Note di Matematica **27,** n. 2, 2007, 131–154.

On large bending deformations of transversely isotropic rectangular elastic blocks

Fotios Kassianidis
Department of Mathematics, University of Glasgow,
University Gardens, Glasgow G12 8QW, UK
fk@maths.gla.ac.uk

Raymond W. Ogden
Department of Mathematics, University of Glasgow,
University Gardens, Glasgow G12 8QW, UK
rwo@maths.gla.ac.uk

Received: 20/04/2006; accepted: 25/07/2006.

Abstract. In this paper we examine the classical problem of finite bending of a rectangular block of elastic material into a sector of a circular cylindrical tube in respect of compressible *transversely isotropic* elastic materials. More specifically, we consider the possible existence of *isochoric* solutions. In contrast to the corresponding problem for *isotropic* materials, for which such solutions do not exist for a compressible material, we determine conditions on the form of the strain-energy function for which isochoric solutions are possible. The results are illustrated for particular classes of energy function.

Keywords: nonlinear elasticity, transverse isotropy, large elastic deformations

MSC 2000 classification: primary 74B20

1 Introduction

The problem of finite bending of a rectangular elastic block into a sector of a circular cylindrical tube has been examined by many researchers, almost exclusively for isotropic materials. First, in Rivlin [17], necessary and sufficient conditions for the solution of this problem in terms of the boundary data were derived for incompressible Mooney-Rivlin and neo-Hookean materials by assuming that the block remains in its deformed state in the absence of applied tractions on its curved surfaces but with appropriate tractions applied on its other surfaces. Corresponding results for a general incompressible isotropic material were given by Rivlin [18]. A similar analysis was presented by Green and Zerna [6] and by Green and Adkins [5]. Green and Adkins also examined the problem for incompressible transversely isotropic, initially curved incompressible isotropic and compressible isotropic rectangular blocks. Formulation of the governing equilibrium equations in respect of compressible isotropic materials and the derivation of closed-form solutions for the general class of the so-called harmonic materials were given in Ogden [13], wherein the incompressible case is also discussed. Furthermore, several classes of compressible isotropic materials were investigated by Jiang [7], in which it was shown that finite *isochoric* bending of a block is only sustainable for incompressible materials. The problem of bending in the compressible theory was also discussed by Aron and Wang [2], who used constant modified stretches to express the total energy as a function of the deformed volume V and to deduce that (under plane strain) it attains a minimum at a certain value V_0 of V. In addition, a stability analysis for semi-linear harmonic materials (under plane strain) was discussed by the same authors in [1]. In a recent paper, Bruhns et al. [3] examined the same problem for compressible and incompressible isotropic Hencky materials.

In the present analysis, we consider the problem of bending for *transversely isotropic* elastic materials. In Section 2, we introduce the notation and summarize the necessary kinematics for unconstrained transversely isotropic materials. In Section 3, the bending deformation is formulated and, under the appropriate specialization for particular directions of the axis of transverse isotropy, the governing differential equations are derived. As expected for the considered deformation, the expressions obtained have the same structure as for the case of compressible isotropic materials given in [13], except that material properties, expressed in terms of a strain-energy function, are different.

Specialization to *isochoric* bending is then discussed in Section 4, and attention is confined mainly to the case of plane strain. The remaining equilibrium equation identifies necessary and sufficient conditions on the energy function for the considered deformation to be sustainable, and, in particular, restrictions on the classical (linear) elastic constants are imposed. In this connection it is interesting to examine the status of so-called *reinforcing models*, for which an isotropic energy function is augmented by an added function that reflects the transverse isotropy as a basic model representing the influence of reinforcing fibres. The linear specialization of the strong ellipticity inequalities (see, for example, Payton [14] and Merodio and Ogden [12]) shows that the considered bending deformation cannot, in general, be achieved for such materials for realistic forms of the reinforcement model. Along the lines of the work of Jiang and Ogden [8, 9], some general forms of strain-energy functions that admit isochoric bending are derived. Some specific forms of these strain energies are then chosen to illustrate the results, and some closed-form solutions are obtained. Numerical calculations are used to demonstrate the stress distributions in the deformed block for two specific energy functions.

In Section 5 we examine aspects of the stability of the block for the considered deformation as embodied in the notion of strong ellipticity. For plane strain we provide necessary and sufficient conditions for strong ellipticity to hold.

Finally, Section 6 contains a brief discussion of the incompressible counterpart of the analysis presented here.

2 Background and notation

2.1 Kinematics

Let $\mathbf{X} = (X_1, X_2, X_3)$ denote the position vector of a material particle in some stress-free reference configuration $\mathcal{B}_r$ relative to a rectangular Cartesian basis $\{\mathbf{E}_i\}, i \in \{1, 2, 3\}$. Let $\mathcal{B}$ denote the deformed configuration of the body and $\mathbf{x}$ the corresponding position vector of the particle. The deformation gradient tensor, denoted $\mathbf{F}$, is given by

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial X_1} \otimes \mathbf{E}_1 + \frac{\partial \mathbf{x}}{\partial X_2} \otimes \mathbf{E}_2 + \frac{\partial \mathbf{x}}{\partial X_3} \otimes \mathbf{E}_3. \tag{1}$$

The right and left Cauchy-Green deformation tensors, denoted $\mathbf{C}$ and $\mathbf{B}$ respectively, are then defined as

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^2, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{V}^2, \tag{2}$$

wherein $\mathbf{U}$ and $\mathbf{V}$ are, respectively, the right and left stretch tensors that arise in the polar decompositions $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$, $\mathbf{R}$ being a rotation tensor.

For an unconstrained material the principal invariants I_1, I_2, I_3 of $\mathbf{C}$ (also of $\mathbf{B}$) are given by

$$I_1 = \operatorname{tr}\mathbf{C}, \quad I_2 = I_3 \operatorname{tr}\left(\mathbf{C}^{-1}\right), \quad I_3 = \det\mathbf{C} \equiv J^2, \tag{3}$$

where $J = \det\mathbf{F} > 0$, or, in terms of the principal stretches $\lambda_i > 0$, $i \in \{1, 2, 3\}$,

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{4}$$

In the reference configuration $\mathcal{B}_r$ we identify a preferred material direction, characterized by the unit vector $\mathbf{M}$. This generates two additional invariants, denoted I_4 and I_5 and defined by

$$I_4 = \mathbf{M} \cdot (\mathbf{C}\mathbf{M}) \equiv \mathbf{m} \cdot \mathbf{m}, \quad I_5 = \mathbf{M} \cdot (\mathbf{C}^2\mathbf{M}) \equiv \mathbf{m} \cdot (\mathbf{B}\mathbf{m}). \tag{5}$$

These may also be written in component form as

$$I_4 = m_1^2 + m_2^2 + m_3^2 = \lambda_1^2 M_1^2 + \lambda_2^2 M_2^2 + \lambda_3^2 M_3^2, \tag{6}$$
$$I_5 = \lambda_1^2 m_1^2 + \lambda_2^2 m_2^2 + \lambda_3^2 m_3^2 = \lambda_1^4 M_1^2 + \lambda_2^4 M_2^2 + \lambda_3^4 M_3^2, \tag{7}$$

where M_i, $i \in \{1, 2, 3\}$, are the components of $\mathbf{M}$ on the principal axes, $\{\mathbf{u}^{(i)}\}$ say, of $\mathbf{C}$ and m_i, $i \in \{1, 2, 3\}$, those of $\mathbf{m} = \mathbf{FM}$ on the principal axes of $\mathbf{B}$, which we denote by $\{\mathbf{v}^{(i)}\}$, $i \in \{1, 2, 3\}$. Note that, in general, $\mathbf{M}$ varies with position $\mathbf{X}$.

Finally, we emphasize the direct kinematical interpretation of I_4, namely that $\sqrt{I_4}$ represents the stretch in the direction $\mathbf{M}$. We also note that if $\mathbf{M}$ is not a principal direction of $\mathbf{C}$ then $I_5 - I_4^2$ provides a measure of the shearing deformation. For more details we refer to the work of Merodio and Ogden [10,11].

