Wave scattering by small bodies and creating materials with a desired refraction coefficient

Alexander G. Ramm

Published Version Information

Citation: Ramm, A. G. (2011). Wave scattering by small bodies and creating materials with a desired refraction coefficient**.** Afrika Matematika, 22(1), 33-55.

Digital Object Identifier (DOI): doi:10.1007/s13370-011-0004-3

Publisher's Link: http://www.springerlink.com/content/6140t00056137861/

Afrika Matematika, 22, N1, (2011), 33-55.

Wave scattering by small bodies and creating materials with a desired refraction coefficient

Alexander G. Ramm
Department of Mathematics
Kansas State University, Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu

Abstract

In this paper the author's invited plenary talk at the 7-th PACOM (PanAfrican Congress of Mathematicians), is presented.

Asymptotic solution to many-body wave scattering problem is given in the case of many small scatterers. The small scatterers can be particles whose physical properties are described by the boundary impedances, or they can be small inhomogeneities, whose physical properties are described by their refraction coefficients. Equations for the effective field in the limiting medium are derived. The limit is considered as the size a of the particles or inhomogeneities tends to zero while their number $M(a)$ tends to infinity. These results are applied to the problem of creating materials with a desired refraction coefficient. For example, the refraction coefficient may have wave-focusing property, or it may have negative refraction, i.e., the group velocity may be directed opposite to the phase velocity. This paper is a review of the author's results presented in MR2442305 (2009g:78016), MR2354140 (2008g:82123), MR2317263 (2008a:35040), MR2362884 (2008j:78010), and contains new results.

MSC: 35J05, 65R20, 65Z05, 74Q10
Key words: wave scattering,small scatterers, wave focusing, negative refraction, metamaterials

1 Introduction

In this paper the author's invited plenary talk at the 7-th PACOM (PanAfrican Congress of Mathematicians), is presented. This PACOM was held in August 3-8, 2009, in Yamoussoukro, Ivory Coast.

Wave scattering by small bodies is a classical area of research which goes back to Rayleigh (1871) (see, e.g., [10],[11]) who understood that the main part of the field scattered by a small body is the dipole radiation. For spherical and ellipsoidal bodies the dipole moment can be calculated analytically. For small bodies of arbitrary shapes analytical formulas for the S-matrix for acoustic and electromagnetic wave scattering were derived by the author ([15]-[18]). Wave scattering by many bodies was studied extensively because of high interest to this problem and its importance in applications. A review of this research area is given in [11]. It contains 1386 references, but the bibliography is far from complete. Monograph [18] contains a systematic presentation of the author's results on wave scattering by many small bodies of arbitrary shapes, formulas for polarizability tensors for dielectric and conducting bodies of arbitrary shapes, for electrical capacitances of perfect conductors of arbitrary shapes, and for S-matrices for acoustic and electromagnetic (EM) wave scattering by small bodies of arbitrary shapes.

In recent works [19]–[34] the author has developed an asymptotically exact theory of wave scattering by many small bodies (particles) embedded in an inhomogeneous medium. The medium can be dielectric and conducting. The particles can also be dielectric and/or conducting. Acoustic and EM wave scattering theory was developed in the cited papers. This theory is presented in Sections 2 and 4.

The novel feature of the author's approach is to seek not some boundary functions on the surface of the small scatterers, but rather some numbers Q_m, $1 \leq m \leq M$, where M is the total number of the scatterers, and we are especially (but not only) interested in the case $M \gg 1$. This approach not only simplifies the derivations drastically and allows the asymptotic treatment of the many-body wave scattering problem, but also has a clear physical meaning. Namely, the numbers Q_m can be interpreted as "total charges" of the $m-$th small body (particle), as will become clear in Section 2. This approach allows one to avoid solving boundary integral equations for the unknown boundary functions (analogous to the surface charge distributions or surface currents), and to find the main terms of Q_m asymptotically, when the characteristic size a of the particles tends to zero, while their total number $M = M(a)$ tends to infinity at an appropriate rate. The numbers Q_m define the effective field in the medium with many embedded particles. Another novel feature of our theory is the treatment of the scattering by many small particles embedded in an inhomogeneous medium, rather than in a free space or in a homogeneous space.

In Section 3 a method is given for creating materials with a desired refraction coefficient by embedding many small particles into a given material. The embedded small particles may be balls without loss of generality, because using small balls with a suitable boundary impedance one can already create material with a desired refraction coefficient. The physical properties of the embedded small balls are described by their boundary impedances. The radius of these small balls is a. The smallness of the particles is described by the assumption $ka \ll 1$, where k is the wave number in the original material. We formulate a recipe for creating materials with a desired refraction coefficient and, also, two technological problems, which have to be solved in order to implement our recipe practically.

We do not discuss in this paper possible applications of the materials with a desired refraction coefficient. These applications are, probably, numerous. We mention two of these applications:

1) Creating materials with a desired wave-focusing properties (see [24], [25], [27]),

and

2) Creating materials with negative refraction, i.e., materials in which the direction of the group velocity of waves is opposite to the direction of their phase velocity (see [24], [22], [31]).

In Section 5 we develop a theory of wave scattering by small inhomogeneities. The difference between a small inhomogeneity and a particle in this paper can be explained as follows: physical properties of a small particle are described by its boundary impedance, while these properties of an inhomogeneity are described by its refraction coefficient. One may hope that the advantage of using small inhomogeneities, rather than small particles, for creating materials with a desired refraction coefficient, consists of relative ease in preparing small inhomogeneities with a desired constant refraction coefficient, which may have a desired absorption property and a desired tensorial character. A disadvantage of using small inhomogeneities comes from the fact that their number is $O(\frac{1}{a^3})$, which is much larger than the number $O(\frac{1}{a^{2-\kappa}})$, $\kappa \in (0,1)$, of small particles, used for preparing material with a desired refraction coefficient.

In Section 6 some auxiliary results are given. These results include a justification of a version of the collocation method for solving operator equations of the type $(I+T)u = f$ in a Banach space X. Here I is the identity operator and T is a linear compact operator in X. These results are used in our paper for a justification of the limiting procedure $a \to 0$, and for a derivation of the integral equation for the effective field in the limiting medium obtained by embedding $M = M(a)$ small particles as $M \to \infty$. In Section 6 we also derive two lemmas which allow one to pass to the limit in certain sums as $a \to 0$.

Numerical results, based on our theory, are not presented here. The reader can find these results in [3], [35], [4], [8], [2].

Our work can be considered as a work in the area of the homogenization theory. The homogenization theory was discussed in many papers and books, see [5, 9, 12, 14] and references therein. On the other hand, our theory differs from the earlier ones in several respects: we do not assume periodic structure in the medium, which is often assumed in the cited literature, our differential expressions are non-selfadjoint, in contrast to the usual assumptions, and our boundary conditions are non-selfadjoint as well, so we treat non-selfadjoint operators.

2 Wave scattering by small bodies embedded in an inhomogeneous medium

Let us assume that a bounded domain $D \subset \mathbb{R}^3$ is filled with a material with refraction coefficient $n_0^2(x)$. The scattering problem consists of solving the

Helmholtz equation

$$L_0u_0 := [\nabla^2 + k^2n_0^2(x)]u_0 = 0 \quad \text{in } \mathbb{R}^3,\ k = const > 0, \tag{1}$$

$$u_0 = e^{ik\alpha\cdot x} + v_0, \tag{2}$$

$$\frac{\partial v_0}{\partial r} - ikv_0 = o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \tag{3}$$

where the radiation condition (3) holds uniformly in directions $\frac{x}{r} := \beta$. Here k is wave number, $n_0^2(x) = 1$ in $D' := \mathbb{R}^3 \setminus D$, Im $n_0^2(x) \geq 0$ in D, $\alpha \in S^2$ is the direction of the incident plane wave, S^3 is the unit sphere, $n_0^2(x)$ is a Riemann-integrable function, that is, a bounded function whose discontinuities form a set of Lebesgue measure zero. It is known that problem (1)-(3) has a unique solution (see [22]). A proof of this can be obtained by considering the following integral equation

$$u_0(x) = e^{ik\alpha\cdot x} + k^2\int_D g(x,y,k)[n_0^2(x) - 1]u_0(y)dy, \quad g(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}. \tag{4}$$

Equation (4) is a Fredholm-type equation in the Banach space $C(D)$, and the corresponding homogeneous equation has only the trivial solution. By the Fredholm alternative equation (4) is uniquely solvable. It is easy to check that equation (4) is equivalent to problem (1)-(3). By $C(D)$, $L^2(D)$ and $H^\ell(D)$, $C^s(D)$, we denote the usual functional spaces of continuous functions, square-integrable functions, Sobolev spaces, and Hölder spaces, respectively.

