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Some Nonlinear Inequalities and Applications

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Abstract. Sufficient conditions are given for the relation $\lim_{t\to\infty} y(t) = 0$ to hold, where $y(t)$ is a continuous nonnegative function on $[0,\infty)$ satisfying some nonlinear inequalities. The results are used for a study of large time behavior of the solutions to nonlinear evolution equations. Example of application is given for a solution to some evolution equation with a nonlinear partial differential operator.

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1 Introduction

The stability study of many evolution equations is a study of large time behavior of the solutions to these equations. In this paper we reduce such a study to a study of the behavior of a solution $y(t)$ to some nonlinear inequalities. Assume that a nonnegative continuous function $y(t)$ satisfies the following conditions

$$\int_0^\infty \omega(y(t))\frac{1}{(1+t)^\alpha}dt < \infty, \qquad 0 \le \alpha \le 1, \tag{1.1}$$

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and

$$y(t) - y(s) \le \int_s^t f(x, y(x))dx, \qquad 0 \le s \le t, \tag{1.2}$$

where $f(x,y)$ is a nonnegative and locally integrable function on $[0,\infty)\times[0,\infty)$, $0 \le \omega(t)$ is a non-decreasing continuous function, and $\omega(t) = 0$ implies $t = 0$.

The question arises:

Under what condition on $f(y,t)$ does it follow that

$$\lim_{t\to\infty} y(t) = 0? \tag{1.3}$$

There is a very large literature on inequalities (see, e.g., [1], [2] and references therein). The Barbalat's lemma is an integral inequality used in applied nonlinear control (see [12]). The inequalities, derived in this paper, are new and are useful in many applications. In [13, p.227] inequality (1.1) is studied for $\omega(t) = t$ and $\alpha = 0$. In this case condition (1.1) becomes $y \in L^1[0,\infty)$. In [13] it is proved that (1.3) holds if $y \in L^1[0,\infty)$ and the following two conditions hold:

$$y(t) - y(s) \le \int_s^t f(y(x))dx + \int_s^t h(x)dx, \qquad \int_0^\infty h(x)dx < \infty. \tag{1.4}$$

Here f and h are nonnegative functions, and f is continuous and non-decreasing. Proofs of this result can be found in [13] and in [5]. Applications of this result to the stability study of evolution equations can be found in [13] and references therein. This result is not applicable if $y(t) = O(\frac{1}{t^\beta})$ as $t \to \infty$, where $\beta \in (0,1)$, because then $y(t)$ is not in $L^1[0,\infty)$. Also, this result is not applicable if (1.2) holds instead of (1.4) and f depends on x.

The second nonlinear inequality we study is the following one:

$$\dot{g}(t) \le -a(t)f(g(t)) + b(t), \qquad t \ge 0, \tag{1.5}$$

where a, b and g are nonnegative functions on $[0,\infty)$, $g \in C^1([0,\infty))$, $a \in C([0,\infty))$ and $b \in L^1_{loc}([0,\infty))$. Sufficient conditions for the relation $\lim_{t\to\infty} g(t) = 0$ to hold are proposed and justified in [5], see also [7], [8], [9], and [10]. In our paper inequality (1.5) is studied by a different method and some new sufficient conditions for (1.3) to hold are proposed and justified.

The paper is organized as follows. In Theorems 2.1, 2.4 and 2.7 and their corollaries, sufficient conditions for (1.3) to hold are formulated and justified. In Theorems 2.11, 2.13 and 2.14 sufficient conditions for the relation $\lim_{t\to\infty} g(t) = 0$ to hold are proposed and justified under the assumption that $f(t)$ is a continuous and non-decreasing function on $[0,\infty)$. In Section 3 applications of the new results to the stability study of evolution equations are given.

2 Main results

Throughout the paper we assume that the following assumption holds.

Assumption A)

1. *$\omega(t) \geq 0$ is a locally integrable function.*

2. *There exists $\epsilon_0 > 0$ such that ω is non-decreasing on $(0,\epsilon_0)$ and*

$$\omega(t) \geq \omega(\epsilon_0), \quad \forall t \geq \epsilon_0. \tag{2.1}$$

3. *If $\omega(t) = 0$ then $t = 0$.*

This assumption is standing and is not repeated.

Theorem 2.1. *Let $y(t) \geq 0$ be a continuous function on $[0,\infty)$,*

$$\int_0^\infty \omega(y(t))dt < \infty, \tag{2.2}$$

and

$$y(t) - y(s) \leq \int_s^t f(\xi, y(\xi))d\xi, \qquad 0 \leq s \leq t, \tag{2.3}$$

where $f(t,y)$ is a nonnegative function. Assume that $f(t,y)$ is continuous with respect to y and that the function $G_v(\xi) := \sup_{0\leq\zeta\leq v} f(\xi,\zeta)$ is locally integrable on $[0,\infty)$ for any fixed $v > 0$. Let

$$F(t,v) := \int_0^t \sup_{0\leq\zeta\leq v} f(\xi,\zeta)d\xi, \qquad v,t \geq 0. \tag{2.4}$$

If there exists a constant $a > 0$ such that the function $F(t,a)$ is uniformly continuous with respect to t on $[0,\infty)$, then

$$\lim_{t\to\infty} y(t) = 0. \tag{2.5}$$

Proof. If (2.5) does not hold, then there exists an $\epsilon > 0$ and a sequence $(t_n)_{n=1}^\infty$ such that

$$0 < t_n \nearrow \infty, \qquad y(t_n) \geq \epsilon, \qquad \forall n \geq 1. \tag{2.6}$$

Without loss of generality we assume that

$$\epsilon < \min(a, 2\epsilon_0). \tag{2.7}$$

Since $F(t,a)$ is uniformly continuous with respect to t, there exists $\delta > 0$ such that

$$\int_t^{t+\delta} \sup_{0\leq y\leq a} f(\xi,y)d\xi = F(t+\delta,a) - F(t,a) < \frac{\epsilon}{2}, \qquad \forall t \geq 0. \tag{2.8}$$

Let us prove that

$$y(t) \geq \frac{\epsilon}{2}, \qquad \forall t \in [t_n - \delta, t_n], \qquad \forall n \geq 1. \tag{2.9}$$