2.2 The strain-energy function and the stresses

We now consider the material to be transversely isotropic with preferred direction $\mathbf{M}$ (locally). This leads to a strain-energy function that depends on the invariants (3) and (5), and we adopt the notation

$$W = \bar{W}(I_1, I_2, I_3, I_4, I_5) \tag{8}$$

to represent this (defined per unit volume in $\mathcal{B}_r$).

The general forms of the nominal and Cauchy stress tensors, denoted $\mathbf{S}$ and $\boldsymbol{\sigma}$ respectively, are given by

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}, \quad J\boldsymbol{\sigma} = \mathbf{FS}, \tag{9}$$

expansion of which for the considered transverse isotropy yields

$$\begin{aligned} \mathbf{S} = 2\bar{W}_1\mathbf{F}^{\mathrm{T}} + 2\bar{W}_2(I_1\mathbf{I} - \mathbf{C})\mathbf{F}^{\mathrm{T}} + 2I_3\bar{W}_3\mathbf{F}^{-1} + 2\bar{W}_4\mathbf{M} \otimes \mathbf{FM} \\ + 2\bar{W}_5(\mathbf{M} \otimes \mathbf{FCM} + \mathbf{CM} \otimes \mathbf{FM}) \end{aligned} \tag{10}$$

and

$$\begin{aligned} J\boldsymbol{\sigma} = 2\bar{W}_1\mathbf{B} + 2\bar{W}_2(I_1\mathbf{I} - \mathbf{B})\mathbf{B} + 2I_3\bar{W}_3\mathbf{I} + 2\bar{W}_4\mathbf{m} \otimes \mathbf{m} \\ + 2\bar{W}_5(\mathbf{m} \otimes \mathbf{Bm} + \mathbf{Bm} \otimes \mathbf{m}), \end{aligned} \tag{11}$$

where $\mathbf{I}$ is the identity tensor and $\bar{W}_r = \frac{\partial \bar{W}}{\partial I_r}$ for $r \in \{1, 2, 3, 4, 5\}$. Note that $\boldsymbol{\sigma}$ is symmetric while, in general, $\mathbf{S}$ is not, but satisfies the connection $\mathbf{FS} = \mathbf{S}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}$ arising from the symmetry of $\boldsymbol{\sigma}$.

On specialization to the undeformed configuration $\mathcal{B}_r$, we have

$$I_1 = I_2 = 3, \quad I_3 = I_4 = I_5 = 1. \tag{12}$$

On noting that the tensors $\mathbf{I}$ and $\mathbf{M} \otimes \mathbf{M}$ are independent, and that the strain-energy function and stress should vanish in $\mathcal{B}_r$, we see that

$$\bar{W} = \bar{W}_1 + 2\bar{W}_2 + \bar{W}_3 = \bar{W}_4 + 2\bar{W}_5 = 0 \tag{13}$$

when evaluated for (12).

3 The bending deformation

3.1 Kinematics of the problem

Consider a rectangular block defined, in its reference configuration $\mathcal{B}_r$, by

$$-A \leq X_1 \leq A, \quad -B \leq X_2 \leq B, \quad -C \leq X_3 \leq C, \tag{14}$$

and suppose that the body is deformed so that the planes $X_1 =$ constant become sectors of the cylindrical surface $r =$ constant, planes $X_2 =$ constant become planes $\theta =$ constant and planes $X_3 =$ constant become planes $z =$ constant, where (r, θ, z) are cylindrical polar coordinates.

The equations describing the deformation may be written

$$r = f(X_1), \quad \theta = g(X_2), \quad z = \lambda X_3, \tag{15}$$

where λ is a constant and the functions f and g are to be determined. We assume that the deformation is symmetric about the X_1 axis, so that $g(-X_2) = -g(X_2)$, and, for definiteness, we take $f(A) > f(-A)$. For convenience we set the notations

$$f(-A) = a_1, \quad f(A) = a_2, \quad g(B) = \alpha, \tag{16}$$

so that $a_2 > a_1$.

If we let $\{\,\mathbf{e}_a\,\}, a \in \{\,r, \theta, z\,\}$ be the cylindrical polar basis vectors in the deformed configuration, then the position vector of a particle in this configuration is given by $\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z$, and the deformation gradient tensor (1) takes the form

$$\mathbf{F} = f'(X_1)\mathbf{e}_r \otimes \mathbf{E}_1 + f(X_1)g'(X_2)\mathbf{e}_\theta \otimes \mathbf{E}_2 + \lambda\mathbf{e}_z \otimes \mathbf{E}_3. \tag{17}$$

Equivalently, $\mathbf{F}$ can be decomposed as $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$, where

$$\mathbf{U} = f'\mathbf{E}_1 \otimes \mathbf{E}_1 + fg'\mathbf{E}_2 \otimes \mathbf{E}_2 + \lambda\mathbf{E}_3 \otimes \mathbf{E}_3, \tag{18}$$
$$\mathbf{V} = f'\mathbf{e}_r \otimes \mathbf{e}_r + fg'\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda\mathbf{e}_z \otimes \mathbf{e}_z, \tag{19}$$
$$\mathbf{R} = \mathbf{e}_r \otimes \mathbf{E}_1 + \mathbf{e}_\theta \otimes \mathbf{E}_2 + \mathbf{e}_z \otimes \mathbf{E}_3. \tag{20}$$

From equations (18)–(20) we deduce that the Lagrangian principal axes coincide with the Cartesian basis vectors $\{\,\mathbf{E}_i\,\}$, while the Eulerian principal axes are aligned with the cylindrical polar basis vectors $\{\,\mathbf{e}_i\,\}$. The associated principal stretches can therefore be identified as

$$\lambda_1 = f'(X_1), \quad \lambda_2 = f(X_1)g'(X_2), \quad \lambda_3 = \lambda. \tag{21}$$

3.2 Some restrictions on the constitutive law

For an *isotropic* elastic solid, as is well known, the Cauchy stress tensor $\boldsymbol{\sigma}$ is coaxial with $\mathbf{V}$. Therefore, all the non-zero components of $\boldsymbol{\sigma}$ can be expressed in terms of the principal stresses, denoted $\sigma_1, \sigma_2, \sigma_3$, with respect to the Eulerian principal axes. By contrast, for a transversely isotropic material $\boldsymbol{\sigma}$ is not in general coaxial with $\mathbf{V}$. However, from (11), we note that $\boldsymbol{\sigma}$ is coaxial with $\mathbf{V}$ if and only if $\mathbf{m}$ is an eigenvector of $\mathbf{B}$ or, equivalently, $\mathbf{M}$ is an eigenvector of $\mathbf{C}$. (This equivalence is easy to see from (2) and the connection $\mathbf{m} = \mathbf{FM}$.)

Thus, if $\mathbf{M}$ lies in the (X_1, X_2)-plane and is directed along the X_1 axis, equations (17), (21) and $(2)_1$ yield $\mathbf{CM} = \lambda_1^2\mathbf{M}$ for all $\lambda_1 > 0$. The invariants (5) then specialize to

$$I_4 = \lambda_1^2, \quad I_5 = I_4^2. \tag{22}$$

As a result, the Cauchy stress tensor can be written in the spectral form

$$\boldsymbol{\sigma} = \sigma_1 \mathbf{e}_r \otimes \mathbf{e}_r + \sigma_2 \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_3 \mathbf{e}_z \otimes \mathbf{e}_z, \tag{23}$$

with

$$\sigma_i = J^{-1}\lambda_i \frac{\partial W}{\partial \lambda_i}, \quad i \in \{1, 2, 3\}, \tag{24}$$

and now we may represent W as a function of $\lambda_1, \lambda_2, \lambda_3$. We write

$$W = \bar{W}(I_1, I_2, I_3, I_4, I_5) = \hat{W}(\lambda_1, \lambda_2, \lambda_3), \tag{25}$$

wherein the notation $\hat{W}$ is introduced. We emphasize that the representation $\hat{W}(\lambda_1, \lambda_2, \lambda_3)$, in contrast to $\bar{W}(I_1, I_2, I_3, I_4, I_5)$, applies for the specific deformation considered here and is not in general valid. Note that shear stresses are not required to achieve the considered deformation.