Suppose now that $M = M(a)$ small bodies D_m, $1 \leq m \leq M$, are embedded in D, $a = \frac{1}{2}\max_{1\leq m\leq M}\text{diam } D_m$, $n_0 := \max_{x\in D}|n_0(x)|$.

Assume that

$$kan_0 \ll 1, \quad d \gg a, \tag{5}$$

where d is the minimal distance between two neighboring particles. The distribution of the embedded particles is described as follows. Let $x_m \in D_m$, D_m is the m-th small particle, $1 \leq m \leq M$, x_m is a point in D_m. If D_m is a ball of radius a, as we assume for simplicity, then x_m is the center of this ball. Let $\triangle \subset D$ be an arbitrary open set in D, $N(x) \geq 0$ is a given continuous function, and

$$\mathcal{N}(\triangle) = \frac{1}{a^{2-\kappa}}\int_\triangle N(x)dx[1 + o(1)], \quad 0 < \kappa < 1, \tag{6}$$

is the number of points x_m in $\triangle$, in other words, the number of the particles embedded in $\triangle$. The numerical parameter κ we can choose as we wish. The distribution law (6) is quite natural and general. It includes the case of uniform distribution of particles if $N(x)$ is independent of x, and periodic structures, if $N(x)$ is, for example, a periodic sequence of narrow smooth pulses (mollified delta-functions supported at the locations of the vertices of periodic cell).

On the surface S_m of the $m-$th particle D_m an impedance boundary condition holds. The scattering problem can be formulated as follows:

$$L_0u_M = 0 \quad \text{in } \mathbb{R}^3 \setminus \cup_{m=1}^M D_m, \tag{7}$$

$$\frac{\partial u_M}{\partial \nu} = \zeta_m u_M \quad \text{on } S_m, \quad 1 \le m \le M, \tag{8}$$

$$u_M = u_0 + v_M, \tag{9}$$

where u_0 solves problem (1)-(3), the operator L_0 is defined in (1), the boundary impedance $\zeta_m = \frac{h(x_m)}{a^\kappa}$, where $h(x)$ is a continuous function in D, it is assumed that $\text{Im}h(x) \le 0$ in D, u_M is the total field, v_M is the scattered field satisfying the radiation condition (3), and ν is the normal to S_m pointing out of D_m.

By $\delta(x)$ the delta-function will be denoted.

Lemma 1 ([22])**.** *If Im $n_0^2(x) \ge 0$ and Im $h(x) \le 0$, then problem* (7)-(9) *has a unique solution, and this solution can be found in the form*

$$u_M(x) = u_0(x) + \sum_{m=1}^{M} \int_S G(x,s)\sigma_m(s)ds, \tag{10}$$

where G is the Green's function of the operator L_0:

$$L_0 G(x,y) = -\delta(x-y) \quad \textit{in } \mathbb{R}^3, \tag{11}$$

and $\sigma_m(s)$, $1 \le m \le M$, are some functions.

The function $\sigma_m(s)$ solves the following equation which comes from the boundary condition (8):

$$u_{e\nu} - \zeta_m u_e + \frac{A_m\sigma_m - \sigma_m}{2} - \zeta_m T_m \sigma_m = 0 \quad \text{on } S_m. \tag{12}$$

The regularity of the function σ_m depends on the regularity of the boundary S_m and on the regularity of u_e. In this paper S_m and u_e are smooth, and so are σ_m.

In equation (12) $u_e(x)$ is the effective field acting on the m-th particle. This field is defined by the formula:

$$u_e(x) := u_M(x) - \int_{S_m} G(x,s)\sigma_m(s)ds. \tag{13}$$

This definition is used in equation (12) when $|x - x_m| \sim a$. Thus, the field u_e, defined in (13), depends on m, when x is in a neighborhood of D_m, and on a, $u_e(x) = u_e^{(m)}(x,a)$. On the other hand, if $|x - x_m| \gg a$, then $u_e(x) \sim u_M(x)$ as $a \to 0$, because

$$\left| \int_{S_m} G(x,s)\sigma_m(s)ds \right| \le \frac{ca^{2-\kappa}}{|x - x_m|} = o(1), \quad |x - x_m| \ge a, \ a \to 0, \tag{14}$$

where $c > 0$ is a constant independent of a, and $\kappa \in (0,1)$ is a constant from (6). The operators A_m and T_m in equation (12) are defined as follows:

$$A_m\sigma_m := 2\int_{S_m} \frac{\partial G(s,t)}{\partial \nu_s}\sigma_m(t)dt, \quad T_m\sigma_m := \int_{S_m} G(s,t)\sigma_m(t)dt. \tag{15}$$

Thus, A_m is a normal derivative on S_m of a potential of single layer, and T_m is a potential of single layer. It is proved in [22] that

$$G(x,y) = \frac{1}{4\pi|x-y|}[1 + O(|x-y|)], \quad |x-y| \to 0, \tag{16}$$

and one can differentiate this formula. Let

$$A\sigma_m := \int_{S_m} \frac{\partial}{\partial \nu_s} \frac{1}{2\pi|s-t|}\sigma_m(t)dt, \quad T\sigma_m := \int_{S_m} \frac{1}{4\pi|s-t|}\sigma(t)dt. \tag{17}$$

Then

$$\|A_m - A\| = o(\|A\|), \ \|T - T_m\| = o(\|T\|), \quad a \to 0, \tag{18}$$

where the norm is in $C(S_m)$ or $L^2(S_m)$, and $\|T\| = O(a)$, $\|A\| = O(a)$ if S_m are smooth surfaces uniformly with respect to m, $1 \le m \le M$, and we make this assumption. Let us also assume that

$$\zeta_m = \frac{h(x_m)}{a^\kappa}, \quad d = O(a^{\frac{2-\kappa}{3}}), \quad M = O(\frac{1}{a^{2-\kappa}}), \quad \kappa \in (0,1), \tag{19}$$

where d is the distance between neighboring particles, $h \in C(D)$ is an arbitrary given function, Im $h \le 0$, and the points $x_m \in D$, $1 \le m \le M$, are distributed in D according to formula (6).

Our first goal is to find asymptotic formulas for σ_m, for $u_e(x)$, and for $Q_m := \int_{S_m} \sigma_m(t)dt$ as $a \to 0$.

Our second goal is to derive an integral equation for the limiting field $u(s) = \lim_{a\to 0} u_e(x)$, and to prove the existence of this limit assuming (6) and (19). Since k and n_0 are fixed, the first condition (5) is satisfied when $a \to 0$.

The usual approach to finding σ_m consists of solving M boundary integral equations for the unknown functions σ_m in (10). If M is large, this approach is not possible to use numerically or theoretically in order to achieve our goals. By this reason we have developed a new approach. We are looking for M numbers Q_m,

$$Q_m := \int_{S_m} \sigma_m(t)dt,$$

for which we derive an asymptotic formula. We prove that these numbers determine the main term of the asymptotics of the scattered field v_M as $a \to 0$. Compared with the standard approach, when one is looking for the unknown functions $\sigma_m(s)$, rather than numbers Q_m, our approach allows one to solve the many-body scattering problem when the scatterers are small.

To find asymptotics of Q_m, let us rewrite (10) as

$$u_M(x) = u_0(x) + \sum_{m=1}^{M} G(x,x_m)Q_m + \sum_{m=1}^{M} \int_{S_m} [G(x,s) - G(s,x_m)]\sigma_m(s)ds. \tag{20}$$

We will show that the term

$$J_m := \int_{S_m} [G(x,s) - G(s,x_m)]\sigma_m(s)ds$$

is negligible compared to the term $I_m := |G(x, x_m)Q_m|$ as $a \to 0$, that is,

$$|J_m| \ll |I_m|, \qquad 1 \le m \le M.$$

If this is proved, then

$$u_M(x) \sim u_0(x) + \sum_{m=1}^{M} G(x, x_m)Q_m, \quad a \to 0. \tag{21}$$

Formula (21) is valid for $x \in \mathbb{R}^3 \setminus \cup_{m=1}^{M} D_m$. Consequently, the solution to the scattering problem (7)-(9) is reduced, as $a \to 0$, to *finding the numbers Q_m rather than the functions $\sigma_m(s)$.*

In the following lemma we give an asymptotic formula for Q_m and $\sigma_m = \sigma(s)$ as $a \to 0$. It turns out that the main term of the asymptotics of $\sigma(s)$ as $a \to 0$ does not depend on s when S_m are spheres.