Assume that (2.9) does not hold. Then there exist $\tilde{n} > 0$ and $\xi \in [t_{\tilde{n}} - \delta, t_{\tilde{n}})$ such that

$$y(\xi) < \frac{\epsilon}{2}. \tag{2.10}$$

Let

$$\nu = \min\{x : \xi < x \le t_{\tilde{n}}, y(x) \ge \epsilon\}. \tag{2.11}$$

It follows from the continuity of y, (2.6), and (2.10) that this ν exists, it satisfies the following inequality

$$t_{\tilde{n}} - \delta \le \xi < \nu \le t_{\tilde{n}}, \qquad y(\nu) = \epsilon, \tag{2.12}$$

and

$$0 \le y(x) \le y(\nu) = \epsilon, \qquad \xi \le x \le \nu. \tag{2.13}$$

From (2.3), (2.10), (2.12), and (2.13) one obtains

$$\begin{aligned} \frac{\epsilon}{2} < y(\nu) - y(\xi) &\le \int_{\xi}^{\nu} f(x, y(x))dx \le \int_{\xi}^{\nu} \sup_{0\le\zeta\le\epsilon} f(x,\zeta)dx \\ &\le \int_{t_n-\delta}^{t_n} \sup_{0\le\zeta\le a} f(x,\zeta)dx < \frac{\epsilon}{2}. \end{aligned} \tag{2.14}$$

This contradiction proves that (2.9) holds.

From (2.7), (2.9), and Assumption A) one gets

$$\int_{t_n-\delta}^{t_n} \omega(y(x))dx \ge \delta\omega(\frac{\epsilon}{2}) > 0, \qquad \forall n \ge 1. \tag{2.15}$$

This contradicts the Cauchy criterion for the convergence of the integral (2.2). Thus, (2.5) holds.

Theorem 2.1 is proved. □

Remark 2.2. If $F(t,a)$ is uniformly continuous with respect to t on $[0,\infty)$, then $F(t,v)$ is uniformly continuous with respect to t on $[0,\infty)$ for all $v \in [0,a]$. However, $F(t,v)$ may be not uniformly continuous with respect to t on $[0,\infty)$ for some $v > a$. Here is an example:

Let

$$f(x,y) := \begin{cases} 1 & \text{if} \quad 0 \le y \le 1 \\ 1 + (y-1)t & \text{if} \quad y \ge 1. \end{cases} \tag{2.16}$$

By a simple calculation one gets

$$F(t,u) = \begin{cases} t & \text{if} \quad 0 \le u \le 1 \\ t + (u-1)\frac{t^2}{2} & \text{if} \quad u \ge 1. \end{cases} \tag{2.17}$$

It follows from (2.17) that $F(t,u)$ is uniformly continuous with respect to t on $[0,\infty)$ if and only if $u \in [0,1]$.

From Theorem 2.1 one derives the following corollary.

Corollary 2.3. *Assume that $y(t) \ge 0$ be a continuous function satisfying inequality* (2.2)*,*

$$y(t) - y(s) \le \int_s^t [g(\xi)\varphi(y(\xi)) + h(\xi)]d\xi, \qquad 0 \le s \le t, \tag{2.18}$$

where g and h are nonnegative locally integrable functions on $[0,\infty)$, $\varphi \ge 0$ is a continuous function on $[0,\infty)$, and the functions $\int_0^t g(x)dx$ and $\int_0^t h(x)dx$ are uniformly continuous with respect to t on $[0,\infty)$. Then (2.5) *holds.*

Proof. Let

$$f(x,y) := g(x)\varphi(y) + h(x), \qquad x \ge 0, \qquad y \ge 0.$$

It follows from the uniform continuity of $\int_0^t g(x)dx$ and $\int_0^t h(x)dx$ and the local boundedness of $\varphi(t)$ that the function $F(x,a)$, defined in (2.4), is uniformly continuous on $[0,\infty)$ for any fixed $a > 0$. Thus, (2.5) follows from Theorem 2.1. □

Theorem 2.4. *Assume that $y(t) \ge 0$ is a continuous function on $[0,\infty)$,*

$$\int_0^\infty \omega(y(t))\varphi(t)dt < \infty, \tag{2.19}$$

where $\varphi(t) \ge 0$ is a continuous function on $[0,\infty)$, and there exists a constant $C > 0$ such that

$$\lim_{t\to\infty}\Big(t - \frac{C}{\varphi(t)}\Big) = \infty, \qquad M := \limsup_{t\to\infty} \frac{\max_{\xi\in[t-\frac{C}{\varphi(t)},t]}\varphi(\xi)}{\min_{\xi\in[t-\frac{C}{\varphi(t)},t]}\varphi(\xi)} < \infty. \tag{2.20}$$

Let $f(x,y) \ge 0$ be a locally integrable function on $\mathbb{R}^2$ satisfying condition (2.3)*. If there exist constants $a > 0$ and $\theta > 0$ such that the following condition holds:*

$$\int_s^t \sup_{0\le\zeta\le a} f(x,\zeta)dx \le (t-s)\theta a \max_{\xi\in[s,t]}\varphi(\xi), \qquad \theta = const > 0, \quad t > s \gg 1, \tag{2.21}$$

then (2.5) *holds.*

Remark 2.5. In (2.21) and below the notation $s \gg 1$ means "for all sufficiently large $s > 0$".

Proof. Let us assume first that $0 < \theta < 1$, and call this Case 1. Later we consider Case 2, namely, the assumption $\theta \ge 1$, and reduce it to Case 1.

Assume that (2.5) does not hold. Then there exists an $\epsilon > 0$ and a sequence $(t_n)_{n=1}^\infty$ such that

$$0 < t_n \nearrow \infty, \qquad y(t_n) \ge \epsilon, \qquad \forall n \ge 1, \tag{2.22}$$

and without loss of generality one assumes that

$$\epsilon \le \min\Big(2aMC, \frac{\epsilon_0}{1-\theta}\Big). \tag{2.23}$$

Let us prove that

$$y(t) \ge (1-\theta)\epsilon, \qquad \forall t \in [\tilde{t}_n, t_n], \qquad \forall n \gg 1, \tag{2.24}$$

where

$$\tilde{t}_n := t_n - \frac{\epsilon}{2aM\varphi(t_n)} < t_n. \tag{2.25}$$

Assume that (2.24) does not hold. Then there exists a sufficiently large $\tilde{n} > 0$ and a $\xi \in [\tilde{t}_{\tilde{n}}, t_{\tilde{n}})$ such that

$$y(\xi) < (1-\theta)\epsilon. \tag{2.26}$$

Let

$$\nu = \min\{x : \xi \le x \le t_{\tilde{n}},\, y(x) \ge \epsilon\}. \tag{2.27}$$

