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DYNAMICAL SYSTEMS METHOD (DSM) FOR SOLVING NONLINEAR OPERATOR EQUATIONS IN BANACH SPACES

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Abstract. Let $F(u) = h$ be a solvable operator equation in a Banach space X with a Gateaux differentiable norm. Under minimal smoothness assumptions on F, sufficient conditions are given for the validity of the Dynamical Systems Method (DSM) for solving the above operator equation. It is proved that the DSM (Dynamical Systems Method)

$$\dot{u}(t) = -A^{-1}_{a(t)}(u(t))[F(u(t)) + a(t)u(t) - f], \quad u(0) = u_0,$$

converges to y as $t \to +\infty$, for $a(t)$ properly chosen. Here $F(y) = f$, and $\dot{u}$ denotes the time derivative.

1 Introduction

Consider an operator equation

$$F(u) = f, \tag{1.1}$$

where F is an operator in a Banach space X with a Gateaux-differentiable norm. Assume that F is continuously Fréchet differentiable, $F'(u) := A(u)$. Denote by $A_a := A + aI$, where I is the identity operator, and by c_j, $j = 0, 1, 2, 3$, various positive constants. Let L be a smooth path on the complex plane $\mathbb{C}$ joining the origin and some point a_0, $0 < |a_0| < \epsilon_0$, where $\epsilon_0 > 0$ is a small fixed number independent of u.

The following assumptions **A1- A3** are valid throughout the paper.

A1. *Assume that*

$$\|A(u) - A(v)\| \le c_0\|u - v\|^{\kappa}, \quad \kappa \in (0, 1], \tag{1.2}$$

where κ is a constant.

A2. *Assume that*

$$\|A^{-1}_a(u)\| \le \frac{c_1}{|a|^b}; \;\; \forall a \in L, \;\; 0 < |a| < \epsilon_0. \tag{1.3}$$

Assumption (1.3) holds if there is a smooth path L on a complex a-plane, consisting of regular points of the operator $A(u)$, such that the norm of the resolvent $A^{-1}_a(u)$ grows, as $a \to 0$, not faster than a power $|a|^{-b}$. Thus, assumption (1.3) is a weak assumption. For example, assumption (1.3) is satisfied for the class of linear operators A, satisfying *the spectral assumption,* introduced in [10], Chapter 8. This spectral assumption says, that the set $\{a : |\arg a - \pi| \le \phi_0,\ 0 < |a| < \epsilon_0\}$ consists of the regular points of the operator A. This assumption implies the estimate $||A_a^{-1}|| \le \frac{c_1}{a}$, $0 < a < \epsilon_0$, that is, estimate (1.3) with $b = 1$ and $a \in (0, \epsilon_0)$.

A3. *Assume that the equation*

$$F(w_a) + aw_a - f = 0, \quad a \in L, \tag{1.4}$$

is uniquely solvable for any $f \in X$, and

$$\lim_{a \to 0, a \in L} \|w_a - y\| = 0, \quad F(y) = f. \tag{1.5}$$

Assume that there exists a constant $c > 0$ such that

$$|\dot{a}(t)| \le c|\dot{r}(t)|, \qquad r(t) := |a(t)|. \tag{1.6}$$

In formula (2.1) (see below) inequality $|\dot{r}(t)| \le |\dot{a}(t)|$ is established. Thus,

$$|\dot{r}(t)| \le |\dot{a}(t)| \le c|\dot{r}(t)|. \tag{1.7}$$

We formulate the main result at the end of the paper for convenience of the reader, because some additional assumptions, used in the proof of Theorem 2.1 are flexible and will arise naturally in the course of the proof.

One of the goals in this paper is to demonstrate the methodology for establishing the convergence results of the type obtained in Theorem 2.1.

All our assumptions are satisfied, for example, if F is a monotone operator in a Hilbert space H and L is a segment $[0, \epsilon_0]$. In this case $c_1 = 1$ and $b = 1$. Our assumptions are satisfied for the class of operators satisfying a spectral assumption, mentioned above, which was studied in [10] in connection to the Dynamical System Method (DSM) for solving operator equations. Sufficient conditions for (1.5) to hold are given in [10].