Lemma 2. *If assumptions* (19) *hold, then*

$$Q_m = -4\pi h(x_m)u_e(x_m)a^{2-\kappa}[1 + o(1)], \quad a \to 0, \tag{22}$$

$$\sigma_m = -h(x_m)u_e(x_m)a^{-\kappa}[1 + o(1)], \quad a \to 0. \tag{23}$$

The quantities $u_m := u_e(x_m)$ in (22) and (23) are not known. They can be found from a linear algebraic system (LAS):

$$\begin{aligned} u_j &= u_{0j} - 4\pi \sum_{m=1, m\neq j}^{M} G(x_j, x_m)h_m u_m a^{2-\kappa}, \\ u_{0j} &:= u_0(x_j), \ h_m := h(x_m), \quad 1 \le j \le M, \end{aligned} \tag{24}$$

where $u_j := u_e(x_j)$. This LAS is uniquely solvable for all sufficiently small a, as follows from the results in Section 6 on the convergence of the collocation method. These results are taken from [33]. If assumption (6) holds, then the limiting form of the LAS (24) as $a \to 0$ is the integral equation

$$u(x) = u_0(x) - 4\pi \int_D G(x, y)N(y)h(y)u(y)dy, \quad x \in \mathbb{R}^3, \tag{25}$$

where $N(y)$ is the function from (6).
Proof of Lemma 2. Let us integrate equation (12) over S_m and use the divergence theorem. The result is

$$\begin{aligned} \int_{D_m} \nabla^2 u_e(y)dy - \frac{h_m}{a^\kappa}\int_{S_m} u_e(s)ds - \frac{Q_m}{2} \\ + \frac{1}{2}\int_{S_m} A_m\sigma_m ds - \frac{h_m}{a^\kappa}\int_{S_m} ds \int_{S_m} G(s, t)\sigma_m(t)dt = 0. \end{aligned} \tag{26}$$

The solution u to equation (25) is in $C^2(\mathbb{R}^3)$, if, for example, $N(x)h(x)$ is Hólder-continuous. We prove that u is the limit in $C(D)$ of u_e as $a \to 0$. If $u_e \in C^2(\mathbb{R}^3)$, then

$$\int_{D_m} \nabla^2 u_e(y)dy = O(a^3), \qquad \int_{S_m} u_e(s)ds = O(a^2).$$

Using (18), we may replace A_m and T_m by A and T with negligible error as $a \to 0$. It is known that

$$\int_{S_m} A\sigma_m ds = -\int_{S_m} \sigma_m ds = -Q_m.$$

Changing the order of integration and replacing Green's function $G(x,y)$ by $g(x,y) := \frac{1}{4\pi|x-y|}$, which is possible by (18) if $a \to 0$, one gets

$$\int_{S_m} dt\sigma_m(t) \int_{S_m} dsg(s,t) = aQ_m, \tag{27}$$

where we have assumed for simplicity that D_m is a ball B_m centered at x_m and of radius a, and used the formula

$$\int_{|x_m-t|=a} \frac{dt}{4\pi|s-t|} = a, \qquad |s-x_m| = a.$$

If $D_m = B_m$, then

$$\int_{S_m} u_e(s)ds = 4\pi a^2 u_e(x_m)[1+o(1)].$$

Collecting this information, we rewrite (26) as

$$\left(1 + \frac{h_m a}{a^\kappa}\right) Q_m = -4\pi a^{2-\kappa} h_m u_e(x_m) + O(a^3), \quad 0 < \kappa < 1. \tag{28}$$

If $a \to 0$, then (28) implies (22).

To prove (23) one may argue as follows. As $a \to 0$, the main term of the asymptotic of $\sigma_m(s)$ is a constant σ_m independent of s, and

$$Q_m = \sigma_m \int_{S_m} dt = 4\pi a^2 \sigma_m.$$

This and (22) imply (23). The physical meaning of the above argument consists of the following: if $a \to 0$ then $\zeta_m = \frac{h_m}{a^\kappa} \to \infty$, so that the boundary condition (8) tends to the Dirichlet condition, see [1] for a detailed study of this limiting process. Therefore the body B_m can be considered as a perfect conductor and $\sigma_m(s)$ is its charge distribution on S_m. If D_m is a ball B_m, and its suraface is kept under constant potential, then the surface charge distribution on its surface S_m, $\sigma_m(s) = \sigma_m$, is a constant. If D_m would have an arbitrary shape and S_m is smooth, then $0 < \sigma^{(0)} \leq \sigma_m(s) \leq \sigma^{(1)}$, where $\sigma^{(0)}$ and $\sigma^{(1)}$ are constants. In this case the order of $\sigma_m(s)$ is $O(a^{-\kappa})$. We will use only the order of $\sigma_m(s)$ as $a \to 0$.

Lemma 2 is proved. □

Lemma 3. *If assumptions (19) hold, then $|J_m| \ll |I_m|$ in the region $\mathbb{R}^3 \setminus \cup_{m=1}^M B_m(x_m, r(a))$, where $\lim_{a\to 0} \frac{a}{r(a)} = 0$.*

Proof. One has

$$\begin{aligned} I_m &= |G(x,x_m)Q_m| \le O\left(\frac{a^{2-\kappa}}{r(a)}\right), \\ |J_m| &\le \int_{S_m} |G(x,s)-G(x,x_m)||\sigma_m(s)|ds \le O\left(\frac{aa^{2-\kappa}}{r^2(a)}\right), \end{aligned} \tag{29}$$

where the estimates

$$|G(x,s)-G(x,x_m)| \le \frac{c|s-x_m|}{|x-x_m|^2} = O\left(\frac{a}{r^2(a)}\right), \tag{30}$$

and

$$\int_{S_m} |\sigma_m(s)|ds = O(a^{2-\kappa}), \quad a \to 0, \tag{31}$$

were used. Note that

$$\int_{S_m} |\sigma_m(s)|ds \sim |\int_{S_m} \sigma_m(s)ds|$$

as $a \to 0$, because $\sigma_m(s) \sim \sigma_m$, and σ_m does not depend on s.

From the estimates for I_m and J_m one gets

$$|J_m||I_m|^{-1} = O(\frac{a}{r(a)}) = o(1), \ a \to 0, \tag{32}$$

as claimed. One can also prove under the assumptions (19) that

$$|\sum_{m=1}^{M} J_m| \le |\sum_{m=1}^{M} G(x,x_m)Q_m|o(1), \quad a \to 0. \tag{33}$$

Lemma 3 is proved. □

Lemma 3 justifies formula (21).

Theorem 1. *If assumptions* (6) *and* (19) *hold, then there exists the limit*

$$\lim_{a\to 0} \|u_e(x) - u(x)\|_{C(\mathbb{R}^3)} = 0, \tag{34}$$

and $u(x)$ is the unique solution to equation (25).

Proof. First, we prove that equation (25) is uniquely solvable because the integral operator $Tu = 4\pi \int_D G(x,y)h(y)N(y)u(y)dy$ is compact in the Banach space $X = C(D)$, and the corresponding homogeneous equation $u = -Tu$ has only trivial solution in $C(D)$. Indeed, if $u = -Tu$, then, applying L_0 to this equation, one gets

$$L_0u - p(x)u = [\nabla^2 + k^2 - q(x)]u = 0, \tag{35}$$

$$p(x) = 4\pi h(x)N(x), \quad q(x) = q_0(x) + p(x), \quad q_0(x) = k^2[1 - n_0^2(x)]. \tag{36}$$

The operator $L_0 = \nabla^2 + k^2 n_0^2(x) = \nabla^2 + k^2 - q_0(x)$. If $u = -Tu$, then u satisfies the radiation condition and solves equation (35), where $q(x) = 0$ in $D' := \mathbb{R}^3 \setminus D$, $\text{Im}q(x) \leq 0$. Therefore $u = 0$ (see [22]). This and the Fredholm alternative imply that equation (25) has a unique solution in $C(D)$. This solution is uniquely extended to the unique solution to equation (25) in $C(\mathbb{R}^3)$, because the right-hand side of (25) defines a function in $C(\mathbb{R}^3)$ which solves equation (35) in $\mathbb{R}^3$.