Then

$$\tilde{t}_{\tilde{n}} \le \xi < \nu \le t_{\tilde{n}}, \tag{2.28}$$

and

$$0 \le y(x) \le y(\nu) = \epsilon, \qquad \xi \le x \le \nu. \tag{2.29}$$

It follows from (2.3), (2.21), (2.25), (2.26), (2.28), and (2.29) that

$$\begin{aligned}\theta\epsilon < y(\nu) - y(\xi) &\le \int_\xi^\nu f(x, y(x))dx \le \int_\xi^\nu \sup_{0\le\zeta\le\epsilon} f(x,\zeta)dx \\ &\le \int_{\tilde{t}_{\tilde{n}}}^{t_{\tilde{n}}} \sup_{0\le\zeta\le a} f(x,\zeta)dx \le (t_{\tilde{n}} - \tilde{t}_{\tilde{n}})\theta a \sup_{\xi\in[\tilde{t}_{\tilde{n}}, t_{\tilde{n}}]} \varphi(\xi) \\ &= \theta a \frac{\epsilon \max_{\xi\in[\tilde{t}_{\tilde{n}}, t_{\tilde{n}}]} \varphi(\xi)}{2Ma\varphi(t_{\tilde{n}})} \le \theta\epsilon.\end{aligned} \tag{2.30}$$

This contradiction proves (2.24). In the derivation of (2.30) we have used the following inequality:

$$\frac{\max_{\xi\in[\tilde{t}_{\tilde{n}}, t_{\tilde{n}}]} \varphi(\xi)}{\varphi(t_{\tilde{n}})} \le \frac{\max_{\xi\in[t_{\tilde{n}} - C\varphi^{-1}(t_{\tilde{n}}), t_{\tilde{n}}]} \varphi(\xi)}{\min_{\xi\in[t_{\tilde{n}} - C\varphi^{-1}(t_{\tilde{n}}), t_{\tilde{n}}]} \varphi(\xi)} < 2M, \qquad \tilde{n} \gg 1, \tag{2.31}$$

which follows from (2.20) for sufficiently large $t_{\tilde{n}}$, and the factor 2 in (2.31) can be replaced by any fixed factor $1 + q$, where $q > 0$ can be arbitrarily small if $t_{\tilde{n}}$ is sufficiently large.

It follows from (2.23) and (2.24) and Assumption A) that

$$\begin{aligned}\int_{\tilde{t}_n}^{t_n} \omega(y(x))\varphi(x)dx &\ge (t_n - \tilde{t}_n)\omega((1-\theta)\epsilon) \min_{\tilde{t}_n\le\xi\le t_n} \varphi(\xi) \\ &\ge \omega((1-\theta)\epsilon) \frac{\epsilon}{2aM} \frac{\min_{\tilde{t}_n\le\xi\le t_n} \varphi(\xi)}{\max_{\tilde{t}_n\le\xi\le t_n} \varphi(\xi)} \\ &\ge \omega((1-\theta)\epsilon) \frac{\epsilon}{2aM(M+q)} > 0,\end{aligned} \tag{2.32}$$

where $q > 0$ is arbitrarily small for all sufficiently large n. From (2.23), (2.20), and (2.22), one gets

$$\lim_{n\to\infty} \left(t_n - \frac{\epsilon}{2aM\varphi(t_n)} \right) \ge \lim_{n\to\infty} \left(t_n - \frac{C}{\varphi(t_n)} \right) = \infty. \tag{2.33}$$

Inequalities (2.32) and (2.33) contradict the Cauchy criterion for the convergence of integral (2.19). Thus, (2.5) holds.

Consider Case 2, namely $\theta \ge 1$. In this case one replaces θ by $\theta_1 = \frac{1}{2}$, C by $C_1 = 2\theta C$, M by $M_1 = M$, defined in (2.20) with the C_1 in place of C, and, therefore, one reduces the problem to Case 1 with $\theta = \frac{1}{2} < 1$.

Let us give a more detailed argument. Let $\varphi_1(t) := 2\theta\varphi(t)$ and $C_1 := 2\theta C$. Then

$$\frac{C_1}{\varphi_1(t)} = \frac{C}{\varphi(t)}, \qquad \forall t \ge 0. \tag{2.34}$$

This, (2.19), (2.20) and (2.21) imply

$$\int_0^\infty \omega(y(t))\varphi_1(t)dt < \infty, \tag{2.35}$$

$$\lim_{t\to\infty}\Big(t - \frac{C_1}{\varphi_1(t)}\Big) = \infty, \qquad \limsup_{t\to\infty} \frac{\max_{\xi\in[t-\frac{C_1}{\varphi_1(t)},t]}\varphi_1(\xi)}{\min_{\xi\in[t-\frac{C_1}{\varphi_1(t)},t]}\varphi_1(\xi)} = M < \infty \tag{2.36}$$

$$\int_s^t \sup_{0\le\zeta\le a} f(x,\zeta)dx \le (t-s)\frac{a}{2}\max_{\xi\in[s,t]}\varphi_1(\xi), \qquad t > s \gg 1. \tag{2.37}$$

Theorem 2.4 is proved. □

Remark 2.6. (i) Conditions (2.20) hold if

$$\liminf_{t\to\infty} t\varphi(t) > 0, \qquad \limsup_{t\to\infty}\frac{\max_{\xi\in[(1-\epsilon)t,t]}\varphi(\xi)}{\min_{\xi\in[(1-\epsilon)t,t]}\varphi(\xi)} < \infty, \tag{2.38}$$

for a sufficiently small $\epsilon > 0$.

(ii) If $y(t)$ is differentiable, then (2.2) is equivalent to

$$y'(t) \le f(t,y(t)), \qquad t \ge 0. \tag{2.39}$$

(iii) Theorem 2.4 holds if in place of (2.21) one assumes that

$$\sup_{0\le\zeta\le a} f(t,\zeta) \le \tilde{C}\varphi(t), \qquad t \gg 1, \qquad \tilde{C} = const > 0. \tag{2.40}$$

Indeed, if (2.40) hold then

$$\int_s^t \sup_{0\le\zeta\le a} f(\xi,\zeta)d\xi \le \int_s^t \tilde{C}\varphi(\xi)d\xi \le \tilde{C}(t-s)\max_{s\le\xi\le t}\varphi(\xi).$$

(iv) If $\varphi(t)$ is non-increasing, then the second relation in (2.20) becomes

$$M := \limsup_{t\to\infty}\frac{\varphi(t-\frac{C}{\varphi(t)})}{\varphi(t)} < \infty. \tag{2.41}$$

From Theorem 2.4 we derive the following theorem.

