity vector is $\hat{\omega}_i = i\ \varepsilon_{imn} k_m \hat{u}_n$. Notice now that vector pairs $[\Re(\hat{u}_i), \Im(\hat{\omega}_i)]$ and $[\Im(\hat{u}_i), \Re(\hat{\omega}_i)]$ represent two orthogonal bases for a complex two-dimensional space locally perpendicular to the wave vector $\mathbf{k}$.

The subgrid component of the nonlinear term can be expressed as a linear combination of basis vectors

$$\hat{\zeta}_i^{>} = a\,\Re(\hat{u}_i) + b\,\Im(\hat{\omega}_i) + i\,c\,\Re(\hat{u}_i) + i\,d\,\Im(\hat{\omega}_i)$$

Here, coefficients $a$, $b$, $c$, and $d$ are real and depend on the wave vector, $\mathbf{k}$, and possibly time, $t$.

We now assume that the turbulence flow under consideration is isotropic and that $a(\mathbf{k}), b(\mathbf{k}), c(\mathbf{k})$ and $d(\mathbf{k})$ are functions of the wave number magnitude only. Due to the realizability condition, which requires that both spectral energy density and spectral helicity density are real-valued functions, parameters $b(\mathbf{k})$ and $c(\mathbf{k})$ must vanish. Using the evolution equations for the spherically-integrated spectral densities and assuming that the statistical equilibrium has been reached, the two equations above can be solved for closure parameters $a(k)$ and $d(k)$

$$a(k) = \frac{(2\nu k^2 E + T_e^{<})k^2 E - (2\nu k^2 H + T_h^{<})H}{k^4 E^2 - H^2}$$

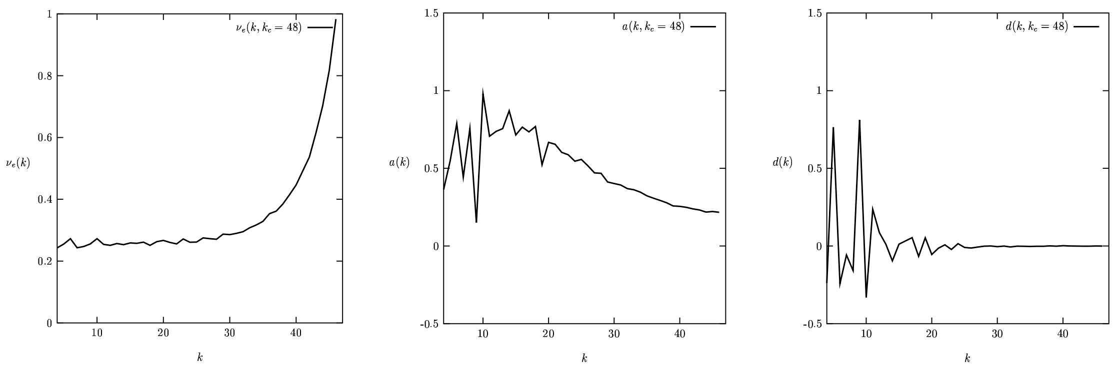


Figure 1: Spectral eddy viscosity, $\nu_e(k|k_c, t)$, model parameter $d(k|k_c, t)$.

$$d(k) \quad = \quad \frac{(2\nu k^2 H + T_h^<)E - (2\nu k^2 E + T_e^<)H}{k^4 E^2 - H^2}$$

Here, $H$ is the spectral helicity density, while $T_h^<$ and $T_h^>$ are resolved and subgrid helicity transfer functions, respectively. Similarly, $T_e^<$ and $T_e^>$ are resolved and subgrid energy transfer functions.

While the one-parameter spectral eddy viscosity (Heisenberg, 1948) is not sufficient to model the subgrid component $\zeta_i^>$ of the nonlinear term in the momentum equation, the two-parameter model is complete. Various statistical theories of turbulence have been used to determine spectral eddy viscosity defined by Heisenberg (1948). Kraichnan (1976) used the test field method and predicted a so called "cusp" in spectral eddy viscosity at the wave number cut-off. The existence of such a cusp was also confirmed using direct numerical simulation (DNS) of Navier-Stokes equations by Domaradzki *et al.* (1993). For comparison to the spectral eddy viscosity, we compute parameters $a(k)$ and $d(k)$ of the proposed spectral turbulence model using the results of DNS. We filtered DNS results obtained at 256 grid-point resolution using a wave cut-off filter with cut-off wave number $k_c = 48$ and computed the subgrid transfer of energy and helicity in addition to energy and helicity spectra. Spectral eddy viscosity $\nu_e(k|k_c, t)$, and two model parameters corresponding to the new proposed spectral turbulence model are given in figures below 1, 2, and 3. The new two-parameter model is complete and does not exhibit the "cusp."

**References**

André, J. C. and M. Lesieur, 1977: Influence of helicity on high Reynolds number isotropic turbulence.*J. Fluid Mech.*, **81**, 187–207.

Chen, Q., S. Chen, and G. L. Eyink, 2003: The joint cascade of energy and helicity in three-dimensional turbulence. *Phys. Fluids*, **15**, 361–374.

Domaradzki, J. A., W. Liu, and M. E. Brachet, 1993: An analysis of subgrid-scale interactions in numerically simulated isotropic turbulence. *Phys. Fluids A*, **5**, 1747–1759.

Heisenberg, W., 1948: Zur statistischen theorie der turbulenz. *Z. Phys.*, **124**, 628–657.

Holm, D. D. and R. Kerr, 2002: Transient vortex events in the initial value problem for turbulence. *Phys. Rev. Lett.*, **88**, 244501.

Kraichnan, R. H., 1976: Eddy viscosity in two and three dimensions. *J. Atmos. Sci.*, **33**, 1521–1536.