Every equation (1.1) with a linear, closed, densely defined in a Hilbert space H operator $F = A$ can be reduced to an equation with a monotone operator A^*A, where A^* is the adjoint to A. The operator $T := A^*A$ is selfadjoint and densely defined in H. If $f \in D(A^*)$, where $D(A^*)$ is the domain of A^*, then the equation $Au = f$ is equivalent to $Tu = A^*f$, provided that $Au = f$ has a solution, i.e., $f \in R(A)$, where $R(A)$ is the range of A. Recall that $D(A^*)$ is dense in H if A is closed and densely defined in H. If $f \in R(A)$ but $f \notin D(A^*)$, then equation $Tu = A^*f$ still makes sense and its normal solution y, i.e., the solution with minimal norm, can be defined as

$$y = \lim_{a \to 0} T_a^{-1} A^* f. \tag{1.8}$$

One proves that $Ay = f$, and $y \perp N(A)$, where $N(A)$ is the null-space of A. These results are proved in [10].

Our aim is to prove convergence of the DSM for solving equation (1.1):

$$\dot{u}(t) = -A_{a(t)}^{-1}(u(t))[F(u(t)) + a(t)u(t) - f], \quad u(0) = u_0, \tag{1.9}$$

where $u_0 \in X$ is an initial element, $a(t) \in C^1[0,\infty)$, $a(t) \in L$. The DSM version (1.9) is a computationally efficient analog of a continuous regularized Newton's method for solving equation (1.1). Other versions of DSM are studied in [10]. In [16] an approach to a justification of the DSM in Banach spaces is developed. The ideas from [16] are used in this paper. Among other things, an important Lemma 1 is formulated in a more general form than in [10], see also [7],[12], [8], [14], [15], [9]. Our main result is formulated in Theorem 5, in Section 2.

The DSM for solving operator equations has been developed in the monograph [10]. It was used as an efficient computational tool in [6], [9]. One of the earliest papers on the continuous analogue of Newton's method for solving well-posed nonlinear operator equations was [4].

The novel points in our paper include the larger class of the operator equations than earlier considered, and the weakened assumptions on the smoothness of the nonlinear operator F: in [10] it was often assumed that $F''(u)$ is locally bounded, in the current paper a much weaker assumption (1.2) is used.

Our proof of Theorem 5 uses the following result.

Lemma 1. *Assume that $g(t) \geq 0$ is continuously differentiable on any interval $[0,T)$, on which it is defined, and satisfies the following inequality:*

$$\dot{g}(t) \leq -\gamma(t)g(t) + \alpha(t,g) + \beta(t), \quad t \in [0,T), \tag{1.10}$$

where $\alpha(t,g)$, $\gamma(t)$ and $\beta(t)$ are rel-valued continuous on $[0,\infty)$ functions of t, $\alpha(t,g)$ is locally Lipschitz with respect to g. Suppose that there exists a $\mu(t) > 0$, $\mu(t) \in C^1[0,\infty)$, such that

$$\alpha(t,\mu^{-1}(t)) + \beta(t) \leq \mu^{-1}(t)[\gamma(t) - \dot{\mu}(t)\mu^{-1}(t)], \qquad t \geq 0, \tag{1.11}$$

and

$$\mu(0)g(0) \leq 1. \tag{1.12}$$

Then $T = \infty$, i.e., g exists on $[0,\infty)$, and

$$0 \leq g(t) \leq \mu^{-1}(t), \quad t \geq 0. \tag{1.13}$$

This lemma generalizes some results from [10], [13]. It is useful in a study of large-time behavior of solutions to evolution problems, which are important in many appications, see, for example, [1], [18], [14]. Lemma 1 is proved at the end of the paper for convenience of the reader and for making this paper essentially self-contained. We apply Lemma 1 with $\alpha(t,g) = \alpha(t)g^p$, $p > 1$ is a constant, and $\alpha(t) > 0$ is a continuous function.

In Section 2 a method is given for a proof of the following conclusions:
There exists a unique solution $u(t)$ to problem (1.9) for all $t \geq 0$, there exists $u(\infty) := \lim_{t\to\infty} u(t)$, and $F(u(\infty)) = f$, that is:

$$\exists! u(t) \quad \forall t \geq 0;\ \exists u(\infty); \quad F(u(\infty)) = f. \tag{1.14}$$

The assumptions on u_0 and $a(t)$ under which conclusions (1.14) hold for the solution to problem (1.9) are formulated in Theorem 5 in Section 2. Theorem 5 in Section 2 is our main result. Roughly speaking, this result says that conclusions (1.14) hold for the solution to problem (1.9), provided that $a(t)$ is suitably chosen.