Theorem 2.7. *Assume that $y(t) \ge 0$ is a continuous on $[0,\infty)$ function,*

$$\int_0^\infty \omega(y(t))\frac{1}{(1+t)^\alpha}dt < \infty, \qquad 0 < \alpha \le 1, \tag{2.42}$$

$$y(t) - y(s) \le \int_s^t f(x,y(x))dx, \qquad 0 \le s \le t, \tag{2.43}$$

and there exist constants $a > 0$ and $\kappa > 0$ such that

$$\int_s^t \sup_{0\le\zeta\le a} f(x,\zeta)dx \le \kappa a\frac{t-s}{s^\alpha}, \qquad \kappa > 0, \quad t > s \gg 1. \tag{2.44}$$

Then,

$$\lim_{t\to\infty} y(t) = 0. \tag{2.45}$$

Proof. Let $\varphi(t) := \frac{1}{(1+t)^\alpha}$, $\alpha \in (0,1]$. Then one can easily verify that conditions (2.20) hold with $C = 1/2$. Condition (2.21) also holds for this choice of φ and $\theta = 2\kappa$. Thus, Theorem 2.7 follows from Theorem 2.4. □

Remark 2.8. The assumption $\alpha \in (0,1]$ in (2.42) is essential: if $\alpha > 1$, then inequality (2.20) does not hold for $\varphi(t) = \frac{1}{(1+t)^\alpha}$ whatever fixed $C > 0$ is.

Corollary 2.9. *Let $y(t) \geq 0$ be a continuous function on $[0,\infty)$ and*

$$\int_0^\infty \omega(y(t))\varphi(t)dt < \infty, \tag{2.46}$$

where $\varphi(t) > 0$ is a continuous function on $[0,\infty)$. Assume that there exists a constant $C > 0$ such that

$$\lim_{t\to\infty}\Big(t - \frac{C}{\varphi(t)}\Big) = \infty, \qquad M := \limsup_{t\to\infty} \frac{\max_{\xi\in[t-\frac{C}{\varphi(t)},t]}\varphi(\xi)}{\min_{\xi\in[t-\frac{C}{\varphi(t)},t]}\varphi(\xi)} < \infty, \tag{2.47}$$

$$y(t) - y(s) \leq \int_s^t h(\xi)d\xi, \qquad 0 \leq s \leq t, \tag{2.48}$$

where $h(t)$ is a nonnegative and locally integrable function on $[0,\infty)$, and

$$A := \limsup_{t\to\infty} \frac{h(t)}{\varphi(t)} < \infty. \tag{2.49}$$

Then,

$$\lim_{t\to\infty} y(t) = 0. \tag{2.50}$$

Proof. Let

$$f(t,y) := h(t), \qquad t \geq 0, \qquad y \geq 0. \tag{2.51}$$

Let us check that condition (2.21) holds for this $f(t,y)$ and $a = 2A$. From (2.51) one gets

$$f(t,y) \leq 2A\varphi(t), \qquad t \gg 1, \qquad \forall y \geq 0. \tag{2.52}$$

This implies

$$\int_s^t \max_{0\leq\zeta\leq 2A} f(\xi,\zeta)d\xi \leq \int_s^t 2A\varphi(\xi)d\xi \leq 2A(t-s)\sup_{\xi\in(s,t)}\varphi(\xi), \qquad t > s \gg 1. \tag{2.53}$$

This and Theorem 2.4 imply (2.50). □

Corollary 2.10. *Let $y(t) \geq 0$ be a continuous function on $[0,\infty)$,*

$$\int_0^\infty \omega(y(t))\frac{1}{(1+t)^\alpha}dt < \infty, \qquad 0 < \alpha \leq 1, \tag{2.54}$$

and

$$y(t) - y(s) \leq \int_s^t h(\xi)d\xi, \qquad 0 \leq s \leq t, \tag{2.55}$$

where $h(t)$ is a nonnegative and locally integrable function on $[0,\infty)$. *If*

$$A := \limsup_{t\to\infty} h(t)t^{\alpha} < \infty, \tag{2.56}$$

then

$$\lim_{t\to\infty} y(t) = 0. \tag{2.57}$$

Proof. Let $\varphi(t) = \frac{1}{(t+1)^{\alpha}}$, $\alpha \in (0,1]$. Then conditions (2.47) hold with $C = \frac{1}{2}$ and $M = 1$, and condition (2.49) also holds. Thus, (2.57) follows from Corollary 2.9. □

Let us study inequality (1.5) and justify sufficient conditions for relation (1.3) to hold. Since inequality (1.5) is a special case of inequality (1.2), some of the results below are obtained by applying results proved earlier for inequality (1.2).

Theorem 2.11. *Assume that $g \geq 0$ is a continuously differentiable function on* $[0,\infty)$*,*

$$\dot{g}(t) \leq -a(t)f(g(t)) + b(t), \tag{2.58}$$

where $f(t)$ is a nonnegative continuous function on $[0,\infty)$*, $f(0) = 0$, $f(t) > 0$ for $t > 0$, and*

$$m(\epsilon) := \inf_{x\geq\epsilon} f(x) > 0, \qquad \forall\epsilon > 0. \tag{2.59}$$

If $a(t) > 0$, $b(t) \geq 0$ are continuous on $[0,\infty)$ *functions, and*

$$\int_0^{\infty} a(s)ds = \infty, \qquad \lim_{t\to\infty} \frac{b(t)}{a(t)} = 0, \tag{2.60}$$

then

$$\lim_{t\to\infty} g(t) = 0. \tag{2.61}$$

Proof. Let

$$s := s(t) := \int_0^t a(\xi)d\xi. \tag{2.62}$$

It follows from (2.60) that the map $t \to s$ maps $[0,\infty)$ onto $[0,\infty)$. Let $t(s)$ be the inverse map and define $w(s) = g(t(s))$. Then (2.58) takes the form

$$w'(s) \leq -f(w) + \beta(s), \qquad w(0) = g_0, \tag{2.63}$$

where

$$w' = \frac{dw}{ds}, \qquad \beta(s) = \frac{b(t(s))}{a(t(s))}, \qquad \lim_{s\to\infty} \beta(s) = 0. \tag{2.64}$$

Assume that (2.61) does not hold. Then there exist $\epsilon > 0$ and $(s_n)_{n=1}^{\infty})$ such that

$$0 < s_n \nearrow \infty, \qquad w(s_n) > \epsilon, \qquad \forall n. \tag{2.65}$$

From the last relation in (2.64) it follows that there exists $T > 0$ such that

$$\beta(s) < \frac{m(\epsilon)}{2}, \qquad \forall s \geq T. \tag{2.66}$$

Since $s_n \nearrow \infty$, there exists $N > 0$ such that $s_n > T$, $\forall n \geq N$. Thus,

$$w'(s_n) \leq -f(w(s_n)) + \beta(s_n) \leq -m(\epsilon) + \frac{m(\epsilon)}{2} < 0, \qquad \forall n \geq N. \tag{2.67}$$

Since $w(s)$ is continuously differentiable on the interval (s_{n-1}, s_n) and $w'(s_n) < 0$, $\forall n \geq N$, there are two possibilities:

Case 1: $w'(s) < 0$, $n \geq N$, for all $s \in (s_{n-1}, s_n)$.