Secondly, let us prove the existence of the limit (34). The asymptotic of $u_e(x)$ as $a \to 0$ follows from (13), (21) and (22). This asymptotics is of the form:

$$u_e(x) = u_0(x) - 4\pi {\sum}'_{1\leq m\leq M} G(x, x_m)h(x_m)u_e(x_m)a^{2-\kappa}[1 + o(1)], \ a \to 0, \tag{37}$$

where ${\sum}'_{1\leq m\leq M}$ means that if $|x - x_j| \leq a$, then the term with $m = j$ is dropped in the sum. Therefore $u_e(x)$ is defined for all $x \in \mathbb{R}^3$. The quantity $u_e(x_m) := u_m$ is found from the LAS (24). The sum (37) is of the type (107) (see this equation in Section 6) with $\varphi(a) = a^{2-\kappa}$ and $f(x_m) = -4\pi G(x, x_m)h(x_m)u_e(x_m)$. Using Lemma 7 in Section 6, one concludes that this sum converges, as $a \to 0$, to the integral $\int_D G(x, y)p(y)u_e(y)dy$, where $p(x) := 4\pi N(x)h(x)$, and formula (37) in the limit $a \to 0$ yields equation (25), which has a unique solution $u(x)$ as we have already proved. This solution $u \in H^2(D)$, if $p(x) \in L^2(D)$, where $H^\ell(D)$ are the Sobolev spaces, and if $p(x) \in C^s(D)$, $s > 0$, then $u \in C^{2+s}(D)$, by the Schauder's estimates (see [7]). Equation (25) is of the form of equation (83) in Section 6 with

$$Tu = \int_D G(x, y)p(y)u(y)dy, \quad p(x) = 4\pi h(x)N(x). \tag{38}$$

Applying the collocation method to equation (25) one gets a linear algebraic system

$$u_j = u_{0j} - 4\pi \sum_{p=1, p\neq j}^{P} G(y_j, y_p)h(y_p)N(y_p)u_p|\triangle_p|, \tag{39}$$

where $\cup_{p=1}^P \triangle_p = D$ and $\text{diam}\triangle_p \gg a$, for example, diam $\triangle = O(a^{1/2})$, $a \to 0$. It follows from the assumption (6) that

$$\frac{1}{a^{2-\kappa}} N(y_p)|\triangle| = \sum_{x_m\in\triangle_p} 1, \qquad G(y_j, y_p)h(y_p) = G(y_j, x_m)h(x_m)(1 + \delta_p),$$

where $\delta_p \to 0$ as $a \to 0$, $j \neq p$. Therefore one may rewrite (39) as

$$u(x_i) = u_0(x_i) - 4\pi \sum_{m=1, m\neq i}^{M} G(x_i, x_m)h(x_m)u(x_m)a^{2-\kappa}, \tag{40}$$

which is equation (37) with $x = x_i$ and the term $1 + o(1)$ replaced by 1. The points x_i in (40) are distributed in D so that (6) holds, and the points x_i depend on a.

LAS (40) is obtained from LAS (39), which is a particular case of system (86) with $T_{jp} = 4\pi G(y_j, y_p)|\triangle_p|h(x_p)N(y_p)$, where $|\triangle_p|$ is the volume of $\triangle_p$, and $f_j = u_{0j}$. By Theorem 3 one obtains convergence in $C(D)$ of the sequence $u^{(n)}(x)$, defined in formula (88) of Section 6, to the function $u(x)$, the solution to equation (25). By Lemma 5 of Section 6 there is a one-to-one correspondence between the solution $\{u_j\}_{j=1}^P$ to LAS (39) and the function $u^{(n)}(x)$. The role of the parameter n in formula (92) is played by the parameter a. Therefore, the function $u_e(x)$ converges in $C(D)$ as $a \to 0$ to the solution of equation (25), because condition (84) of Section 6 holds for the kernel $T(x, y) = 4\pi G(x, y)h(y)N(y)$, as follows from the estimate

$$|\nabla_x G(x, y)| \leq \frac{c}{|x - y|^2},$$

where $c > 0$ is a constant. This inequality implies condition (84).
Theorem 1 is proved. □

Let us summarize some of the basic results of this Section.

We are given a material, possibly inhomogeneous, with a refraction coefficient $n_0^2(x)$, in which the waves are described by equation (1), or, equivalently, by the equation

$$L_0 u_0 = [\nabla^2 + k^2 - q_0(x)]u_0 = 0, \quad q_0(x) := k^2[1 - n_0^2(x)], \tag{41}$$

where $q_0(x) = 0$ in $D' = \mathbb{R}^3 \setminus D$.

We embed into D small particles D_m, namely, balls of radius a centered at the points x_m, where the points x_m are distributed in D according to formula (6). The total number of the embedded particles is $M = M(a) = O\left(\frac{1}{a^{2-\kappa}}\right)$.

We solve the scattering problem (7)-(9) asymptotically, as $a \to 0$, and proved that the effective field $u(x)$ in the limiting material, obtained in the limit $a \to 0$, is the unique solution of equation (25).

Applying to equation (25) the operator L_0 and using the equation $L_0 G(x, y) = -\delta(x - y)$, we obtain the following equation for u:

$$Lu = 0 \text{ in } \mathbb{R}^3, \qquad L := \nabla^2 + k^2 - q(x), \quad q(x) = q_0(x) + p(x), \tag{42}$$

where $p(x) = 4\pi h(x)N(x)$.

Equation (42) can be written as

$$Lu := [\nabla^2 + k^2 n^2(x)]u = 0 \quad \text{in } \mathbb{R}^3, \tag{43}$$

where

$$\begin{gathered} n^2(x) = 1 - k^{-2}q(x), \; n_0^2(x) = 1 - k^{-2}q_0(x), \\ k^2[n_0^2(x) - n^2(x)] = p(x) = 4\pi h(x)N(x). \end{gathered} \tag{44}$$

We conclude that any desired refraction coefficient $n^2(x)$ of the limiting material can be created by choosing suitable function $p(x)$. Any suitable function $p(x)$ can be created by choosing functions $h(x)$ and $N(x)$, such that $p(x) = 4\pi N(x)h(x)$. The functions $h(x)$ and $N(x)$ are at our disposal.

A recipe for creating materials with a desired refraction coefficient is formulated and discussed in Section 3.

At the end of this Section let us formulate some observations which are of interest for physicists.

Claim 1. The limit, as $a \to 0$, of the total volume of the embedded particles is zero.

Proof. The volume of a single particle is $O(a^3)$, the number of the embedded particles is $O\left(\frac{1}{a^{2-\kappa}}\right)$, the total volume of the particles is $O(a^{3-2+\kappa}) \to 0$ as $a \to 0$ since $\kappa > 0$. □

Claim 2. The order of the distance $d = d(a) = O(a^{\frac{2-\kappa}{3}})$ between neighboring particles is uniquely determined by the order of $M = M(a)$ as $a \to 0$.

Proof. Assume (without loss of generality) that D is a unit cube. The number of particles along a side of this cube is $O(\frac{1}{d(a)})$. Therefore, the total number $M(a)$ of particles embedded in D, is $O(\frac{1}{d^3(a)}) = M(a) = O(a^{2-\kappa})$. Thus, $d = O(a^{\frac{2-\kappa}{3}})$, see formula (19). □

3 Recipe for creating materials with a desired refraction coefficient

Step 1. Given the original refraction coefficient $n_0^2(x)$ and the desired refraction coefficient $n^2(x)$, calculate $p(x)$ by formula (44),

$$p(x) = k^2[n_0^2(x) - n^2(x)].$$

This step is trivial.

Step 2. Given $p(x) = 4\pi h(x)N(x)$, calculate the functions $h(x)$ and $N(x)$. These functions satisfy the following restrictions:

$$Im\, h(x) \leq 0, \qquad N(x) \geq 0.$$

This step has many solutions. For example, one can fix $N(x) > 0$ arbitrary, and find $h(x) = h_1(x) + ih_2(x)$, where $h_1 =$Re h, $h_2 =$ Im h, by the formulas

$$h_1(x) = \frac{p_1(x)}{4\pi N(x)}, \quad h_2(x) = \frac{p_2(x)}{4\pi N(x)}, \tag{45}$$

where $p_1 =$Re p, $p_2 =$Im p. The condition Im$h \leq 0$ holds if Im$p \leq 0$, see formula (44).

Step 3. Prepare $M = \frac{1}{a^{2-\kappa}} \int_D N(x)dx[1+o(1)]$ small balls B_m of radius a with the boundary impedances $\zeta_m = \frac{h(x_m)}{a^\kappa}$, where the points x_m, $1 \le m \le M$, are distributed in D according to formula (6). Embed ball B_m with boundary impedance ζ_m so that its center is at the point $x_m \in D$, $1 \le m \le M$. The material, obtained after the embedding of these M small balls will have the desired refraction coefficient $n^2(x)$ with an error that tends to zero as $a \to 0$. This follows from Theorem 1.

Step 3 is the only difficult step in this recipe.

Two technological problems should be solved in order to implement this step.

The first technological problem is:

How can one embed many, namely $M = M(a)$, small balls in a given material so that the centers of the balls are points x_m distributed according to (6)?

Possibly, the stereolitography process can be used.

The second technological problem is:

How does one prepare a ball B_m of small radius a with large boundary impedance $\zeta_m = \frac{h(x_m)}{a^\kappa}$?