2 Proofs

Let $|a(t)| := r(t) > 0$. If $a(t) = a_1(t) + ia_2(t)$, where $a_1(t) = \operatorname{Re} a(t)$, $a_2(t) = \operatorname{Im} a(t)$, then

$$|\dot{r}(t)| \le |\dot{a}(t)|. \tag{2.1}$$

Indeed,

$$|\dot{r}(t)| = \frac{|a_1\dot{a}_1 + a_2\dot{a}_2|}{r(t)} \le \frac{r(t)|\dot{a}(t)|}{r(t)}, \tag{2.2}$$

and (2.2) implies (2.1).

Let

$$g(t) := \|z(t)\|, \qquad z(t) := u(t) - w_a(t), \tag{2.3}$$

where $u(t)$ solves (1.9) and $w_a(t)$ solves (1.4) with $a = a(t)$. By the assumption, $w_a(t)$ exists for every $t \ge 0$. The local existence of $u(t)$, the solution to (1.9), is the conclusion of Lemma 2. Let $\psi(t) \in C^1([0,\infty); X)$. In the following lemma a proof of local existence of the solution to problem (1.9) is given by a novel argument. The right-hand side of (1.9) is a nonlinear function of u, which does not, in general, satisfy the Lipschitz condition. This condition is the standard condition in the usual proofs of the local existence of the solution to an evolution problem. Our argument uses an abstract inverse function theorem. *This argument is valid under the minimal assumption that $F'(u)$ depends continuously on u.*

Lemma 2. *If assumption (1.3) holds and (1.4) is uniquely solvable for any $f \in X$, then the solution $u(t)$ to (1.9) exists locally.*

Proof. Differentiate equation (1.4) with $a = a(t)$ with respect to t. The result is

$$A_{a(t)}(w_a(t))\dot{w}_a(t) = -\dot{a}(t)w_a(t), \tag{2.4}$$

or

$$\dot{w}_a(t) = -\dot{a}(t)A^{-1}_{a(t)}(w_a(t))w_a(t). \tag{2.5}$$

Denote

$$\psi(t) := F(u(t)) + a(t)u(t) - f. \tag{2.6}$$

For any $\psi \in H$ equation (2.6) is uniquely solvable for $u(t)$ by our assumption (1.4), which is used with $f + \psi(t)$ in place of f in (1.4). By the inverse function theorem, which holds due to our assumption (1.3), and by assumption (1.2), the solution $u(t)$ to (2.6) is continuously differentiable with respect to t provided $\psi(t)$ is. One may solve (2.6) for u and write $u = G(\psi)$, where the map G is continuously Fréchet differentiable because F is.

Differentiate (2.6) and get

$$\dot{\psi}(t) = A_{a(t)}(u(t))\dot{u}(t) + \dot{a}(t)u. \tag{2.7}$$

If one wants the solution to (2.6) to be a solution to (1.9), then one has to require that

$$A_{a(t)}(u(t))\dot{u} = -\psi(t). \tag{2.8}$$

If (2.8) holds, then (2.7) can be written as

$$\dot{\psi}(t) = -\psi + \dot{a}(t)G(\psi), \quad G(\psi) := u(t), \tag{2.9}$$

where $G(\psi)$ is continuously Fréchet differentiable. Thus, equation (2.9) is equivalent to (1.9) at all $t \geq 0$ if

$$\psi(0) = F(u_0) + a(0)u_0 - f. \tag{2.10}$$

Indeed, if u solves (1.9) then ψ, defined in (2.6), solves the Cauchy problem (2.9)-(2.10). Conversely, if ψ solves (2.9)-(2.10), then $u(t)$, defined as the unique solution to (2.6), solves (1.9). Since the right-hand side of (2.9) is Fréchet differentiable, it satisfies a local Lipschitz condition. Thus, problem (2.9)-(2.10) is locally, solvable. Therefore, problem (1.9) is locally solvable.
Lemma 2 is proved. □

It is known (see, for example, [10]) that the solution $u(t)$ to (1.9) exists globally if the following estimate holds:

$$\sup_{t\geq 0} \|u(t)\| < \infty. \tag{2.11}$$

Lemma 3. *Estimate* (2.11) *holds.*

Proof. Denote

$$z(t) := u(t) - w(t), \tag{2.12}$$

where $u(t)$ solves (1.9) and $w(t) = w_{a(t)}$ solves (1.4) with $a = a(t)$. When $t \to \infty$, the function $w(t)$ tends to the limit y by (1.5), and, therefore, is uniformly bounded. If one proves that

$$\lim_{t\to\infty} \|z(t)\| = 0, \tag{2.13}$$

then (2.11) follows from (2.13) and the boundedness of $w(t)$. Indeed,

$$\sup_{t\geq 0} \|u(t)\| \leq \sup_{t\geq 0} \|z(t)\| + \sup_{t\geq 0} \|w(t)\| < \infty. \tag{2.14}$$