Case 2: there exists a point $t_n \in (s_{n-1}, s_n)$ such that $w'(s) < 0$, $\forall s \in (t_n, s_n)$ and $w'(t_n) = 0$ where $n \geq N$.

We claim that Case 2 cannot happen if $n \geq N$ is sufficiently large, namely so large that $\beta(t_n) < m(\epsilon)$. Indeed, if Case 2 holds for such n, then

$$w'(t_n) = 0, \qquad w(t_n) > w(s_n) > \epsilon. \tag{2.68}$$

This and (2.63) imply

$$0 = w'(t_n) \leq -f(w(t_n)) + \beta(t_n) \leq -m(\epsilon) + \beta(t_n), \tag{2.69}$$

i.e., $0 < m(\epsilon) \leq \beta(t_n)$. This contradicts the assumption $\lim_{t\to\infty}\beta(t) = 0$ because if n is sufficiently large then t_n is so large that $\beta(t_n) < m(\epsilon)$.

Since Case 2 cannot happen for all sufficiently large n, there exists $N_1 > 0$ sufficiently large so that

$$w'(s) < 0, \qquad \forall s \in (s_{n-1}, s_n), \qquad n \geq N_1. \tag{2.70}$$

This and (2.67) imply

$$w'(t) < 0, \qquad \forall t \geq s_{N_1}. \tag{2.71}$$

Therefore $w(t)$ decays monotonically for all sufficiently large t. Since $w(t) \geq 0$, one concludes that the following limit $W \geq 0$ exists and is finite

$$\lim_{t\to\infty} w(t) = W < \infty. \tag{2.72}$$

This and (2.63) imply

$$\limsup_{t\to\infty} w'(t) \leq \lim_{t\to\infty}[-f(w(t)) + \beta(t)] \leq -m(W). \tag{2.73}$$

If $W \neq 0$, then $m(W) > 0$ and

$$\limsup_{t\to\infty} w'(t) \leq -m(W) < 0. \tag{2.74}$$

Thus, $w'(t) \leq -m(W) < 0$ for all t sufficiently large. This is impossible since $w(t) \geq 0$, $\forall t$. This contradiction implies that $W = 0$, so (2.61) holds.

Theorem 2.11 is proved. □

Remark 2.12. Theorem 2.11 is proved in [5] under the assumption that $f \in Lip_{loc}[0, \infty)$ and

$$f(0) = 0, \qquad f(u) > 0 \quad \text{for} \quad u > 0, \quad f(u) \geq c > 0 \quad \text{for} \quad u \geq 1, \tag{2.75}$$

where $c = const$. The assumption $f \in Lip_{loc}[0, \infty)$ was used in [5] in order to prove the global existence of $g(t)$. In this paper we assume the global existence of $g(t)$, and give a new simple proof of Theorem 2.11.

Theorem 2.13. *Assume that* $g \geq 0$ *is a* $C^1([0,\infty))$*–function,*

$$\dot{g}(t) \leq -a(t)f(g(t)) + b(t), \qquad g(0) = g_0, \tag{2.76}$$

where $f(t) \geq 0$ *is a non-decreasing function on* $[0,\infty)$*,* $f(0) = 0$*,* $f(t) > 0$ *if* $t > 0$*. If* $a(t) > 0$*,* $b(t) \geq 0$ *are continuous on* $[0,\infty)$ *functions, and*

$$\int_0^\infty a(s)ds = \infty, \qquad \int_0^\infty \beta(s)ds < \infty, \qquad \beta(t) := \frac{b(t)}{a(t)}, \tag{2.77}$$

then

$$\lim_{t\to\infty} g(t) = 0. \tag{2.78}$$

Proof. Let s be defined in (2.62) and $w(s) = g(t(s))$. From (2.63) one gets

$$w(t) - w(0) + \int_0^t f(w(s))ds \leq \int_0^t \beta(s)ds \leq \int_0^\infty \beta(s)ds < \infty, \qquad \forall t \geq 0. \tag{2.79}$$

This and the assumption that $w \geq 0$ imply

$$\int_0^\infty f(w(s))ds < \infty. \tag{2.80}$$

From (2.63) one obtains

$$w'(s) \leq \beta(s), \qquad \forall s \geq 0. \tag{2.81}$$

Since $\int_0^\infty \beta(s)ds < \infty$, the function $\psi(t) := \int_0^t \beta(s)ds < \infty$ is uniformly continuous with respect to t on $[0,\infty)$. This, relation (2.80), inequality (2.81), and Theorem 2.1 imply

$$\lim_{s\to\infty} w(s) = 0. \tag{2.82}$$

This and the relation $w(s) = g(t(s))$ imply (2.78).

Theorem 2.13 is proved. □

Theorem 2.14. *Assume that* $g \geq 0$ *is a* $C^1([0,\infty))$ *function,*

$$\dot{g}(t) \leq -a(t)f(g(t)) + b(t), \qquad g(0) = g_0, \tag{2.83}$$

where $f(t) \geq 0$ *is a non-decreasing continuous function on* $[0,\infty)$*,* $f(0) = 0$*,* $f(t) > 0$ *if* $t > 0$*,* $a(t) > 0$ *and* $b(t) \geq 0$ *are continuous functions on* $[0,\infty)$*, and there exists a constant* $C > 0$ *such that*

$$\lim_{t\to\infty}\Big(t - \frac{C}{a(t)}\Big) = \infty, \qquad \limsup_{t\to\infty} \frac{\max_{\xi\in[t-\frac{C}{a(t)},t]} a(\xi)}{\min_{\xi\in[t-\frac{C}{a(t)},t]} a(\xi)} < \infty. \tag{2.84}$$

If

$$K := \limsup_{t\to\infty} \frac{b(t)}{a(t)} < \infty, \qquad \int_0^\infty b(s)ds < \infty, \tag{2.85}$$

then

$$\lim_{t\to\infty} g(t) = 0. \tag{2.86}$$

Proof. From (2.83) one gets for all $t \geq 0$ the following inequalities

$$g(t) - g(0) + \int_0^t a(s)f(g(s))ds \leq \int_0^t b(s)ds \leq \int_0^\infty b(s)ds < \infty. \tag{2.87}$$

Thus

$$\int_0^\infty a(s)f(g(s))ds < \infty. \tag{2.88}$$

This relation, (2.85), (2.84), and Corollary 2.9 imply (2.86). Theorem 2.14 is proved. □

Remark 2.15. If $a(t) = O(\frac{1}{(1+t)^\gamma})$, $\gamma \in [0,1)$, then conditions (2.84) hold for any $C > 0$. If $a(t) = O(\frac{1}{1+t})$ then conditions (2.84) hold if $C > 0$ is sufficiently small.