4 Scattering by small inhomogeneities

Consider the following scattering problem:

$$L_M u_M := [\nabla^2 + k^2 - p_M(x)]u_M = 0 \quad \text{in } \mathbb{R}^3, \tag{46}$$

$$u = e^{ik\alpha\cdot x} + v_M, \quad \frac{\partial v_M}{\partial |x|} - ikv_M = o\left(\frac{1}{|x|}\right), \quad |x| \to \infty, \tag{47}$$

where

$$p_M(x) = \sum_{m=1}^{M} q_m(x), \quad q_m(x) = A_m \chi_m(x), \quad A_m = const, \; M = M(a), \tag{48}$$

$$\chi_m(x) = 1 \quad x \in B_m := \{x \;:\; |x - x_m| \le a\}, \qquad \chi_m(x) = 0 \quad x \notin B_m. \tag{49}$$

Problem (46)-(48) has a unique solution. The points x_m are distributed in a bounded domain D according to the formula:

$$\mathcal{N}(\triangle) = \frac{1}{V(a)} \int_\triangle N(x)dx[1 + o(1)], \quad a \to 0, \tag{50}$$

where $\mathcal{N}(\triangle)$ is the number of points x_m in the domain $\triangle$, $V(a) = \frac{4\pi a^3}{3}$ is the volume of a ball $B_m := \{x \;:\; |x - x_m| \le a\}$, and $N(x) \ge 0$, $N(x) \in C(D)$. Since $V(a)\mathcal{N}(\triangle)$ is the total volume of the balls of radius a, embedded in the domain $\triangle$ so that these balls do not have common interior points, one has $\int_\triangle N(x)dx < \mathcal{P}|\triangle|$, where $|\triangle|$ is the volume of $\triangle$, and $0 < \mathcal{P} < 1$. Here $\mathcal{P}$ is the ratio of the total volume of the packed spheres divided by $|\triangle|$. Since the domain $\triangle$ is arbitrary, one concludes that $N(x) \le \mathcal{P} < 1$. It is conjectured that $\mathcal{P} < 0.74$, see [36].

There is a large literature on optimal packing of spheres (see, e.g., [6], [36]). For us the maximal value of $\mathcal{P}$ is not important. What is important is the following conclusion: *one can choose $N(x) \ge 0$ as small as one wishes, and still create any desired potential $q(x)$ by choosing suitable $A(x) > 0$, where $A(x_m) = A_m$.*

The problem we are interested in can now be formulated:

Under what conditions there exists the limit

$$\lim_{a\to 0} u_M := u_e(x), \tag{51}$$

and

$$Lu_e := [\nabla^2 + k^2 - q(x)]u_e = 0 \quad \text{in } \mathbb{R}^3, \tag{52}$$

where $q(x)$ is the desired potential?

Our answer is formulated in Theorem 2.

Theorem 2. *Let $q(x)$ be an arbitrary given Riemann-integrable function in D, where $D \subset \mathbb{R}^3$ is an arbitrary large fixed bounded domain, and two functions $A(x)$ and $N(x)$ are such that $q(x) = A(x)N(x)$, where $N(x)$ is the function from* (50) *and $A(x)$ satisfies the conditions $A(x_m) = A_m$, where the constants A_m are from* (48). *Then there exists the limit* (51) *and this limit solves equation* (52) *with the given $q(x)$.*

Proof. The function $u_M(x)$ is the unique solution to the equation

$$u_M(x) = u_0(x) - \int_D g(x,y)p_M(y)u_M(y)dy, \; g(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \; u_0(x) = e^{ik\alpha\cdot x}. \tag{53}$$

Equation (53) can be written as

$$u_M(x) = u_0(x) - {\sum}'_{1\le m\le M} g(x,x_m)A_m u_M(x_m)V(a)[1+o(1)], \quad a\to 0, \tag{54}$$

where ${\sum}'_{1\le m\le M}$ denotes the sum in which the term with $m = j$ is dropped if $x \in B_j$. Equation (54) leads to the following linear algebraic system (LAS) for the unknown $u_M(x_m) := u_m$

$$u_j = u_{0j} - \sum_{m=1, m\ne j}^{M} g(x_i, x_m)A_m u_m V(a), \quad 1 \le j \le M. \tag{55}$$

This system is a collocation-type system corresponding to the integral equation

$$u(x) = u_0(x) - \int_D g(x,y)A(y)N(y)u(y)dy, \tag{56}$$

provided that the points x_m, $1 \le m \le M = M(a)$, are distributed in D according to the formula (50). The solution $u(x)$ to equation (56) is the effective field $u_e(x)$, defined in formula (51).

Convergence of the collocation method is established in Section 6. Equation (56) is similar to equation (83) of Section 6, and the operator $Tu := \int_D g(x,y)A(y)N(y)u(y)dy$ is compact in $C(D)$. Moreover, the operator $I+T$ is injective, and condition (84) holds. Therefore, the results of Section 6 are applicable. These results yield the existence of the limit (51). This limit $u_e(x) := u(x)$ solves equation (56). Applying to equation (56) the operator $\nabla^2 + k^2$, one obtains equation (52) with $q(x) = A(x)N(x)$.

Theorem 2 is proved. $\Box$

Remark 1. *In the one-dimensional case, when $x \in \mathbb{R}$, the role of B_m is played by the interval $\{x : \ |x - x_m| \le a\}$. In this case $V(a) = 2a$, $g(x,y) = -\frac{e^{ik|x-y|}}{2ik}$, $\mathcal{N}(\triangle) = \frac{1}{2a}\int_\triangle N(x)dx[1+o(1)]$, $q_m(x) = \begin{cases} A_m, & x \in B_m, \\ 0, & x \notin B_m, \end{cases}$ and an analog of Theorem 2 holds.*

We can formulate the following *recipe for creating a desired refraction coefficient $n^2(x)$ by embedding small inhomogeneities into a given domain D.*

Recall that $n^2(x) = 1 - k^{-2}q(x)$, so $n^2(x)$ and $q(x)$ are in one-to-one correspondence. Therefore we formulate the *recipe for creating a desired $q(x)$.* Here is our recipe:

Step 1. *Given $q(x)$, find $A(x)$ and $N(x)$ from the relation $q(x) = A(x)N(x)$. This can be done non-uniquely. For example, one may fix an arbitrary function $N(x) > 0$ and find $A(x) = \frac{q(x)}{N(x)}$.*

Step 2. *Given $A(x)$ and $N(x)$, embed small inhomogeneities $q_m(x) = A(x_m)$ in $B_m := \{x \ : \ |x - x_m| \le a\}$, $q_m(x) = 0$, $x \notin B_m$, where the points x_m, $1 \le m \le M$, are distributed in D according to (50). The resulting potential $p_M(x) = \sum_{m=1}^M q_m(x)$ approximates the desired potential $q(x)$ with an error that tends to zero as $a \to 0$.*

The convergence rate of the $\sup_{x\in D}|p_M(x) - q(x)|$ as $a \to 0$ can be estimated: it is the rate of approximation of $q(x)$ by a piecewise-constant functions $p_M(x)$. If the modulus of continuity of $q(x)$ is $\omega_q(\delta)$, then

$$\sup_{x\in D}|q(x) - \sum_{m=1}^M q(x_m)\chi_m(x)| \le \max_{1\le m\le M}\sup_{x\in B_m}|q(x) - q(x_m)| \le c\omega_q(a).$$

5 Electromagnetic wave scattering by small particles embedded in an inhomogeneous medium

Consider a scattering problem for electromagnetic (EM) waves:

$$\nabla \times E = i\omega\mu_0 H, \quad \nabla \times H = -i\omega\epsilon' E \quad \text{in } \mathbb{R}^3, \tag{57}$$

where $\omega > 0$ is the frequency, μ_0 is the magnetic constant for the free space, $\epsilon' = \epsilon(x) + i\frac{\sigma(x)}{\omega}$, $\sigma(x) \ge 0$ is the conductivity, $\epsilon(x) > 0$ is the dielectric parameter,

$\epsilon(x) = \epsilon_0$ in $D' = \mathbb{R}^3 \setminus D$, $\sigma(x) = 0$ in D', $D \subset \mathbb{R}^3$ is a bounded domain, $K^2(x) = \omega^2 \epsilon' \mu$ is the wave number. Let the incident wave be a plane wave $E_0(x) = \mathcal{E} e^{ik\alpha \cdot x}$, where $\alpha \in S^2$ is the direction of the incident wave, S^2 is the unit sphere, $\mathcal{E}$ is a constant vector, $\mathcal{E} \cdot \alpha = 0$, $k = \omega\sqrt{\epsilon_0 \mu_0} = \frac{\omega}{c}$, c is the wave velocity in D', $H_0 = \frac{\nabla \times E_0}{i\omega\mu_0}$, and

$$E(x) = E_0(x) + v, \quad \frac{\partial v}{\partial |x|} - ikv = o\left(\frac{1}{|x|}\right), \ |x| \to \infty, \tag{58}$$

where v is the scattered field. Problem (57)-(58) is equivalent to the following equations:

$$\nabla \times \nabla \times E = K^2(x) E, \qquad H = \frac{\nabla \times E}{i\omega\mu_0}, \tag{59}$$

and E has to satisfy the radiation condition (58). Let us write equation (59) for E as

$$-\nabla^2 E - k^2 E = p(x) E - \nabla\nabla \cdot E \quad \text{in} \quad \mathbb{R}^3; \ p(x) := K^2(x) - k^2. \tag{60}$$