When $\mathbf{M}$ is chosen as above then for consistency with the classical linear theory of transversely isotropic elasticity the conditions

$$\bar{W}_{11} + 4\bar{W}_{12} + 4\bar{W}_{22} + 4\bar{W}_{23} + 2\bar{W}_{13} + \bar{W}_{33} = \frac{c_{22}}{4}, \tag{26}$$

$$\bar{W}_{14} + 2\bar{W}_{15} + 2\bar{W}_{24} + 4\bar{W}_{25} + \bar{W}_{34} + 2\bar{W}_{35} = \frac{c_{12} - c_{23}}{4}, \tag{27}$$

$$\bar{W}_{44} + 4\bar{W}_{55} + 4\bar{W}_{45} + 2\bar{W}_5 = \frac{c_{11} - c_{22} + 2c_{23} - 2c_{12}}{4}, \tag{28}$$

$$\bar{W}_1 + \bar{W}_2 + \bar{W}_5 = \frac{c_{55}}{2}, \tag{29}$$

$$\bar{W}_2 + \bar{W}_3 = \frac{c_{23} - c_{22}}{4}, \tag{30}$$

should be satisfied. Here, the derivatives of $\bar{W}$ are evaluated in the reference configuration and the constants $c_{11}, \ldots, c_{55}$ constitute the standard notation for the elastic constants used in the classical theory of transverse isotropy for the case in which $\mathbf{E}_1$ is the direction of transverse isotropy. We mention that the counterparts of (26)–(30) with $\mathbf{E}_3$ as the axis of transverse isotropy were given in [11].

Finally, we note that the inequalities

$$c_{11} > 0, \quad c_{22} > 0, \quad c_{55} > 0, \quad c_{22} > c_{23} \tag{31}$$

and

$$| \, c_{12} + c_{55} \, | < c_{55} + \sqrt{c_{11} c_{22}}, \tag{32}$$

are necessary and sufficient conditions for strong ellipticity to hold in the classical theory (see, for example, [12, 14]).

If we consider that the preferred direction is parallel to the X_2 axis then $\mathbf{CM} = \lambda_2^2 \mathbf{M}$ for all $\lambda_2 > 0$, while equations (22) are replaced by

$$I_4 = \lambda_2^2, \quad I_5 = I_4^2, \tag{33}$$

and appropriate changes are needed in the subscripts in (31) and (32) for this case.

3.3 Reduction of the equilibrium equations

For the considered deformation, the equilibrium equation $\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}$ (in the absence of body forces) yields the two scalar equations

$$\frac{\partial \sigma_1}{\partial r} + \frac{1}{r} (\sigma_1 - \sigma_2) = 0, \quad \frac{\partial \sigma_2}{\partial \theta} = 0. \tag{34}$$

Since λ_3 is a constant and λ_1 depends only on X_1 it follows from equations $(34)_2$ and (21) that

$$\frac{\partial \sigma_2}{\partial \lambda_2} g''(X_2) = 0. \tag{35}$$

Hence, assuming that $\frac{\partial \sigma_2}{\partial \lambda_2} \neq 0$, which, in view of the above inequalities, certainly holds in the reference configuration, we deduce that

$$g(X_2) = \beta X_2, \tag{36}$$

where β is a constant, which will be determined through the boundary conditions $(16)_3$ such that $\beta = \frac{\alpha}{B} > 0$. (Note that if, instead of the X_1 axis, the X_2 axis is chosen as the axis of transverse isotropy then g has the same form.)

Combination of (36) and (21) leads to

$$\lambda_2 = \beta f(X_1), \quad \frac{\mathrm{d}\lambda_2}{\mathrm{d}X_1} = \beta \lambda_1. \tag{37}$$

Hence, each of λ_1 and λ_2 depends only on X_1. Then, through use of $(21)_1$, (24), (37) and some manipulation, equation $(34)_1$ simplifies to

$$\frac{\mathrm{d}\hat{W}_1}{\mathrm{d}X_1} = \beta \hat{W}_2, \tag{38}$$

where we are using the representation $W = \hat{W}(\lambda_1, \lambda_2, \lambda_3)$ and the subscripts 1 and 2 on $\hat{W}$ signify differentiation with respect to λ_1 and λ_2, respectively. It follows, on use of $(37)_2$, that

$$\frac{\mathrm{d}}{\mathrm{d}X_1}(\lambda_1 \hat{W}_1) = \frac{\mathrm{d}}{\mathrm{d}X_1}(\hat{W}). \tag{39}$$

Since $\hat{W} = \hat{W}_1 = 0$ in the reference configuration and the result of integrating (39) must hold for all deformations of the considered form, we obtain

$$\hat{W} = \lambda_1 \hat{W}_1, \tag{40}$$

which is an (implicit) first-order differential equation for $f(X_1)$ for any given form of strain-energy function. This equation is the same as that arising for an isotropic material except that here $\hat{W}$ does not in general possess the symmetry in $(\lambda_1, \lambda_2, \lambda_3)$ that holds in the isotropic situation (see, for example, [7,13]).

As we have already mentioned, it was first shown by Rivlin [17,18] and then by several other authors (see, for example, [5,6,13]) that it is possible to hold the body in its current configuration even if there are no tractions on the curved surfaces $r = a_1, a_2$ of the deformed block. This requires $\sigma_1 = 0$ on $X_1 = \pm A$, which, because of (24) and (40), can be expressed in terms of the strain-energy function as

$$W = 0 \quad \text{on} \quad X_1 = \pm A, \tag{41}$$

where W is either $\bar{W}$ or $\hat{W}$, as appropriate.

4 Isochoric specialization

If the deformation is considered to be isochoric then $\lambda_1\lambda_2\lambda_3 = 1$ and from (21) we therefore have

$$f'(X_1) f(X_1) g'(X_2) \lambda_3 = 1. \tag{42}$$

As discussed by Rivlin [17,18], solution of the preceding equation leads to

$$f(X_1)^2 = \frac{2X_1}{\beta\lambda_3} + a, \quad g(X_2) = \beta X_2,$$

where the constant $\beta > 0$ is again given via $(16)_3$ and

$$a = \frac{a_1^2 + a_2^2}{2}, \quad a_2^2 = \frac{4A}{\beta\lambda_3} + a_1^2. \tag{43}$$

It then follows that the deformation can be described by

$$r = \sqrt{a + \frac{2X_1}{\beta\lambda_3}}, \quad \theta = \beta X_2, \quad z = \lambda_3 X_3, \tag{44}$$

and we deduce that the principal stretches can be written as

$$\lambda_1 = \frac{1}{\beta\lambda_3 r}, \quad \lambda_2 = \beta r, \quad \lambda_3 = \lambda. \tag{45}$$

Application of (42) and (45) to (24) shows that σ_1 and σ_2 depend only on X_1 while θ depends only on X_2. Therefore, for an isochoric deformation, $(34)_2$ is satisfied identically while $(34)_1$ again leads to the equation $\hat{W} = \lambda_1 \hat{W}_1$. We emphasize that we are considering here an isochoric deformation in a compressible material, not an incompressible material.

Furthermore, we note that the solutions (44) arising from the kinematical restriction (42) are universal solutions since they apply independently of the constitutive law. Thus, in this case, the radial equilibrium equation (40) serves to identify the possible forms of W that admit the isochoric bending deformation.