3 Applications

Let H be a real Hilbert space. Consider the following problem

$$\dot{u} = A(t,u) + f(t), \qquad u(0) = u_0; \qquad f \in C([0,\infty);H), \tag{3.1}$$

where $u_0 \in H$, $A(t,u) : [0,\infty)\times H \to H$ is continuous with respect to t and u. Assume that

$$A(t,0) = 0, \qquad \forall t \geq 0, \tag{3.2}$$

$$\langle A(t,u) - A(t,v), u - v\rangle \leq -\gamma(t)\|u - v\|\omega(\|u - v\|), \qquad u,v \in H, \tag{3.3}$$

where $\gamma(t) > 0$ for all $t \geq 0$ is a continuous function and $\omega(t) \geq 0$ is continuous and strictly increasing function on $[0,\infty)$, $\omega(0) = 0$. Here, $\langle\cdot,\cdot\rangle$ denotes the inner product in H and $\|\cdot\|$ denotes the norm in H.

The above assumptions are standing and are not repeated. Assumption (3.3) means that A is a dissipative operator. Existence of the solution to problem (3.1) with such operators was discussed in the literature (see, e.g., [4], [5], and [11]).

Let

$$\beta(t) := \|f(t)\|.$$

Consider the following three assumptions:

- Assumption B)

$$\int_0^\infty \gamma(t)dt = \infty, \qquad \lim_{t\to\infty} \frac{\beta(t)}{\gamma(t)} = 0. \tag{3.4}$$

- Assumption C)

$$\int_0^\infty \gamma(t)dt = \infty, \qquad \int_0^\infty \frac{\beta(t)}{\gamma(t)}dt < \infty. \tag{3.5}$$

- Assumption D)

$$\gamma(t) = O(\frac{1}{(1+t)^\alpha}), \qquad \int_0^\infty \beta(t)dt < \infty, \qquad \limsup_{t\to\infty} \frac{\beta(t)}{\gamma(t)} < \infty, \tag{3.6}$$

where $\alpha = const \in (0,1]$.

Remark 3.1. Assumption (3.2) is not an essential restriction: if it does not hold, define $f_1(t) := f(t) + A(t,0)$, and $A_1(t,u) := A(t,u) - A(t,0)$. Then $A_1(t,u)$ satisfies assumptions (3.2) and (3.3) and $f_1(t)$ plays the role of $f(t)$.

Lemma 3.2. *If assumptions* (3.2)–(3.3) *hold, then there exists a unique global solution* $u(t)$ *to* (3.1).

Proof. Let us first prove the uniqueness of solution to (3.1).

Assume that u and v are two solutions to (3.1). Then one gets

$$\dot{u} - \dot{v} = A(t,u) - A(t,v), \qquad t \geq 0. \tag{3.7}$$

Multiply (3.7) by $u - v$ and use (3.3) to obtain

$$\frac{1}{2}\frac{d}{dt}\|u - v\|^2 = \langle A(t,u) - A(t,v), u - v\rangle \leq 0. \tag{3.8}$$

Integrating (3.8) one gets

$$\frac{1}{2}\Big(\|u(t) - v(t)\|^2 - \|u(0) - v(0)\|^2\Big) \leq 0, \qquad \forall t \geq 0. \tag{3.9}$$

This implies $u(t) = v(t)$, $\forall t \geq 0$, since $u(0) = v(0)$.

Let us prove the local existence of a solution to (3.1).

In this proof an argument similar to the one in [3] or [5] is used. Let $u_n(t)$, called Peano's approximation of u, solve the following equation

$$u_n(t) = u_0 + \int_0^t [A(s, u_n(s - \frac{1}{n})) + f(s)]ds, \qquad t \geq 0, \tag{3.10}$$

and $u_n(t) = u_0, \qquad \forall t \leq 0.$

Fix some positive numbers $r > 0$ and $t_1 > 0$. Let

$$B(u_0, r) := \{u : \|u - u_0\| \leq r\},$$

and

$$c := \sup_{(t,u)\in[0,t_1]\times B(u_0,r)} (\|A(t,u)\| + \|f(t)\|) < \infty. \tag{3.11}$$

Let

$$T_1 := \min(t_1, \frac{r}{c}) > 0, \qquad T_2 := \sup\{t \geq 0 : \|u_n(\xi) - u_0\| \leq r, \forall \xi \in (0,t]\}. \tag{3.12}$$

Let us prove that $T_2 \geq T_1$. Arguing by contradiction, we assume that $T_2 < T_1$. Then it follows from (3.10), (3.11), and (3.12) that

$$\begin{aligned}
\|u_n(t) - u_0\| &\leq \int_0^t (\|A(s, u_n(s - \frac{1}{n}))\| + \|f(s)\|)ds \\
&\leq \int_0^{T_2+\zeta} \sup_{(t,u)\in[0,t_1]\times B(u_0,r)} (\|A(s,v)\| + \|f(s)\|)ds \\
&\leq c(T_2 + \zeta) \\
&\leq cT_1 \leq r, \qquad 0 \leq t \leq T_2 + \zeta,
\end{aligned} \tag{3.13}$$

where $\zeta = \min(\frac{1}{n}, T_1 - T_2) > 0$. This contradicts the definition of T_2. Thus, $T_2 \geq T_1$ and, therefore,

$$\|u_n(t) - u_0\| \leq r, \qquad \forall t \in [0, T_1]. \tag{3.14}$$

Define

$$w_{nm} := u_n(t) - u_m(t), \qquad g_{mn} := \|w_{mn}(t)\|, \qquad t \geq 0. \tag{3.15}$$