One has Im $K^2(x) \geq 0$, so

$$\operatorname{Im} p(x) \geq 0. \tag{61}$$

From the second equation (57) one gets

$$0 = \nabla \cdot (\epsilon' E), \quad \nabla \cdot E = -\frac{\nabla \epsilon' \cdot E}{\epsilon'}. \tag{62}$$

Using (62), rewrite (60) as

$$-\nabla^2 E - k^2 E = p(x) E + \nabla(q(x) \cdot E), \quad q(x) := \frac{\nabla K^2(x)}{K^2(x)}. \tag{63}$$

Let us assume that $K^2(x) \in C^2(\mathbb{R}^3)$, $q = 0$ on $\partial D = S$ and

$$\inf_{x \in \mathbb{R}^3} |K^2(x)| > 0. \tag{64}$$

Then problem (63), (58) is equivalent to the integral equation

$$E(x) = E_0(x) + \int_D g(x,y)[p(y)E(y) + \nabla_y (q(y) \cdot E(y))] dy, \tag{65}$$

where $g(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$. Integrate by parts the last term in (65) and use the formula $-\nabla_y g(x,y) = \nabla_x g(x,y)$, to get

$$E(x) = E_0(x) + \int_D g(x,y) p(y) E(y) dy + \nabla_x \int_D g(x,y) q(y) \cdot E(y) dy. \tag{66}$$

Equation (66) is new. It is convenient in applications. The scattering problem is reduced to finding just one vector field E. If E is found, then H is found by the formula $H = \frac{\nabla \times E}{i\omega\mu_0}$. Vector E is found from equation (66) uniquely, and this equation can be used also for constructing numerical and asymptotic methods for finding E. This important simplification in solving the EM wave scattering problem is possible because of our assumption $\mu = \mu_0$ in $\mathbb{R}^3$. An asymptotic treatment of the solution to equation (66) as $a \to 0$ is given at the end of this Section.

Equation (66) is of the type (83) in Section 6, and the operator T is defined by the formula:

$$TE = -\left(\int_D g(x,y)p(y)E(y)dy + \nabla_x \int_D g(x,y)q(y) \cdot E(y)dy \right). \tag{67}$$

This operator is compact in $C(D)$ and in $H^1(D)$, so equation (66) is a Fredholm-type equation in these spaces.

Lemma 4. *Equation* (66) *has a unique solution in* $H^1(D)$.

Proof. Since equation (66) is of Fredholm-type, it is sufficient to prove that the homogeneous equation (66) has only the trivial solution $E = 0$. If $E = -TE$, then

$$(-\triangle - k^2)E = p(x)E + \nabla(q(x) \cdot E), \tag{68}$$

or

$$\nabla \times \nabla \times E - \nabla \frac{K^2(x)\nabla \cdot E + \nabla K^2(x) \cdot E}{K^2(x)} = K^2(x)E. \tag{69}$$

Denote

$$\psi := \frac{\nabla \cdot (K^2(x)E)}{K^2(x)}.$$

Take divergence of (69) and get

$$0 = -\nabla^2\psi - K^2(x)\psi = -\nabla^2\psi - k^2\psi - p(x)\psi, \quad k^2 > 0. \tag{70}$$

Since $p(x)$ is compactly supported and Im $p \geq 0$, it follows that $\psi = 0$ (see [29] for details). Thus, $\psi = 0$, and $\nabla \cdot (K^2(x)E) = 0$. Therefore equation (68) reduces to

$$(\nabla^2 + k^2 + p(x))E = 0 \quad \text{in } \mathbb{R}^3, \tag{71}$$

where E satisfies the radiation condition, and Im$p \geq 0$. Consequently, $E = 0$. Lemma 4 is proved. □

Let us write equation (66) as

$$\begin{aligned} E(x) = E_0(x) + g(x,x_m) \int_D p(y)E(y)dy \\ + \nabla_x g(x,x_m) \int_D q \cdot E dy + \int_D [g(x,y) - g(x,x_m)]p(y)E(y)dy \\ + \nabla_x \int_D [g(x,y) - g(x,x_m)]q(y) \cdot E(y)dy. \end{aligned} \tag{72}$$

Here $x_m \in D$ and we assume that $\text{diam}D \sim a$ is small and $ka \ll 1$. If $d = |x - x_m|$ and $\frac{a}{d} \to 0$ as $a \to 0$, then we prove that asymptotically, as $a \to 0$, the last two terms in (72) are negligible compared with the second and third terms on the right-hand side of (72).

Denote

$$V_m := \int_D p(y)E(y)dy, \quad W_m := \int_D q(y) \cdot E(y)dy. \tag{73}$$

It is proved in [29] that, up to the terms of higher order of smallness as $a \to 0$, one has:

$$V_m = \frac{V_{0m}}{1 - a_m} + \frac{A_m}{1 - a_m} \frac{(1 - a_m)W_{0m} + B_m \cdot V_m}{(1 - a_m)(1 - b_m) - B_m \cdot A_m}, \tag{74}$$

$$W_m = \frac{(1 - a_m)W_{0m} + B_m \cdot V_{0m}}{(1 - a_m)(1 - b_m) - B_m \cdot A_m}, \tag{75}$$

where

$$V_{0m} = \int_D p(x)E_0(x)dx, \quad a_m = \int_D p(x)g(x, x_m)dx, \tag{76}$$

$$B_m = \int_D q(x)g(x, x_m)dx, \quad b_m = \int_D q(x)\nabla_x g(x, x_m)dx. \tag{77}$$

One obtains the following formula for the electric field, scattered by a small body $D = D_m$:

$$E(x) = E_0(x) + g(x, x_m)V_m + A_m W_m. \tag{78}$$

This formula is accurate up to the terms of higher order of smallness as $a \to 0$. The error of the formula (78) is of the order $O\left(\frac{a}{d} + ka\right)$.

If there are $M \gg 1$ small bodies D_m, and $\text{diam}D_m = 2a$, then the formula for the field E, accurate up to the term of the higher order of smallness as $a \to 0$, is

$$\begin{aligned} E(x) = E_0(x) &+ \sum_{m=1}^{M} g(x, x_m)E(x_m) \int_{D_m} p(y)dy \\ &+ \sum_{m=1}^{M} \nabla_x g(x, x_m) \int_{D_m} \frac{\nabla_y K^2(y)}{K^2(y)} \cdot E(y)dy. \end{aligned} \tag{79}$$

The last integral in equation (79) can be approximated by the formula:

$$\int_{D_m} \frac{\nabla_y K^2(y)}{K^2(y)} \cdot E(y)dy = E(x_m) \cdot \int_{D_m} \frac{\nabla_y K^2(y)}{K^2(y)} dy.$$

This formula and equation (79) can be used for a derivation of a linear algebraic system (LAS) for finding $E(x_m)$, similar to the system (56). The resulting LAS can be used for numerical solution of the problem of EM wave scattering by many small bodies.

We assume here that the small bodies are embedded in the free space. If they are embedded in an inhomogeneous medium, then one has to replace Green's function $g(x,y)$ by the Green's function $G(x,y)$, corresponding to this inhomogeneous medium.

Suppose that D_m, $1 \le m \le M$, are balls, $D_m = B_m := \{x \, : \, |x - x_m \le a\}$, and the points $\{x_m\}_{m=1}^M$ are distributed as in (50) with $V(a) = a^{3-\kappa}$, $0 < \kappa < 3$, and $M = O\left(\frac{1}{a^{3-\kappa}}\right)$ as $a \to 0$. Choose

$$p(y) = p(r,a), \quad r = |y - x_m| \quad in\ B_m, \quad 1 \le m \le M, \qquad \int_{B_m} p(r,a)dy = O(a^{3-\kappa}).$$

Let

$$p(r,u) = 0 \text{ if } t = \frac{r}{a} > 1, \quad p(r,a) = p(r) = \frac{\gamma_m}{4\pi a^\kappa}(1-t)^2, \quad t \le 1. \tag{80}$$

Then, as $a \to 0$, $E = E_M$ tends to the effective field $E_e(x)$, which is the unique solution to the equation

$$E_e(x) = E_0(x) + \int_D g(x,y)C(y)E_e(y)dy, \tag{81}$$

where

$$C(x) = \frac{\gamma_m N(x_m)}{30} a^{3-\kappa}[1 + o(1)], \qquad a \to 0,$$

see [29] for details. A continuous function $C(y)$ is uniquely defined by its values at the set of points x_m as $a \to 0$, because this set tends to a dense set in D as $a \to 0$. Therefore, by choosing the numbers γ_m, $1 \le m \le M$, one can create any desired continuous function $C(y)$. Applying the operator $\nabla^2 + k^2$ to equation (81), one gets

$$[\nabla^2 + \mathcal{K}^2(x)]E_e = 0, \quad \mathcal{K}^2(x) = k^2 + C(x) = k^2 n^2(x), \tag{82}$$

where $n^2(x) = 1 + k^{-2}C(x)$. Since $C(x)$ is determined by the function $N(x)$ and numbers γ_m, and the numbers γ_m and the function $N(x)$ are at our disposal, it is possible to create any continuous function $C(y)$, prescribed a priori.