4.1 Plane strain specialization

Henceforth, we confine our analysis to the plane strain specialization. We consider that the deformation is in the (X_1, X_2) coordinate plane, such that $z = X_3$ with (r, θ) being independent of X_3, and the direction $\mathbf{M}$ is parallel to the considered plane. The components of $\mathbf{F}$ and $\mathbf{C}$ satisfy $F_{33} = C_{33} = 1$, and the out-of-plane principal stretch is now $\lambda_3 = 1$. The principal invariants (4) reduce to

$$I_1 = \lambda_1^2 + \lambda_2^2 + 1, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^2 + \lambda_2^2, \quad I_3 = \lambda_1^2\lambda_2^2, \tag{46}$$

and we rewrite (22) and (33) together compactly as

$$I_4 = \lambda_\kappa^2, \quad I_5 = I_4^2, \quad \kappa \in \{1, 2\}, \tag{47}$$

wherein the subscript κ has been introduced to identify the orientation of the unit vector field $\mathbf{M}$ in the undeformed configuration.

For either $\kappa = 1$ or 2 we deduce from (46) and (47) the connections

$$I_2 = I_1 + I_3 - 1, \quad I_5 = (I_1 - 1)I_4 - I_3, \tag{48}$$

and it follows that, for plane strain, the function $\bar{W}$ introduced in (8) depends only on the invariants I_1, I_3 and I_4. Accordingly, we may introduce a reduced strain-energy function, denoted $\bar{\bar{W}}$ and defined by

$$\bar{\bar{W}}(I_1, I_3, I_4) = \bar{W}(I_1, I_1 + I_3 - 1, I_3, I_4, (I_1 - 1)I_4 - I_3), \tag{49}$$

for either value of κ.

Now, if $\mathbf{F}$ denotes the corresponding in-plane deformation gradient, the associated restricted expressions for the nominal and the Cauchy stress tensors are given by

$$\mathbf{S} = 2\bar{\bar{W}}_1\mathbf{F}^{\mathrm{T}} + 2I_3\bar{\bar{W}}_3\mathbf{F}^{-1} + 2\bar{\bar{W}}_4\mathbf{M} \otimes \mathbf{FM}, \tag{50}$$

$$J\boldsymbol{\sigma} = 2\bar{\bar{W}}_1\mathbf{B} + 2I_3\bar{\bar{W}}_3\mathbf{I} + 2\bar{\bar{W}}_4\mathbf{m} \otimes \mathbf{m}. \tag{51}$$

Note that in order to maintain the plane strain deformation the out-of-plane stress components S_{33} and σ_{33} are in general non-zero. These are not given by (50), (51), but, if needed, they may be calculated from (10) and (11), respectively, evaluated for the considered plane strain specialization.

The counterparts of (13) for $\bar{\bar{W}}$ are

$$\bar{\bar{W}} = \bar{\bar{W}}_1 + \bar{\bar{W}}_3 = \bar{\bar{W}}_4 = 0 \tag{52}$$

for $\kappa = 1, 2$, while (26)–(30) specialize to

$$\bar{\bar{W}}_{11} + 2\bar{\bar{W}}_{13} + \bar{\bar{W}}_{33} = \frac{c_{22}}{4}, \quad \bar{\bar{W}}_{44} - 2\bar{\bar{W}}_3 = \frac{c_{11} + c_{22} - 2c_{12}}{4}, \tag{53}$$

$$2\bar{\bar{W}}_{14} + 2\bar{\bar{W}}_{34} + \bar{\bar{W}}_{44} = \frac{c_{11} - c_{22}}{4}, \quad \bar{\bar{W}}_3 = -\frac{c_{55}}{2}, \tag{54}$$

with the derivatives of $\bar{\bar{W}}$ being evaluated for $(I_1, I_3, I_4) = (3, 1, 1)$ (see also [11]). Finally, the properties (31) that the elastic constants should satisfy are reduced to

$$c_{11} > 0, \quad c_{22} > 0, \quad c_{55} > 0, \tag{55}$$

while (32) is still in place. We recall that the properties (53)–(55) and (32) correspond to $\kappa = 1$.

4.2 Certain classes of materials

As discussed in Jiang [7], a finite isochoric bending deformation of a rectangular block is not sustainable for compressible isotropic materials. Here, we present two specific classes of transversely isotropic compressible materials, depending on the choice of κ, for which the considered isochoric bending deformation can be achieved.

For this purpose, we substitute (49) into (40) to obtain

$$\bar{\bar{W}}\left(I_1, I_3, I_4\right) = 2\lambda_1^2 \bar{\bar{W}}_1\left(I_1, I_3, I_4\right) + 2I_3 \bar{\bar{W}}_3\left(I_1, I_3, I_4\right) + 2\lambda_1^2 \bar{\bar{W}}_4\left(I_1, I_3, I_4\right)\delta_{1\kappa}, \tag{56}$$

where $\delta_{1\kappa}$ is the Kronecker delta, with $\kappa \in \{1, 2\}$.

Since the deformation is considered to be isochoric, equation (56) for $\kappa = 1$ applies for

$$I_1 = \lambda_1^2 + \lambda_1^{-2} + 1, \quad I_3 = 1, \quad I_4 = \lambda_1^2 \tag{57}$$

for all $\lambda_1 > 0$. Differentiation of (56) with respect to λ_1 yields

$$\begin{aligned} 2\left(\lambda_1 - \lambda_1^{-3}\right)&\left(2\lambda_1^2 \bar{\bar{W}}_{11} + 2\bar{\bar{W}}_{13} + 2\lambda_1^2 \bar{\bar{W}}_{14} - \bar{\bar{W}}_1\right) \\ &+ 4\lambda_1^3\left(\bar{\bar{W}}_{14} + \bar{\bar{W}}_{44}\right) + 2\lambda_1\left(2\bar{\bar{W}}_1 + \bar{\bar{W}}_4 + 2\bar{\bar{W}}_{34}\right) = 0, \end{aligned} \tag{58}$$

for all $\lambda_1 > 0$.

On use of (52)–(54), we see that in the limit $\lambda_1 \to 1$ equation (58) holds if and only if

$$c_{11} = c_{12}. \tag{59}$$

This is a necessary restriction on the class of compressible transversely isotropic materials for which an isochoric bending deformation of a rectangular block can be achieved. We note that for this class of materials the conditions (55) and (32) can be reduced to

$$c_{22} > c_{11} > 0, \quad c_{55} > 0. \tag{60}$$

For $\kappa = 2$, equation (56) with $I_4 = \lambda_1^{-2}$ gives, analogously to (58), the connection

$$\begin{aligned} 2\lambda_1^{-3}\left(\bar{\bar{W}}_4 - 2\bar{\bar{W}}_{34}\right) &+ 4\lambda_1\left(\bar{\bar{W}}_1 - \lambda_1^{-2}\bar{\bar{W}}_{14}\right) \\ &+ 2\left(\lambda_1 - \lambda_1^{-3}\right)\left(2\lambda_1^2 \bar{\bar{W}}_{11} + 2\bar{\bar{W}}_{13} - \bar{\bar{W}}_1\right) = 0, \end{aligned}$$

for all $\lambda_1 > 0$. This leads again to the restriction (59), and the corresponding inequalities that the elastic constants satisfy are

$$c_{22} > c_{11} > 0, \quad c_{44} > 0. \tag{61}$$

It is also worth noting that the inequalities $(60)_1$ show that for $\kappa = 1$ the materials have “low” anisotropy, since the elastic modulus c_{11}, which is directly related to the stiffness in the X_1 direction, is less than c_{22}, which is defined with respect to the isotropic planes of the body.

4.2.1 A note on reinforcing models

Without referring to any details, we remark that several authors (see, for example, [10,11,15,16,19]) have considered a decomposition of the strain-energy function of the form

$$\bar{\bar{W}} = \bar{\bar{W}}_{\mathrm{iso}}(I_1, I_3) + \bar{\bar{W}}_{\mathrm{fib}}(I_4), \tag{62}$$

in which the first term $\bar{\bar{W}}_{\mathrm{iso}}$ represents the isotropic base material, while the additional term $\bar{\bar{W}}_{\mathrm{fib}}$ represents the reinforcement associated with a family of fibres whose referential direction is the preferred direction $\mathbf{M}$.