From (3.10) one obtains

$$\begin{aligned} g_{mn}(t)\dot{g}_{mn}(t) =& \langle A(t, u_n(t - \frac{1}{n})) - A(t, u_m(t - \frac{1}{m})), u_n(t) - u_m(t)\rangle \\ =& \langle A(t, u_n(t - \frac{1}{n})) - A(t, u_m(t - \frac{1}{m})), u_n(t - \frac{1}{n}) - u_m(t - \frac{1}{m})\rangle \\ &+ \langle A(t, u_n(t - \frac{1}{n})) - A(t, u_m(t - \frac{1}{m})), u_n(t) - u_n(t - \frac{1}{n})\rangle \\ &+ \langle A(t, u_n(t - \frac{1}{n})) - A(t, u_m(t - \frac{1}{m})), u_m(t - \frac{1}{m}) - u_m(t)\rangle. \end{aligned} \tag{3.16}$$

From (3.16), (3.11), and (3.3), one obtains

$$\frac{1}{2}\frac{d}{dt}g^2_{nm}(t) \leq 4c^2(\frac{1}{n} + \frac{1}{m}), \qquad m, n \geq 0, \quad t \in [0, T_1]. \tag{3.17}$$

Integrating (3.17), using the relation $g_{mn}(0) = 0$, and taking the limit as $m, n \to \infty$ one obtains

$$0 \leq \lim_{n,m\to\infty} g^2_{nm}(t) \leq \lim_{n,m\to\infty} t4c^2(\frac{1}{n} + \frac{1}{m}) = 0, \qquad \forall t \in [0, T_1]. \tag{3.18}$$

It follows from (3.18) and the Cauchy criterion for convergence of a sequence that the following limit exists

$$u(t) := \lim_{n\to\infty} u_n(t), \qquad 0 \leq t \leq T_1. \tag{3.19}$$

Passing to the limit $n \to \infty$ in equation (3.10) and using the continuity of $A(t, u)$ on $[0, \infty) \times H$ and (3.19) one concludes that $u(t)$ solves the equation

$$u(t) = u_0 + \int_0^t [A(s, u(s)) + f(s)]ds, \qquad \forall t \in [0, T_1]. \tag{3.20}$$

Thus, the local existence of the solution $u(t)$ to equation (3.1) is proved.

Let us prove the global existence of $u(t)$*.*

Assume that $u(t)$ does not exist globally. Let $[0, T]$ be the maximal existence interval of $u(t)$. Then, $0 < T < \infty$. By similar arguments as in the proof of Theorem (3.3) (see (3.32) below) one gets

$$\frac{d}{dt}\|u(t)\| \leq -\gamma(t)w(\|u(t)\|) + \|f(t)\| \leq \|f(t)\|, \qquad 0 \leq t < T.$$

This implies

$$\|u(t)\| \leq \|u(0)\| + \int_0^T \|f(t)\|dt, \qquad 0 \leq t < T.$$

Thus

$$\|u(t)\| = g(t) < c = const < \infty, \qquad \forall t \in [0, T). \tag{3.21}$$

Let us prove the existence of the finite limit

$$\lim_{t\to T_-} u(t) = u_T. \tag{3.22}$$

Let $z_h(t) := u(t+h) - u(t)$, $0 < t \le t+h < T$. From (3.1) one gets

$$\dot{z}_h(t) = A(t+h, u(t+h)) - A(t, u(t)) + f(t+h) - f(t). \tag{3.23}$$

Multiply (3.23) by $z_h(t)$ and get

$$\frac{1}{2}\frac{d}{dt}\|z_h(t)\|^2 \le -\gamma(\|z_h(t)\|)\|z_h(t)\|\omega(\|z_h(t)\|) + \|z_h(t)\|\|f(t+h) - f(t)\|. \tag{3.24}$$

This implies

$$\frac{d}{dt}\|z_h(t)\| \le \|f(t+h) - f(t)\|, \qquad 0 < t \le t+h < T. \tag{3.25}$$

Integrating (3.25) one gets

$$\begin{aligned} \|z_h(t)\| &\le \|z_h(0)\| + \int_0^t \|f(x+h) - f(x)\| dx \\ &\le \|z_h(0)\| + T \max_{0\le t\le T-h} \|f(t+h) - f(t)\|. \end{aligned} \tag{3.26}$$

Since $\lim_{h\to+0} \|u(h) - u(0)\| = 0$ and $\lim_{h\to+0} \max_{0\le t\le T-h} \|f(t+h) - f(t)\| = 0$, one concludes that

$$\lim_{h\to 0} \|u(t+h) - u(t)\| = 0, \tag{3.27}$$

and this relation holds uniformly with respect to t and $t+h$ such that $t < t+h < T$. Relation (3.27) and the Cauchy criterion for convergence imply the existence of the finite limit in (3.22).

Consider the following Cauchy problem

$$\dot{u} = A(t,u) + f(t), \qquad u(T) = u_T. \tag{3.28}$$

By the arguments similar to the given above one derives that there exists a unique solution $u(t)$ to (3.28) on $[T, T+\delta]$, where $\delta > 0$ is a sufficiently small number. From the continuity of $f(t)$ and $u(t)$ in t and $A(t,u)$ in both t and u one gets

$$\lim_{t\to T_-} \dot{u}(t) = \lim_{t\to T_-} A(t,u) + f(t) = \lim_{t\to T_+} A(t,u) + f(t) = \lim_{t\to T_+} \dot{u}(t), \tag{3.29}$$

and the above limits are finite. Thus, the solution to (3.1) can be extended to the interval $[0, T+\delta]$. This contradicts the definition of T. Thus, $T = \infty$ i.e., $u(t)$ exists globally.

Lemma 3.2 is proved. □

The main result of this Section is the following theorem.

Theorem 3.3. *Let assumptions* (3.2) *and* (3.3) *hold. If $u(t)$ is the solution to problem* (3.1) *and at least one of the assumptions B), C), or D) holds, then*

$$\lim_{t\to\infty} \|u(t)\| = 0. \tag{3.30}$$

Proof. The global existence of $u(t)$ follows from Lemma 3.2 under the assumptions of this lemma, or from the results in [4] and [11].

Let us prove relation (3.30).