To prove (2.13) *we use Lemma 1.* □

Rewrite (1.9) as

$$\dot{z} = -\dot{w} - A^{-1}_{a(t)}(u(t))[F(u(t)) - F(w(t)) + a(t)z(t)]. \tag{2.15}$$

Lemma 4. *If the norm $\|w(t)\|$ in X is differentiable, then*

$$\left|\frac{d\|w(t)\|}{dt}\right| \leq \|\dot{w}(t)\|. \tag{2.16}$$

Proof. The triangle inequality implies:

$$\frac{\|w(t+s)\| - \|w(t)\|}{s} \leq \frac{\|w(t+s) - w(t)\|}{s}, \quad s > 0. \tag{2.17}$$

Passing to the limit $s \searrow 0$ and using the assumption concerning the differentiability of the norm in X, one gets $\frac{d\|w(t)\|}{dt} \leq \|\dot{w}(t)\|$. Similarly, one gets $-\frac{d\|w(t)\|}{dt} \leq \|\dot{w}(t)\|$. These two inequalities yield (2.16). □

Various necessary and sufficient conditions for the Gateaux or Fréchet differentiability of the norm in Banach spaces are known in the literature (see, for example, [2] and [3]), starting with Shmulian's paper of 1940, see [17].

Hilbert spaces, $L^p(D)$ and ℓ^p-spaces, $p \in (1,\infty)$, and Sobolev spaces $W^{\ell,p}(D)$, $p \in (1,\infty)$, $D \subset \mathbb{R}^n$ is a bounded domain, have Fréchet differentiable norms. These spaces are uniformly convex and they have the following property: if $u_n \rightharpoonup u$ and $||u_n|| \to ||u||$ as $n \to \infty$, then $\lim_{n\to\infty} ||u_n - u|| = 0$.

From (2.5) and (1.7) one gets

$$\|\dot{w}\| \le c_1|\dot{a}(t)|r^{-b}(t)\|w(t)\|, \quad r(t) = |a(t)|, \tag{2.18}$$

where $w(t) := w_a(t)$. Since we assume that $\lim_{t\to\infty} |a(t)| = 0$, one concludes that (1.5) and (2.18) imply the following inequality:

$$\|\dot{w}\| \le c_2|\dot{a}(t)|r^{-b}(t), \quad c_2 = const > 0, \tag{2.19}$$

because (1.5) implies the following estimate:

$$c_1\|w(t)\| \le c_2, \quad t \ge 0. \tag{2.20}$$

Inequalities (1.7) and (2.19) imply that

$$\|\dot{w}\| \le c_2|\dot{r}(t)|r^{-b}(t), \quad t \ge 0. \tag{2.21}$$

Recall that $F'(u) := A(u)$ and note that

$$F(u) - F(w) = \int_0^1 F'(w + sz)dsz = A(u)z + \int_0^1 [A(w+sz) - A(u)]dsz. \tag{2.22}$$

Thus, one can write (2.15) as

$$\dot{z}(t) = -z(t) - \dot{w}(t) - A^{-1}_{a(t)}(u(t))\eta(t) := -z(t) + W, \tag{2.23}$$

$$\|\eta(t)\| = O(g^p(t)), \quad p = 1+\kappa, \quad g(t) := \|z(t)\|, \tag{2.24}$$

where estimate (1.2) was used, and W is defined by the formula

$$W := -\dot{w}(t) - A^{-1}_{a(t)}(u(t))\eta(t). \tag{2.25}$$

Let

$$Z(t) := e^t z(t). \tag{2.26}$$

Then (2.23) yields

$$e^{-t}\dot{Z} = W. \tag{2.27}$$

Taking the norm of this equation yields

$$e^{-t}\|\dot{Z}\| = \|W\|. \tag{2.28}$$

One has

$$\|W\| \le c_2|\dot{r}(t)|r^{-b}(t) + c_3 r^{-b}(t)g^p(t), \qquad g(t) := \|z(t)\|, \quad p = 1+\kappa, \tag{2.29}$$

where $c_3 := c_0c_1$, c_0 is the constant from (1.2) and c_1 is the constant from (1.3). Using estimate (2.16), one gets

$$\|\dot{Z}\| \geq \left|\frac{d\|Z(t)\|}{dt}\right| = \left|\frac{d(e^t g(t))}{dt}\right|. \tag{2.30}$$