6 Auxiliary results

6.1. Collocation method

In this subsection the results from [33] are presented.

Consider an equation

$$(I + T)u = f, \quad Tu = \int_D T(x,y)u(y)dy, \tag{83}$$

where $D \subset \mathbb{R}^3$ is a bounded domain. T is compact in a Banach space $X = L^\infty(D)$, and $I + T$ is injective. Then, by the Fredholm alternative, equation (83) has a unique solution for any $f \in X$.

Let us assume that

$$\sup_{x\in D}\int_D (|\nabla_x T(x,y)| + |T(x,y)|)dy = c_0 < \infty, \tag{84}$$

where c_0 is a constant.

Consider a partition of $D = \cup_{j=1}^{j_n} D_j$ into a union of cubes D_j with a side $\frac{1}{n}$ centered at the points $x_j \in D$. In a small boundary strip some of the cubes may contain some points of D', but this is not important for our arguments. Define $\chi_j(x) = \begin{cases} 1, & \text{in } D_j, \\ 0, & \text{in } D'_j. \end{cases}$ Let

$$\omega_u\left(\frac{1}{n}\right) := \sup_{|x-y|\le\frac{1}{n},x,y\in D} |u(x) - u(y)|$$

be the continuity modulus of the function u. If $u \in Lip_a(D)$, then $\omega_u(\delta) \le c\delta^a$, $0 < a \le 1$. Let

$$u_j := u(x_j), \quad T_{ij} := \int_{D_j} T(x_i, y)dy, \quad u_j(x) := u_j\chi_j(x). \tag{85}$$

Consider the following linear algebraic system (LAS)

$$u_i + \sum_{j=1}^{j_n} T_{ij}u_j = f_i, \quad 1 \le i \le j_n, \tag{86}$$

and the corresponding equation in X

$$u_i(x) + \sum_{j=1}^{j_n} \chi_i(x)\int_{D_j} T(x_i, y)u_j(y)dy = f_i(x), \quad 1 \le i \le j_n. \tag{87}$$

Define

$$u^{(n)}(x) := \sum_{j=1}^{j_n} u_j\chi_j(x). \tag{88}$$

Equation (87) is equivalent to the following equation in X:

$$(I + T_n)u^{(n)} = f^{(n)}, \qquad T_n u := \int_D T^{(n)}(x, y)u(y)dy, \tag{89}$$

$$T^{(n)}(x, y) := \sum_{i=1}^{j_n} \chi_i(x)T(x_i, y). \tag{90}$$

Lemma 5. *Equation* (89) *is equivalent to LAS* (86) *in the following sense:*

If $\{u_j\}_{j=1}^{j_n}$ *solves* (86)*, then* $u^{(n)}(x)$ *defined in* (88)*, solves* (89)*.*

Conversely, if $u^{(n)}$*, defined in* (88)*, solves* (89)*, then the numbers* $\{u_j\}_{j=1}^{j_n}$ *solve LAS* (86)*.*

Proof. If $\{u_j\}_{j=1}^{j_n}$ solve (86), multiply (86) by $\chi_i(x)$ and sum over i to get (89):

$$u^{(n)}(x)+\sum_{i=1}^{j_n}\chi_i(x)\sum_{j=1}^{j_n}\int_{D_j}T(x_i,y)dyu_j=u^{(n)}+T^{(n)}u^{(n)}=f^{(n)}. \tag{91}$$

Conversely, if (89) holds, let $x=x_i$ in (89) and use the relation

$$\chi_j(x_i)=\delta_{ij}:=\begin{cases}1, & i=j,\\ 0, & i\neq j,\end{cases}$$

to get (86).
Lemma 5 is proved. □

Theorem 3. *Linear algebraic system (LAS)* (86) *has a unique solution for all $n>n_0$, where n_0 is sufficiently large, and this solution generates by formula* (88) *a function $u^{(n)}(x)$ such that*

$$\|u^{(n)}-u\|\leq O\left(\frac{1}{n}\right)\ \textit{as}\ n\to\infty, \tag{92}$$

provided that assumption (84) *holds.*

Proof. Assumption (84) implies

$$\lim_{n\to\infty}\|T_n-T\|=0. \tag{93}$$

If (93) holds, then $(I+T_n)^{-1}$ exists and is bounded for all $n>n_0$, where n_0 is sufficiently large. This follows from the fact that $(I+T)^{-1}$ exists and is bounded. To prove (92) one derives:

$$\begin{aligned}\|u^{(n)}-u\|&=\|(I+T_n)^{-1}f_n-(I+T)^{-1}f\|\\&\leq\|(I+T)^{-1}\|\|f_n-f\|+\|(I+T_n)^{-1}-(I+T)^{-1}\|\|f_n\|\\&\leq c(\|f_n-f\|+\|(I+T_n)^{-1}-(I+T)^{-1}\|),\end{aligned} \tag{94}$$

where $c>0$ stands for various constants independent of n. If $f\in Lip_a(D)$, then $\|f_n-f\|\leq c\omega_f\left(\frac{1}{n^a}\right)$. Also,

$$\|(I+T_n)^{-1}-(I+T)^{-1}\|\leq c\|T_n-T\|\leq c\sup_{x\in D}\int_D|T(x,y)-T^{(n)}(x,y)|dy:=\epsilon_n. \tag{95}$$

Let $x\in D_j$. Then $|x-x_j|\leq \text{diam}D_j=O(1/n)$. Thus using assumption (84), one gets:

$$\epsilon_n=O\left(\frac{1}{n}\right). \tag{96}$$

Therefore, (84), (94) and (96) imply

$$\|u^{(n)}-u\|\leq O\left(\frac{1}{n}\right), \tag{97}$$

because, in our case, $f=u_0$ is smooth, so $\omega_f\left(\frac{1}{n}\right)=O\left(\frac{1}{n}\right)$.
Theorem 3 is proved. □

Remark 2. *One has*

$$T_{ij} := \int_{D_j} T(x_i, y)dy = T(x_i, y_j)|D_j|, \tag{98}$$

where $|D_j|$ *is the volume of* D_j*, it is assumed that* $T(x_i, y)$ *is continuous in* D_j*, and* y_j *can be interpreted as a point in the mean value theorem,* y_j *is not necessarily the center* x_j *of the cube* D_j*. The LAS* (86) *can be written as*

$$u_i + \sum_{j=1}^{j_n} T(x_i, y_j)u_j|D_j| = f_i, \quad 1 \le i \le j_n. \tag{99}$$

Theorem 3 allows one to claim that the limiting form of the LAS (86) and (99) is the integral equation (83), for which the LAS (99) is a collocation method for solving equation (83). Theorem 3 is a result which proves the convergence of this collocation method under the assumptions (84).

6.2. Limits of certain sums

Suppose now that f is a bounded Riemann-integrable function and there is a set of points $x_j \in D$, distributed according to the following law:

$$\mathcal{N}(\triangle) = \frac{1}{\varphi(a)} \int_{\triangle} N(x)dx[1 + o(1)], \quad a \to 0, \tag{100}$$

where $\triangle \subset D$ is an arbitrary open subset of D, $\varphi(a) > 0$ for $a \ne 0$, $\varphi(0) = 0$, φ is a continuous and strictly monotone function, $N(x) \le 0$ is a continuous function in D. In formula (86) $\varphi(x) = a^{2-\kappa}$ satisfies these assumptions.

Consider the following sum

$$I(a) := \varphi(a) \sum_{x_m \in D} f(x_m). \tag{101}$$

Lemma 6. *If* (100) *holds, then there exists the limit*

$$\lim_{a \to 0} I(a) = \int_D f(x)N(x)dx. \tag{102}$$

Proof. Consider a partition of D into a union of P sets $\triangle_i$, $\mathrm{diam}\triangle_i \le a^{\frac{1}{4}}$, where the sets $\triangle_i$ and $\triangle_j$ do not have common interior points if $i \ne j$, and there are still many points x_m in $\triangle_i$. One can write, using (100),

$$\sum_{x_m \in \triangle_i} 1 = \frac{1}{\varphi(a)} N(y_i)|\triangle_i|[1 + o(1)], \quad a \to 0, \tag{103}$$

where $y_i \in \triangle_i$ is some point, and

$$I(a) = \varphi(a) \sum_{i=1}^{P} f(y_i)(1 + \delta_i) \sum_{x_m \in \triangle_i} 1 = \sum_{i=1}^{P} f(y_i)N(y_i)|\triangle_i|(1 + \delta_i), \tag{104}$$

where $f(x_m) = f(y_i)(1 + \delta_i)$ for $x_m \in \triangle_i$ and $\delta_i \to 0$ as $\text{diam}\triangle_i \to 0$, because f is continuous. The sum in the right-hand side of (104) is the Riemann sum for the integral in (102), and since $\lim_{a\to} \delta_i = 0$, one obtains formula (102). Lemma 6 is proved. □

In our case the sum (24) is of the type (101) with $\varphi(a) = a^{2-\kappa}$, and the corresponding integral equation (25) is of the type (83). However the function $-4\pi G(x, y)N(y)h(y)u(y)$ is not bounded because $|G(x, y)| \to \infty$ as $y \to x$.