Multiplying (3.1) by u, one obtains

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 = \langle A(t,u),u\rangle + \langle f(t),u\rangle \le -\gamma(t)\|u\|\omega(\|u\|) + \|f(t)\|\|u\|. \tag{3.31}$$

Since $u(t)$ is continuously differentiable, so is $\|u(t)\|$ at the points t at which $\|u(t)\| > 0$. At these points inequality (3.31) implies

$$\frac{d}{dt}\|u(t)\| \le -\gamma(t)\omega(\|u(t)\|) + \|f(t)\|, \qquad t \ge 0. \tag{3.32}$$

If $\|u(t)\| = 0$ on an open interval $(a,b) \subset [0,\infty)$, then $\frac{d}{dt}\|u(t)\| = 0$ on (a,b), and inequality (3.32) holds trivially, because $\omega(0) = 0$ and $\|f(t)\| \ge 0$. If $\|u(s)\| = 0$ at an isolated point $s > 0$, then the right-sided derivative of $\|u(t)\|$ at the point s exists, and

$$\frac{d}{dt}\|u(t)\| = \lim_{\tau\to+0}\frac{\|u(s+\tau)\|}{\tau} = \|\frac{d}{dt}u(s)\|,$$

and inequality (3.32) holds for this derivative. In what follows we understand by $\frac{d}{dt}\|u(t)\|$ the right-sided derivative at the points s at which $\|u(s)\| = 0$. The left-sided derivative of $\|u(t)\|$ also exists at such points, and is equal to $-\|\frac{d}{dt}u(s)\|$, but we will not need this left-sided derivative.

Let $g(t) := \|u(t)\|$ and $\beta(t) := \|f(t)\|$. From (3.32) one gets

$$\dot{g}(t) \le -\gamma(t)\omega(g(t)) + \beta(t), \qquad t \ge 0. \tag{3.33}$$

Let $a(t) := \gamma(t)$ and $b(t) := \beta(t)$.

If Assumption B) holds, then (3.30) follows from this assumption and Theorem 2.11.

If Assumption C) holds, then (3.30) follows from this assumption and Theorem 2.13.

If Assumption D) holds, then (3.30) follows from this assumption and Theorem 2.14.

Thus, (3.30) holds.

Theorem 3.3 is proved. □

Example.

Let $D \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary, $H = L^2(D)$, $\|f(t)\|$ is the norm of f in H, $f(t)$ is a locally bounded function of t with values in H. Let H^ℓ, $\ell = 1,2$, be the usual Sobolev spaces. By the Sobolev embedding theorem, the embedding operator from $H^1(D)$ into $L^6(D)$ is bounded, so if $u \in H^1(D)$, then $u^3 \in H$. Consider problem (3.1) with $\gamma(t) = (1+t)^{-\alpha}$, $\alpha = const \in (0,1]$, $\|f(t)\| = O(\frac{1}{(1+t)^k})$, $k > 1$, and $A(t,u) := \gamma(t)L_1$, $L_1 := Lu - u^3$, where L is a second order negative-definite selfadjoint Dirichlet elliptic operator in D, e.g., $L = \Delta$, where Δ is the Dirichlet Laplacian in D. Then

$$\beta(t) = \|f(t)\| \le \frac{c}{(1+t)^k}, \qquad k > 1, \tag{3.34}$$

conditions (3.2) and (3.3) are satisfied for $u, v \in D(A) = D(A(t,u))$, where $D(A)$ is the domain of definition of the operator A, and $D(A) = D(L) = D(L_1)$, $L_1(u) := \Delta u - u^3$, namely, $D(A) =$

$H^2(D) \cap H_0^1(D) \subset C(D)$, where $C(D)$ is the space of continuous in D functions with the sup norm, and the inclusion holds by the Sobolev embedding theorem. The function $\omega(r)$ in (3.3) in this example is $\omega(r) = cr$, where $c > 0$ is a constant. This follows from the known inequality

$$-\langle Lu, u\rangle = \|\nabla u\|^2 \geq c(D)\|u\|^2, \tag{3.35}$$

valid for $u \in D(A)$, $c(D) = const$ does not depend on $u \in H_0^1(D)$. In this example the operator A is not continuous in H, but the global solution to problem (3.1) exists and is unique (see, e.g., [4], [11], and [13]). One checks that Assumption D) is satisfied, and concludes using Theorem 2.14 that (3.30) holds for the solution to (3.1) in this example.

Theorem 2.14 can be applied regardless of the method by which the global existence of the unique solution to problem (3.1) is established and inequality (3.32) is derived for this solution.

Let $\langle\cdot,\cdot\rangle$ denote the inner product and $\|\cdot\|$ denote the norm in $L^2(D)$. Then the usual ellipticity constant $c_1 = \gamma(t)c(D)$ in the inequality

$$c_1\|u\|^2 \leq -\gamma(t)\langle Lu, u\rangle \tag{3.36}$$

tends to zero as $t \to \infty$, so one deals with a degenerate elliptic operator as $t \to \infty$ in problem (3.1) in this example.

One can extend the result in this example to much more general nonlinearities. For instance, if $A(t,u) = \gamma(t)[Lu - h(u)]$, where $uh(u) \geq 0$ for all $u \in \mathbb{R}$, and h satisfies a local Lipschitz condition, then one can derive an a priori bound for the solution $u(t)$ of (3.1) $\sup_{t\geq 0}\|u(t)\| \leq c$, and prove the global existence and uniqueness of the solution $u(t)$ to problem (3.1) using, for instance, the method from [6]. The assumption $uh(u) \geq 0$ for all $u \in \mathbb{R}$ makes it possible to consider nonlinearities $h(u)$ with an arbitrary large speed of growth at infinity. Let us outline the derivation of the above bound. Multiplying equation (3.1) by u, taking real part, using the estimate $\langle Lu, u\rangle \leq -c\|u\|^2$, the assumption $uh(u) \geq 0$, denoting $g := \|u\|^2$, and using the relation $2\mathrm{Re}\langle \dot{u}, u\rangle = \dot{g}$, one gets the following inequality

$$\dot{g} \leq -2c\gamma(t)g + 2\|f\|g^{1/2}, \qquad g(0) = \|u_0\|^2. \tag{3.37}$$

For simplicity and without loss of generality assume that $u_0 = 0$. Then it is not difficult to derive the following estimate:

$$\|u(t)\| \leq \int_0^t \|f(s)\|e^{-c\int_s^t \gamma(\tau)d\tau}ds. \tag{3.38}$$

Using the assumptions

$$\|f(t)\| \leq c(1+t)^{-k},\ k > 1, \quad \gamma(t) = (1+t)^{-\alpha} > 0,\ 0 < \alpha \leq 1,$$

one obtains from this inequality the following estimate:

$$\sup_{t\geq 0}\|u(t)\| \leq c\int_0^\infty (1+s)^{-k}ds \leq \frac{c}{k-1}. \tag{3.39}$$

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