Using formulas (2.27)-(2.30) one gets from (2.23) the following inequality:

$$\dot{g}(t) \leq -g + c_2|\dot{r}(t)|r^{-b}(t) + c_3 r^{-b}(t) g^p, \qquad g(t) = \|z(t)\|, \quad p = 1+\kappa. \tag{2.31}$$

Inequality (2.31) is of the form (1.10) with

$$\gamma(t) = 1, \quad \alpha(t) = c_3 r^{-b}(t), \quad \beta(t) = c_2|\dot{r}(t)|r^{-b}(t). \tag{2.32}$$

Choose

$$\mu(t) = \lambda r^{-k}(t), \quad \lambda = const > 0, \quad k = const > 0. \tag{2.33}$$

Then

$$\dot{\mu}\mu^{-1} = -k\dot{r}r^{-1}. \tag{2.34}$$

Let us assume that, as $t \to \infty$,

$$r(t) \searrow 0, \quad \dot{r} < 0, \quad |\dot{r}| \searrow 0. \tag{2.35}$$

Assumption (1.12) implies

$$g(0)\frac{\lambda}{r^k(0)} < 1, \tag{2.36}$$

and inequality (1.11) holds if

$$\frac{c_3 r^{-b}(t) r^{kp}}{\lambda^p} + c_2|\dot{r}(t)|r^{-b}(t) \leq \frac{r^k(t)}{\lambda}\big(1 - k|\dot{r}(t)|r^{-1}(t)\big), \qquad t \geq 0. \tag{2.37}$$

Inequality (2.37) can be written as

$$\frac{c_3 r^{k(p-1)-b}(t)}{\lambda^{p-1}} + \frac{c_2\lambda|\dot{r}(t)|}{r^{k+b}(t)} + \frac{k|\dot{r}(t)|}{r(t)} \leq 1, \qquad t \geq 0. \tag{2.38}$$

Let us choose k so that $k(p-1) - b = 1$, that is,

$$k = \frac{b+1}{p-1}. \tag{2.39}$$

Choose λ, for example, as follows:

$$\lambda := \frac{r^k(0)}{2g(0)}. \tag{2.40}$$

Then inequality (2.36) holds, and inequality (2.38) can be written as:

$$c_3\frac{r(t)[2g(0)]^{p-1}}{[r^k(0)]^{p-1}} + c_2\frac{r^k(0)}{2g(0)}\frac{|\dot{r}(t)|}{r^{k+b}(t)} + k\frac{|\dot{r}(t)|}{r(t)} \leq 1, \qquad t \geq 0. \tag{2.41}$$

Note that (2.39) implies:

$$k + b = kp - 1. \tag{2.42}$$

Choose $r(t)$ so that relations (2.35) hold and

$$k\frac{|\dot{r}(t)|}{r(t)} \le \frac{1}{2}, \qquad t \ge 0. \tag{2.43}$$

Since $r(0) \ge r(t)$ and (2.43) holds, then inequality (2.41) holds if

$$c_3\frac{[2g(0)]^{p-1}}{r^b(0)} + c_2\frac{r^k(0)}{2g(0)}\frac{|\dot{r}(t)|}{r^{kp-1}} \le \frac{1}{2}, \qquad t \ge 0. \tag{2.44}$$

Denote

$$c_2\frac{r^k(0)}{2g(0)} = c_2\lambda := c_4. \tag{2.45}$$

Let

$$c_4\frac{|\dot{r}(t)|}{r^{kp-1}} = \frac{1}{4}, \qquad t \ge 0, \tag{2.46}$$

and $kp > 2$. Then equation (2.46) implies

$$r(t) = [c_5 + c_6 t]^{-\frac{1}{kp-2}}, \quad c_5 = r^{2-kp}(0), \quad c_6 = \frac{kp-2}{4c_4}, \tag{2.47}$$

where c_5 and c_6 are positive constants. Their explicit values are not used below.This $r(t)$ satisfies conditions (2.35), and equation (2.46) can be rewritten as:

$$k\frac{|\dot{r}(t)|}{r(t)} = \frac{kr^{kp-2}(t)}{4c_4}, \quad t \ge 0. \tag{2.48}$$