Let us generalize Lemma 6 to include such cases. Suppose that

$$|f(x)| \leq \frac{c}{[\rho(x, S)]^b}, \quad b \leq 3, \tag{105}$$

where $\rho(x, S)$ is the Euclidean distance from x to S, S is the set of points in D at which $|f| = \infty$, and meas $S = 0$. Let $D_\delta := \{x \mid x \in D, \ \rho(x, S) \geq \delta\}$. We assume that there exists the limit

$$\lim_{\delta\to 0} \int_{D_\delta} f(x)N(x)dx := \int_D f(x)N(x)dx. \tag{106}$$

Let us define

$$I := \lim_{a\to 0} I(a) := \lim_{\delta\to 0} \lim_{a\to 0} \varphi(a) \sum_{x_m \in D_\delta} f(x_m). \tag{107}$$

Lemma 7. *If $N(x) \in C(D)$, $b \in (0, 3)$, and assumptions* (100) *and* (105) *hold, then there exists the limit* (107)*, and*

$$I = \int_D f(x)N(x)dx. \tag{108}$$

Proof. In D_δ the function f is bounded and Lemma 6 yields

$$\lim_{a\to 0} \varphi(a) \sum_{x_m \in D_\delta} f(x_m) = \int_{D_\delta} f(x)N(x)dx. \tag{109}$$

If $0 < b < 3$, then limit (106) exists and the integral $\int_D f(x)N(x)dx$ is understood as an improper Riemann integral. If $b = 3$, then this integral is understood as a singular integral. Let us recall that the singular integral $Tu = \int_{\mathbb{R}^3} |x - y|^{-3} f(x, \theta)u(y)dy$ where $\theta = \frac{x-y}{|x-y|}$, exists if and only if (see [13], p.221)

$$\int_{S^2} f(x, \theta)d\theta = 0, \tag{110}$$

where S^2 is the unit sphere in $\mathbb{R}^3$ and $d\theta$ is the element of the surface area of S^2. One has (see [13], p.242)

$$\frac{\partial}{\partial x_i} \int_D u(y) \frac{f(x, \theta)}{|x - y|^2} dy = \int_D u(y) \frac{\partial}{\partial x_i} [\frac{f(x, \theta)}{|x - y|^2}]dy - u(x) \int_{S^2} f(x, \theta) \cos(\theta, x_i)d\theta. \tag{111}$$

If the set S consists of one points, and condition (110) holds, then the definition of the singular integral implies the existence of the limit (106) with D_δ being the set $\{y \ : \ y \in D, \ |x-y| \geq \delta\}$, which corresponds to the Cauchy principal value. Since (109) is valid when $b=3$, formula (108) holds when $b=3$, if the singular integral exists.
Lemma 7 is proved. $\Box$

References

[1] H-D.Alber and A.G.Ramm, Asymptotics of the solution to Robin problem, J.Math.Anal.Appl., 349, N1, (2009), 156-164.

[2] M.Andriychuk and A.G. Ramm, Scattering by many small particles and creating materials with a desired refraction coefficient, Internat. Journ. Comp.Sci. and Math, 3, N1-2, (2010), 102-121.

[3] M.Andriychuk and A.G. Ramm, Numerical solution of many-body wave scattering problem for small particles, Proc. DIPED-2009, Lvov, Sept. 21-24, (2009), pp. 77-81.

[4] M.Andriychuk and A.G. Ramm, Numerical modeling in wave scattering problem for small particles, Proc. of MIKON-2010, 18-th Internat. Conf. on microwave radar and wireless communications, Geozandas Ltd, Vilnius, Lithuania, 2010, pp/224-227.

[5] A. Bensoussan, J. Lions and G. Papanicolaou, Asymptotic analysis for periodic structures, North-Holland, Amsterdam, 1978.

[6] J. H. Conway and N. J. A. Sloane, Sphere Packings, Lattices and Groups, Springer-Verlag, New York, 1998.

[7] D.Gilbarg and N.Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, Berlin, 1983.

[8] S.Indratno and A.G. Ramm, Creating materials with a desired refraction coefficient: numerical experiments, Internat. Journ. Comp.Sci. and Math, 3, N1-2, (2010), 76-101.

[9] V. Jikov, S. Kozlov, and O. Oleinik, Homogenization of differential and integral functionals, Springer, Berlin, 1994.

[10] L. Landau, E. Lifschitz and L. Pitaevskii, Electrodynamics of continuous media, Elsevier, Oxford, 1984.

[11] P. Martin, Multiple Scattering, Cambridge Univ. Press, Cambridge, 2006.

[12] V. Marchenko and E. Khruslov, Homogenization of partial differential equations, Birkhäuser, Basel, 2006.

[13] S. Mikhlin and S.Prössdorf, Singular integral operators, Springer-Verlag, Berlin, 1986.

[14] G. Milton, The theory of composites, Cambrige Univ. Press, Cambridge, 2001.

[15] A.G. Ramm, Iterative solution of the integral equation in potential theory, Doklady Acad. Sci. USSR, 186, (1969), 62-65.

[16] A.G. Ramm, Calculation of the scattering amplitude for the wave scattering from small bodies of an arbitrary shape, Radiofisika, 12, (1969), 1185-1197.

[17] A.G. Ramm, Approximate formulas for polarizability tensors and capacitances of bodies of arbitrary shapes and applications, Doklady Acad. Sci. USSR, 195, (1970), 1303-1306.

[18] A.G. Ramm, Wave scattering by small bodies of arbitrary shapes, World Sci. Publishers, Singapore, 2005.

[19] A.G. Ramm, Electromagnetic wave scattering by many small particles, Phys. Lett. A, 360, N6, (2007), 735-741.

[20] A.G. Ramm, Materials with the desired refraction coefficients can be made by embedding small particles, Phys. Lett. A, 370, 5-6, (2007), 522-527.

[21] A.G. Ramm, Scattering by many small bodies and applications to condensed matter physics, Europ. Phys. Lett., 80, (2007), 44001.

[22] A.G. Ramm, Many-body wave scattering by small bodies and applications, J. Math. Phys., 48, N10, (2007), 103511.

[23] A.G. Ramm, Wave scattering by small impedance particles in a medium, Phys. Lett. A 368, N1-2,(2007), 164-172.

[24] A.G. Ramm, Distribution of particles which produces a "smart" material, Jour. Stat. Phys., 127, N5, (2007), 915-934.

[25] A.G. Ramm, Distribution of particles which produces a desired radiation pattern, Physica B, 394, N2, (2007), 253-255.

[26] A.G. Ramm, Many-body wave scattering by small bodies, J. Math. Phys., 48, N2, 023512, (2007).

[27] A.G. Ramm, Creating wave-focusing materials, LAJSS (Latin-American Journ. of Solids and Structures), 5, (2008), 119-127.

[28] A.G. Ramm, Electromagnetic wave scattering by many conducting small particles, J. Phys A, 41, (2008), 212001.

[29] A.G. Ramm, Electromagnetic wave scattering by small bodies, Phys. Lett. A, 372/23, (2008), 4298-4306.

[30] A.G. Ramm, Wave scattering by many small particles embedded in a medium, Phys. Lett. A, 372/17, (2008), 3064-3070.

[31] A.G. Ramm, Preparing materials with a desired refraction coefficient and applications, In the book "Topics in Chaotic Systems: Selected Papers from Chaos 2008 International Conference", Editors C.Skiadas, I. Dimotikalis, Char. Skiadas, World Sci.Publishing, 2009, pp.265-273.

[32] A.G. Ramm, Preparing materials with a desired refraction coefficient, Nonlinear Analysis: Theory, Methods and Appl., 70, N12, (2009), e186-e190.

[33] A.G. Ramm, A collocation method for solving integral equations, Int. J. Comput. Sci. Math. (IJCSM), 2, N3, (2009), 222–228.

[34] A.G. Ramm, Creating desired potentials by embedding small inhomogeneities, J. Math. Phys., 50, N12,123525, (2009).

[35] A.G. Ramm, How to prepare materials with a desired refraction coefficient, Proceedings of ISCMII and EPMESCXII, AIP Conference Proceedings 1233, (2010), pp. 165-168.

[36] N.Sloane, The sphere packing problem, Documenta Mathematika, Vol. III (1998), 387-396.