The properties $(53)_2$ and (54) show that the class of strain-energy functions (62) must satisfy the connection

$$c_{22} - c_{12} = 2c_{55} \tag{63}$$

when $\kappa = 1$, and

$$c_{11} - c_{12} = 2c_{44} \tag{64}$$

for $\kappa = 2$. The first of these requirements is consistent with (59) and the inequalities (60), but the second is not consistent with (59) and (61). However, the conditions $(53)_2$, (54) and $(60)_3$ show that the class of materials (63) can admit the isochoric bending deformation only for $\kappa = 1$ and then such that

$$\bar{\bar{W}}''_{\mathrm{fib}}(1) < 0, \tag{65}$$

implying non-convexity of $\bar{\bar{W}}_{\mathrm{fib}}$ in a neighbourhood of $I_4 = 1$.

This is not consistent with the models adopted in the above-cited papers, where the anisotropic part of the strain-energy has been taken to satisfy

$$\bar{\bar{W}}_{\mathrm{fib}}(1) = \bar{\bar{W}}'_{\mathrm{fib}}(1) = 0, \quad \bar{\bar{W}}''_{\mathrm{fib}}(1) > 0, \tag{66}$$

for $\kappa \in \{1, 2\}$. However, in such cases the models used provide reinforcement, i.e. the stiffness in the preferred direction is larger than in the transverse directions, in contrast to the situation here. It can therefore be concluded that the considered deformation is not possible for strictly reinforcing models.

4.3 Some specific strain-energy functions

In this section we present two examples of strain-energy functions that can admit an isochoric bending deformation. The relevant necessary and sufficient condition is obtained from (56) in each case, with I_1, I_3 and I_4 given by

$$I_1 = \lambda_1^2 + \lambda_1^{-2} + 1, \quad I_3 = 1, \quad I_4 = \lambda_\kappa^2. \tag{67}$$

Since $I_3 = 1$, we may write $I_1 = I_4 + I_4^{-1} + 1$, for $\kappa \in \{1, 2\}$, although in the present analysis we do not make formal use of this connection. The specializations of (56) for $\kappa = 1$ and $\kappa = 2$ may be written as

$$\bar{\bar{W}} = 2I_4\bar{\bar{W}}_1 + 2\bar{\bar{W}}_3 + 2I_4\bar{\bar{W}}_4 \tag{68}$$

and

$$I_4\bar{\bar{W}} = 2\bar{\bar{W}}_1 + 2I_4\bar{\bar{W}}_3, \tag{69}$$

respectively, evaluated for $I_3 = 1$.

In the following we adopt an approach towards the construction of forms of W used by Jiang and Ogden [8, 9], and the particular forms of energy function considered are motivated by those examined in these references.

Case (1): $\kappa = 1$.

First, we consider the class of strain-energy functions for which $\bar{\bar{W}}$ has the form

$$\begin{aligned}\bar{\bar{W}}(I_1, I_3, I_4) = h_1(I_1 - I_4 + I_3)\, g_0(I_3) \\ + h_2(I_1 - I_4)\sqrt{I_3} + C_0 I_4\left(\sqrt{I_3}\right)^{-1},\end{aligned} \tag{70}$$

where h_1 is a function to be determined, C_0 is a material constant and the functions h_2 and g_0 are to be consistent with the requirements (52)–(54).

If, without loss of generality, we set $g_0(1) = 1$ then substitution of (70) into (68) leads to the differential equation

$$h_1'(\bar{I}) + q h_1(\bar{I}) = 0, \tag{71}$$

where q and $\bar{I}$ are defined by

$$2q = 2g_0'(1) - 1, \quad \bar{I} = I_1 - I_4 + 1. \tag{72}$$

The general solution of (71) is

$$h_1(\bar{I}) = C_1 \mathrm{e}^{-q\bar{I}}, \tag{73}$$

where C_1 is a constant. In respect of (70) the requirements (52)–(54) give $C_0 = \frac{c_{55}}{2}$, together with

$$h_2(2) + h_1(3) = -\frac{c_{55}}{2}, \quad h_2'(2) - q h_1(3) = \frac{c_{55}}{2}, \tag{74}$$

$$h_2''(2) + q^2 h_1(3) = \frac{c_{22} - c_{11} - 4c_{55}}{4}, \tag{75}$$

$$4h_2''(2) - h_2(2) + 4h_2'(2) - 8q h_1(3) + 4h_1(3) g_0''(1) = c_{22} - \frac{3c_{55}}{2}. \tag{76}$$

The class of strain-energy functions (70) admitting isochoric bending deformation is now specialized to

$$\bar{\bar{W}}\left(I_1, I_3, I_4\right) = C_1 \mathrm{e}^{-q(I_1 - I_4 + I_3)} g_0\left(I_3\right) \\ + h_2\left(I_1 - I_4\right)\sqrt{I_3} + \frac{c_{55}}{2} I_4 \left(\sqrt{I_3}\right)^{-1} \quad (77)$$

for any choice of the parameter q and non-zero C_1, and for any functions g_0 and h_2 that satisfy (74)–(76). In particular, for any given C_1 and q, which can in general be chosen independently, equations (74)–(76) simply serve to identify the properties that g_0 and h_2 should satisfy in the reference configuration, but no restriction otherwise on the forms of these functions is imposed. The expression (77), together with (74)–(76), represents a large class of functions admitting isochoric bending deformation. Finally, we note that for $C_1 = 0$ the requirements (76) and (75) lead to $c_{11} = -4c_{55}$, in which case the properties (60) are violated. For this reason the possibility $C_1 = 0$ is excluded from consideration.

Note, however, that the specialization $q = 0$ is admissible, in which case the strain-energy function (77) reduces to

$$\bar{\bar{W}}\left(I_1, I_3, I_4\right) = C_1 g_0\left(I_3\right) + h_2\left(I_1 - I_4\right)\sqrt{I_3} + \frac{c_{55}}{2} I_4 \left(\sqrt{I_3}\right)^{-1}, \quad (78)$$

which is valid for all non-zero disposable parameters C_1 and all functions h_2 and g_0 that satisfy the appropriate specializations of (74)–(76).

Case (2): $\kappa = 2$.

Next, we examine the class of strain-energy functions of the form

$$\bar{\bar{W}}(I_1, I_3, I_4) = h_3(I_1 I_4) g_1(I_3) + h_4(I_4)\sqrt{I_3}. \quad (79)$$

Similarly to the previous case, by taking $g_1(1) = 1$ and setting $2p = 2g_1'(1) - 1$, we follow the same procedure in respect of (69) to particularize $h_3(I_1 I_4)$. This leads to

$$h_3(I_1 I_4) = C_2 \mathrm{e}^{-p I_1 I_4}, \quad (80)$$

where again, C_2 is a material parameter and h_3, h_4, g_1 satisfy

$$h_3(3) = -\frac{c_{44}}{2p}, \quad h_3(3) + h_4(1) = 0, \quad h_4'(1) = -\frac{3}{2} c_{44}, \quad (81)$$

$$h_4''(1) + 9p^2 h_3(3) = \frac{c_{22} - c_{11} - 4c_{44}}{4}, \quad (82)$$

$$4p(p+1) h_3(3) - 4h_3(3) g_1''(1) = -c_{11} - \frac{c_{44}}{2p}. \quad (83)$$

Hence, in this case, equation (79) is replaced by

$$\bar{\bar{W}}(I_1, I_3, I_4) = C_2 \mathrm{e}^{-pI_1I_4} g_1(I_3) + h_4(I_4)\sqrt{I_3} \tag{84}$$

for all non-zero parameters p and C_2 that satisfy (81)–(83) in respect of (80). As for the case of the functions (77) these serve to determine the conditions that g_1 and h_4 should satisfy in the reference configuration. In addition, we emphasize that the class of strain-energy functions (79) fails to admit isochoric bending deformation for $C_2 = 0$ and/or $p = 0$ since in either case we deduce that $c_{44} = 0$ and the strong ellipticity condition $(61)_3$ is then violated.