Recall that $r(t)$ decays monotonically. Therefore, inequality (2.43) holds if

$$\frac{kr^{kp-2}(0)}{4c_4} \le \frac{1}{2}. \tag{2.49}$$

Inequality (2.49) holds if

$$\frac{kg(0)}{c_2}r^{k(p-1)-2}(0) = \frac{kg(0)}{c_2}r^{b-1}(0) \le 1, \tag{2.50}$$

because (2.39) implies:

$$k(p-1) - 2 = b - 1. \tag{2.51}$$

Condition (2.50) holds if $g(0)$ is sufficiently small or $r^{b-1}(0)$ is sufficiently large:

$$g(0) \le \frac{c_2}{k}r^{b-1}(0). \tag{2.52}$$

If $b > 1$, then condition (2.52) holds for any fixed $g(0)$ if $r(0)$ is sufficiently large. If $b = 1$, then (2.52) holds if $g(0) \le \frac{c_2}{k}$. If $b \in (0, 1)$ then (2.52) holds either if $g(0)$ is sufficiently small or $r(0)$ is sufficiently small.

If (2.47) and (2.52) hold, then (2.46) holds. Consequently, (2.44) holds if

$$c_3 \frac{[2g(0)]^{p-1}}{r^b(0)} \le \frac{1}{4}. \tag{2.53}$$

It follows from (2.52) that (2.53) holds if

$$c_3 2^{p-1} \left(\frac{c_2}{k}\right)^{p-1} \frac{1}{r^{-1+p+2b-bp}(0)} \le \frac{1}{4}. \tag{2.54}$$

One has $p = 1 + \kappa$, and $\kappa \in (0, 1]$. If $b > 0$ and $\kappa \in (0, 1]$, then

$$-1 + p - pb + 2b = \kappa + (1 - \kappa)b > 0. \tag{2.55}$$

Thus, (2.54) always holds if $r(0)$ is sufficiently large, specifically, if

$$r(0) \ge [4c_3 \left(2c_2 k^{-1}\right)^{p-1}]^{\frac{1}{\kappa+(1-\kappa)b}}. \tag{2.56}$$

We have proved the following theorem.

Theorem 5. *Let the assumptions* **A1, A,2, and A3** *hold. If* $r(t) = |a(t)|$ *is defined in* (2.47)*, and inequalities* (2.52) *and* (2.56) *hold, then*

$$\|z(t)\| < r^k(t)\lambda^{-1}, \qquad \lim_{t\to\infty} \|z(t)\| = 0. \tag{2.57}$$

Thus, problem (1.9) *has a unique global solution* $u(t)$ *and*

$$\lim_{t\to\infty} \|u(t) - y\| = 0, \tag{2.58}$$

where $F(y) = f$.

Proof of Lemma 1. Inequality (1.10) can be written as

$$-\gamma(t)\mu^{-1}(t) + \alpha(t, \mu^{-1}(t)) + \beta(t) \le \frac{d\mu^{-1}(t)}{dt}. \tag{2.59}$$

Let $\phi(t)$ solve the following Cauchy problem:

$$\dot{\phi}(t) = -\gamma(t)\phi(t) + \alpha(t, \phi(t)) + \beta(t), \quad t \ge 0, \quad \phi(0) = \phi_0. \tag{2.60}$$

The assumption that $\alpha(t, g)$ is locally Lipschitz with respect to g guarantees local existence and uniqueness of the solution $\phi(t)$ to problem (2.60). From the known comparison result (see, for instance, [5], Theorem III.4.1) it follows that

$$\phi(t) \le \mu^{-1}(t) \qquad \forall t \ge 0, \tag{2.61}$$

provided that $\phi(0) \le \mu^{-1}(0)$, where $\phi(t)$ is the unique solution to problem (2.60). Let us take $\phi(0) = g(0)$. Then $\phi(0) \le \mu^{-1}(0)$ by the assumption in Lemma 1, and inequality (1.10) implies that

$$g(t) \le \phi(t) \qquad t \in [0, T). \tag{2.62}$$

Inequalities $\phi(0) \le \mu^{-1}(0)$, (2.61), and (2.62) imply

$$g(t) \le \phi(t) \le \mu^{-1}(t), \qquad t \in [0, T). \tag{2.63}$$

By the assumption, the function $\mu(t)$ is defined for all $t \ge 0$ and is bounded on any compact subinterval of the set $[0, \infty)$. Consequently, the functions $\phi(t)$ and $g(t) \ge 0$ are defined for all $t \ge 0$, and estimate (1.13) is established. Lemma 1 is proved. □

When this paper was under consideration, convergence of the DSM for general operator equations was established in [19].

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