4.4 Application of the boundary conditions

As we have already mentioned, the solutions (77), (78), (84) derived in the previous section correspond to large classes of transversely isotropic materials admitting the considered isochoric bending deformation under plane strain. However, these solutions are not necessarily compatible with the boundary conditions (41) imposed on our problem. In this respect, the arbitrary functions h_2 and h_4 involved need to be properly specified to ensure that the deformed body is traction free on the boundaries $X_1 = \pm A$.

For illustration, we now examine the strain-energy functions (78), by taking

$$h_2(I_1 - I_4) = -\frac{c_{55}}{8}\left(I_1 - I_4 - 4\right)^2 - C_1, \tag{85}$$

noting that this is compatible with (74) and (75) for $q = 0$ and the nonlinear algebraic system (41) can be solved analytically in respect of the data a and β. The equilibrium equation (70) is then satisfied identically with

$$\sigma_1 = -\frac{c_{55}}{8}\left[\left(\lambda_1^{-2} - 2\right)^2 - 2\left(\lambda_1^{-2} + 2\lambda_1^2\right) + 5\right], \tag{86}$$

while σ_2 and σ_3 take the forms

$$\begin{aligned} \sigma_2 &= -\frac{c_{55}}{8}\left(4\lambda_1^2 - 18\lambda_1^{-2} + 5\lambda_1^{-4} + 9\right), \\ \sigma_3 &= -\frac{c_{55}}{8}\left(4\lambda_1^2 - 2\lambda_1^{-2} + \lambda_1^{-4} - 3\right). \end{aligned} \tag{87}$$

It should be noted, however, that the properties (74) and (75) impose the further restriction $3c_{55} = c_{22} - c_{11}$, which is compatible with (60). From (76) a condition on $g_0''(1)$ may also be derived but we do not need it here.

By recalling the expressions $(44)_1$ and $(45)_1$ (now with $\lambda_3 = 1$), the system (41) is solved to give

$$a = \frac{40}{9}A^2, \quad \beta = \frac{3}{4A} \tag{88}$$

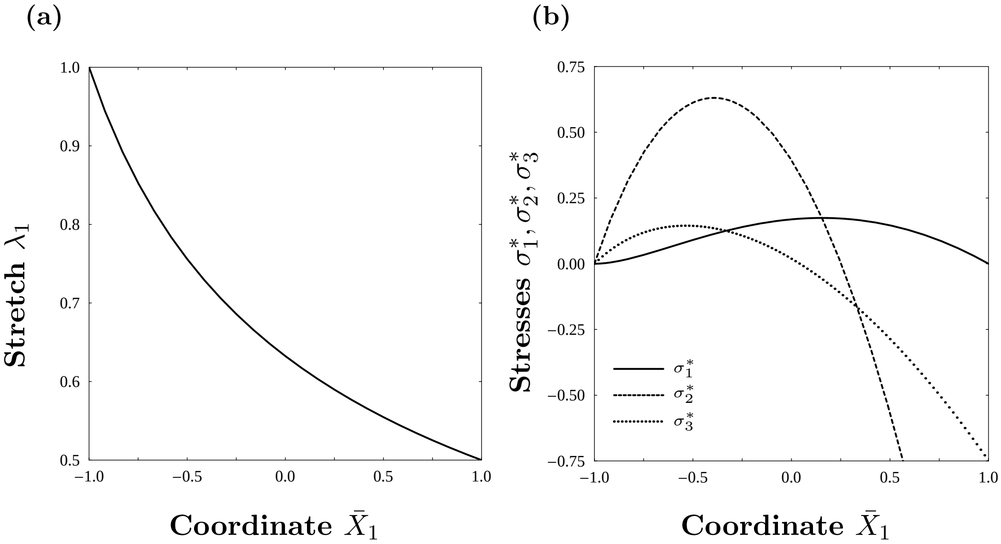

Figure 1. Plots of (a) the stretch λ_1 and (b) the dimensionless stress components σ_1^*, σ_2^* and σ_3^* vs $\bar{X}_1$.

and from $(16)_{1,2}$ we obtain

$$a_1 = 2a_2 = \frac{8}{3}A. \tag{89}$$

Moreover, the range of $\lambda_1 = \lambda_1(X_1)$ for which such a deformation is sustainable may also be identified via (88), $(44)_1$ and $(45)_1$ as

$$\lambda_1(A) = 0.5 \leq \lambda_1(X_1) \leq 1 = \lambda_1(-A), \tag{90}$$

for all $A > 0$ and $-A \leq X_1 \leq A$. Consequently, from $(47)_1$ (i.e. for $\kappa = 1$) and (90) it follows that the material is compressed in the X_1 direction for $X_1 > -A$. The resulting stretch distribution as a function of the dimensionless coordinate $\bar{X}_1 = X_1/A$ is depicted in Figure 1(a). We observe that the inequalities (90) hold independently of the value of A.

In addition, the stress components σ_1, σ_2 and σ_3 are plotted in Figure 1(b) as functions of $\bar{X}$ in dimensionless form $\sigma_i^* = \frac{\sigma_i}{c_{55}}$, $i \in \{1, 2, 3\}$. The non-monotonic nature of $\sigma_1, \sigma_2, \sigma_3$ is now evident. We note that σ_1 vanishes for $\bar{X}_1 = \pm 1$, as prescribed, and takes its maximum value for $\lambda_1 \approx 0.605$ (equivalently, for $\bar{X}_1 \approx 0.155$). Also, σ_2 and σ_3 vanish on the boundary $\bar{X}_1 = -1$ of the block, where $\lambda_1 = 1$, while additionally $\sigma_2 = 0$ for $\lambda_1 \approx 0.589$ ($\bar{X}_1 \approx 0.252$) and $\sigma_3 = 0$ at $\lambda_1 \approx 0.625$ ($\bar{X}_1 \approx 0.041$).

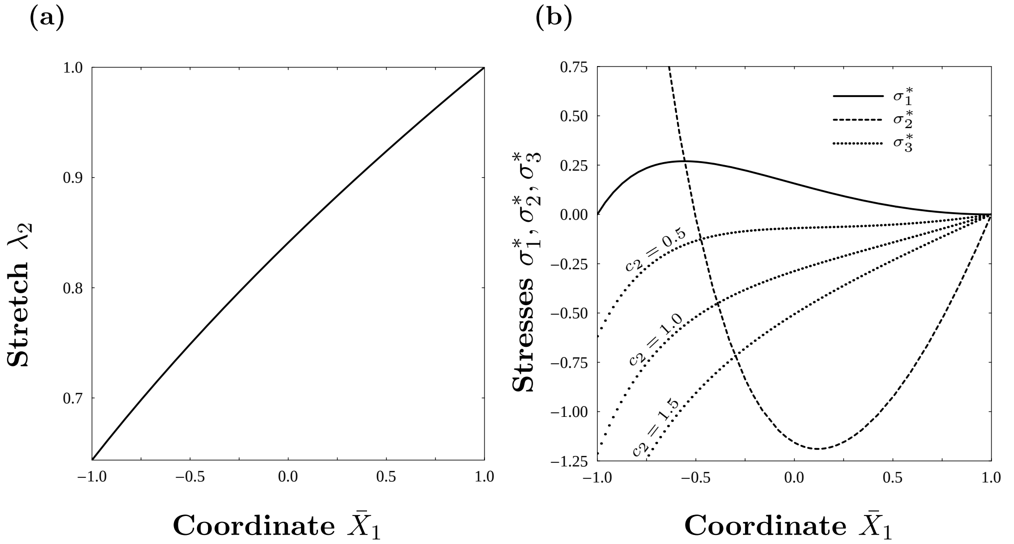

Figure 2. Plots of (a) the stretch λ_2 and (b) the dimensionless stress components σ_1^*, σ_2^* and σ_3^* vs $\bar{X}_1$.

Finally, the moment of the stress σ_2 (about the origin $r = \theta = 0$) that maintains the material in its deformed state is now calculated from the formula

$$M = 2C \int_{-A}^{A} r\lambda_1\sigma_2 dX_1, \tag{91}$$

which can be evaluated explicitly: $M \approx -(0.318/\beta^2)Cc_{55} \approx -0.566A^2Cc_{55}$.

Next we consider the class of materials (84) for the case in which the function h_4 is chosen as

$$h_4(I_4) = \frac{c_{44}}{2p} \mathrm{e}^{p(2-I_4-I_4^2)} - c_1 \left(I_4^{-1} - 2\right)^2 + c_1 I_4^2. \tag{92}$$

The form of (92) satisfies the required restrictions, and the counterparts of (86) and (87) are

$$\sigma_1 = -c_1 \left[(\lambda_1^2 - 2)^2 - \lambda_1^{-4}\right], \tag{93}$$

and

$$\sigma_2 = c_1 \left(3\lambda_1^4 - 4\lambda_1^2 + 5\lambda_1^{-4} - 4\right), \tag{94}$$

$$\sigma_3 = c_{44}\mathrm{e}^{p\left(2-\lambda_1^{-2}-\lambda_1^{-4}\right)} \left(\lambda_1^{-2} - 1\right) - c_1 \left(\lambda_1^4 - 4\lambda_1^2 - \lambda_1^{-4} + 4\right), \tag{95}$$

wherein the notation $c_1 = \frac{c_{22}-c_{11}+3c_{44}}{16}$ has been introduced.

On use of (93), solution of the system (41) yields

$$a = 4\left(4+3\sqrt{2}\right)A^2, \quad \beta = \frac{\left(2-\sqrt{2}\right)}{4A}, \tag{96}$$

with

$$a_1 = \sqrt{2+2\sqrt{2}a_2} - 2\sqrt{2}A = 2\left(2+\sqrt{2}\right)A. \tag{97}$$

We observe that for the particular choice of h_3 the deformation is sustainable only within the range

$$\lambda_1(A) = 1 \leq \lambda_1(X_1) \leq 1.554 \approx \lambda_1(-A), \tag{98}$$

for all $A > 0$ and $-A \leq X_1 \leq A$. We recall, however, that since in this case the direction of transverse isotropy is in the X_2 direction, we have $I_4 = \lambda_2^2 = \lambda_1^{-2}$. The obvious inference is that there is contraction in the X_2 direction for all values of X_1 except $X_1 = A$. The distribution of the stretch $\lambda_2 = \lambda_1^{-1}$ as a function of $\bar{X}_1$ is plotted in Figure 2(a).

Corresponding plots of the stress components $\sigma_1, \sigma_2, \sigma_3$ are given in Figure 2(b) where, analogously to the previous case, we use the dimensionless forms $\sigma_i^* = \frac{\sigma_i}{c_1}$, $i \in \{1,2,3\}$. It can now easily be derived from (93) and (94) that σ_1 and σ_2 are non-monotonic as functions of λ_2 or, equivalently, of X_1. Indeed, σ_1 reaches a maximum value at $\lambda_2 \approx 0.737$, corresponding to $\bar{X}_1 \approx -0.558$ while, σ_2 has a minimum at $\lambda_2 \approx 0.861$, corresponding to $\bar{X}_1 \approx 0.119$. Furthermore, we notice that σ_2 vanishes for the values $\lambda_2 \approx 0.748$ and 1 and hence for $\bar{X}_1 \approx -0.504$ and $\bar{X}_1 = 1$.

We now observe that σ_3 is the only principal stress component that depends on the three parameters c_{44}, p and c_1. Essentially, both the nature and the magnitude of this component are adjusting due to different classes of strain energies and with respect to various extension and shear moduli so that the body can undergo an isochoric deformation while, at the same time, the boundary conditions (41) are satisfied. For illustration, the curves $(\sigma_3^*, \bar{X}_1)$ are presented here for $p = 0.5$ and $c_2 = 0.5, 1, 1.5$, where c_2 is defined as $c_2 = \frac{c_{44}}{c_1}$.

Finally, the moment M is in this case calculated as

$$M = 2C\int_{-A}^{A} r\lambda_1\sigma_2 dX_1 \approx -\frac{0.078}{\beta^2}Cc_1 \approx -3.627A^2Cc_1. \tag{99}$$

5 Strongly elliptic modes of deformation

The issue of stability of modes of deformation such as that considered in the foregoing sections is an important one, and, in particular, the notion of loss of strong ellipticity has a role to play in this regard. In this section we examine the strong ellipticity condition for the considered deformation. For transversely isotropic compressible elastic solids this has been discussed by Merodio and Ogden [11] and, in particular, they gave a general expression for the strong ellipticity condition for plane strain. In general, the strong ellipticity condition may be analyzed in terms of the acoustic tensor $\mathbf{Q}(\mathbf{n})$, whose components are quadratic in the components of the unit vector $\mathbf{n}$ (for a general discussion see, for example, Truesdell and Noll [20]). In two dimensions, in the $(1, 2)$ plane with $\mathbf{n}$ lying in that plane, necessary and sufficient conditions for strong ellipticity are

$$Q_{11}(\mathbf{n}) > 0, \quad Q_{11}(\mathbf{n})Q_{22}(\mathbf{n}) - [Q_{12}(\mathbf{n})]^2 > 0 \tag{100}$$

for all unit vectors $\mathbf{n} = (n_1, n_2, 0)$.

For a compressible material the components of $\mathbf{Q}(\mathbf{n})$ for plane strain ($\lambda_3 = 1$) are, from Merodio and Ogden [11] but in the present notation, given by

$$\begin{aligned} Q_{ij} = {} & 4\bar{\bar{W}}_{11}\lambda_i^2\lambda_j^2 n_i n_j + 4I_3\bar{\bar{W}}_{13}\left(\lambda_i^2 + \lambda_j^2\right) n_i n_j + 4I_3^2\bar{\bar{W}}_{33} n_i n_j \\ & + 4I_3\bar{\bar{W}}_{34}\left(\mathbf{n}\cdot\mathbf{m}\right)\left(n_i m_j + n_j m_i\right) + 4\bar{\bar{W}}_{14}\left(\mathbf{n}\cdot\mathbf{m}\right)\left(\lambda_i^2 n_i m_j + \lambda_j^2 n_j m_i\right) \\ & + 4\bar{\bar{W}}_{44}\left(\mathbf{n}\cdot\mathbf{m}\right)^2 m_i m_j + 2\bar{\bar{W}}_1\delta_{ij}\left(\lambda_1^2 n_1^2 + \lambda_2^2 n_2^2\right) \\ & + 2I_3\bar{\bar{W}}_3 n_i n_j + 2\bar{\bar{W}}_4\delta_{ij}\left(\mathbf{n}\cdot\mathbf{m}\right)^2, \end{aligned} \tag{101}$$

for $i, j \in \{1, 2\}$. When specialized to the considered deformation and on use of the (plane strain) energy function defined by $\hat{\hat{W}}(\lambda_1, \lambda_2) = \hat{W}(\lambda_1, \lambda_2, 1)$, we obtain simply

$$Q_{11} = \lambda_1^2\hat{\hat{W}}_{11}n_1^2 + 2\bar{\bar{W}}_1\lambda_2^2 n_2^2, \tag{102}$$

$$Q_{22} = \lambda_2^2\hat{\hat{W}}_{22}n_2^2 + 2\left(\bar{\bar{W}}_1 + \bar{\bar{W}}_4\right)\lambda_1^2 n_1^2, \tag{103}$$

$$Q_{12} = \lambda_1\lambda_2\hat{\hat{W}}_{12}n_1 n_2 - 2I_3\bar{\bar{W}}_3 n_1 n_2, \tag{104}$$

where $\hat{\hat{W}}_{ij} = \partial^2\hat{\hat{W}}/\partial\lambda_i\partial\lambda_j$.

After a little manipulation using (104) it can be shown that the inequalities (100) lead to

$$\hat{\hat{W}}_{11} > 0, \quad \hat{\hat{W}}_{22} > 0, \quad \bar{\bar{W}}_1 > 0, \quad \bar{\bar{W}}_1 + \bar{\bar{W}}_4 > 0, \tag{105}$$

$$\sqrt{\hat{\hat{W}}_{11}\hat{\hat{W}}_{22}} - \hat{\hat{W}}_{12} + 2\sqrt{\bar{\bar{W}}_1\left(\bar{\bar{W}}_1 + \bar{\bar{W}}_4\right)} + 2\sqrt{I_3}\bar{\bar{W}}_3 > 0, \tag{106}$$

$$\sqrt{\hat{\hat{W}}_{11}\hat{\hat{W}}_{22}} + \hat{\hat{W}}_{12} + 2\sqrt{\bar{\bar{W}}_1\left(\bar{\bar{W}}_1 + \bar{\bar{W}}_4\right)} - 2\sqrt{I_3}\bar{\bar{W}}_3 > 0, \tag{107}$$

which, jointly, are necessary and sufficient conditions on the material properties for strong ellipticity to hold for the considered deformation. Note that both $\hat{\hat{W}}$ and $\bar{\bar{W}}$ are used here since the expressions are simpler in this form.

It is worth noting in passing that for an isotropic material the above inequalities, when expressed entirely in terms of $\hat{\hat{W}}$, reduce to

$$\hat{\hat{W}}_{11} > 0, \quad \hat{\hat{W}}_{22} > 0, \quad \frac{\lambda_1 \hat{\hat{W}}_1 - \lambda_2 \hat{\hat{W}}_2}{\lambda_1^2 - \lambda_2^2} > 0, \tag{108}$$

$$\sqrt{\hat{\hat{W}}_{11}\hat{\hat{W}}_{22}} - \hat{\hat{W}}_{12} + \frac{\hat{\hat{W}}_1 + \hat{\hat{W}}_2}{\lambda_1 + \lambda_2} > 0, \tag{109}$$

$$\sqrt{\hat{\hat{W}}_{11}\hat{\hat{W}}_{22}} + \hat{\hat{W}}_{12} - \frac{\hat{\hat{W}}_1 + \hat{\hat{W}}_2}{\lambda_1 + \lambda_2} > 0, \tag{110}$$

as obtained by [4].

For illustration, the ellipticity status of the strain-energy function (78) undergoing isochoric bending, with h_2 being given by (85), is now discussed. For the considered materials, we deduce via $(60)_1$ that the first and the fourth requirements (105) are automatically satisfied within the range of admissible values of λ_1 as defined in (90). On the other hand, the second of these inequalities fails if and only if the dimensionless quantity $c_3 = \frac{c_{11}}{c_{55}} > 0$ does not exceed the approximate value 21.75. In this connection, the inequality $(105)_2$ fails earlier (for values of λ_1 closer to 1) when c_3 is close to zero, corresponding to $\lambda_1 \approx 0.724$ ($\bar{X}_1 \approx -0.394$). Note that an increase in the ratio c_3 amounts to a decrease in the value of λ_1 for which $(105)_2$ first fails. In the same spirit, it can easily be shown that breakdown of $(105)_3$ occurs when λ_1 reaches the value $\frac{\sqrt{3}}{3}$, independently of the magnitude of the associated elastic material parameters.

It is now interesting that the status of (106) depends on c_3 in a similar way as for $(105)_2$. Once more, small values of c_3 correspond to larger values of λ_1 for which (106) is violated. We emphasize, however, that if c_3 is taken close to zero, (106) fails instantly for λ_1 close to 1 ($\bar{X} \approx -1$) while also the restriction $c_3 \geq 21.75$ is not in this case influential. We further observe that for any fixed value of c_3, violation of (106) occurs for values of λ_1 closer to 1 than for those associated with the failure of $(105)_2$ or $(105)_3$. Finally, bearing in mind (90) we readily deduce that (107) always holds.

Therefore, for a deformation with the considered properties, the inequality (106) alone is sufficient to asses the failure of ellipticity. In that respect, the influence of c_3 on the onset of loss of strong ellipticity is exemplified in Figure 3(a) in terms of the coordinate $\bar{X}_1$. It is worth noting that when c_3 exceeds the approximate value 1.785 loss of strong ellipticity is always expected close to $\lambda_1 \approx 0.75$, or, equivalently at $\bar{X}_1 \approx -0.481$.

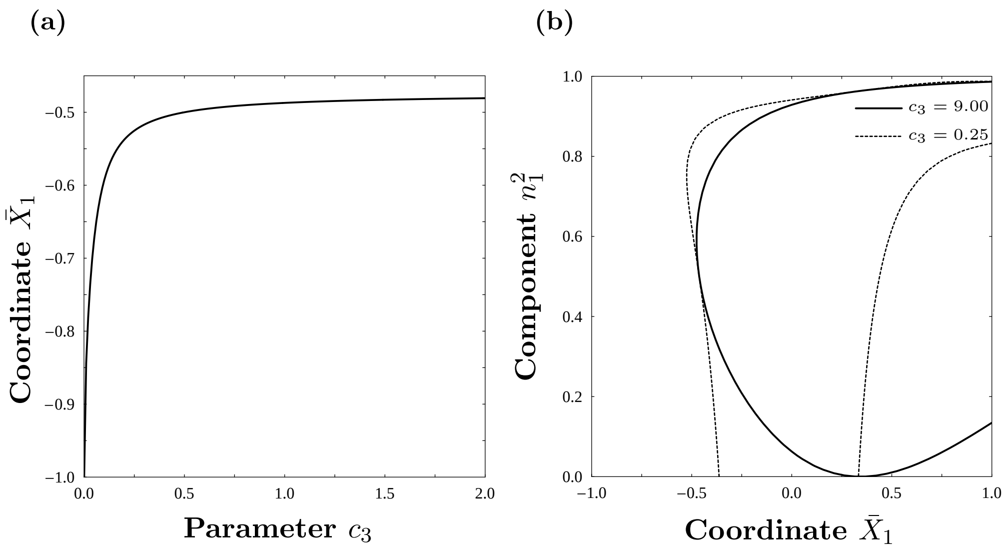

Figure 3. Plots of (a) the dimensionless coordinate $\bar{X}_1$ at which ellipticity is lost as a function of the dimensionless material parameter c_3 and (b) the corresponding value of n_1^2 as a function of $\bar{X}_1$.

It is now evident that, in terms of the components of the acoustic tensor, the onset of failure of ellipticity is strictly associated with breakdown of $(100)_2$. For the considered strain-energy function this gives explicitly

$$\begin{aligned} &4c_3\lambda_1^{10}n_1^4 + \left(3\lambda_1^2 - 1\right)\left(2\lambda_1^6 + 2c_3\lambda_1^4 + 9\lambda_1^2 - 5\right)n_2^4 \\ &\quad + \left[4c_3\lambda_1^{10} + 6\lambda_1^8 + (6c_3 - 2)\lambda_1^6 - 3\left(2c_3 + 3\right)\lambda_1^4 + 6\lambda_1^2 - 1\right]n_1^2n_2^2 = 0. \end{aligned} \tag{111}$$

The implications of (111) are illustrated in Figure 3(b) in which n_1^2 is plotted against $\bar{X}_1$ for two distinct values of c_3. This then identifies the direction of the unit vector $\mathbf{n}$ for which ellipticity fails. Clearly, decrease in the value of the ratio c_3 induces ellipticity to fail first for $\bar{X}_1$ closer to -1 also corresponding to values of n_1 closer to 1. If c_3 exceeds the value 1.785 this fact is only consequential regarding those solutions of (111) lying close to $n_1 = 0$, in which case the smaller c_3 is the closer to $\bar{X}_1 = -1$ ellipticity fails initially. However, it appears that this assertion is valid only for a relatively small range of $c_3 \geq 1.785$ since, as c_3 increases, the solutions of (111) in terms of n_1^2 tend to stabilize. Specifically, in the limit $n_1 \to 0$, (111) reduces to

$$\bar{\bar{W}}_1\hat{\hat{W}}_{22} \equiv \frac{1}{8}c_{55}^2\lambda_2^4\left(3\lambda_1^2 - 1\right)\left(2\lambda_1^6 + 2c_3\lambda_1^4 + 9\lambda_1^2 - 5\right) = 0 \tag{112}$$

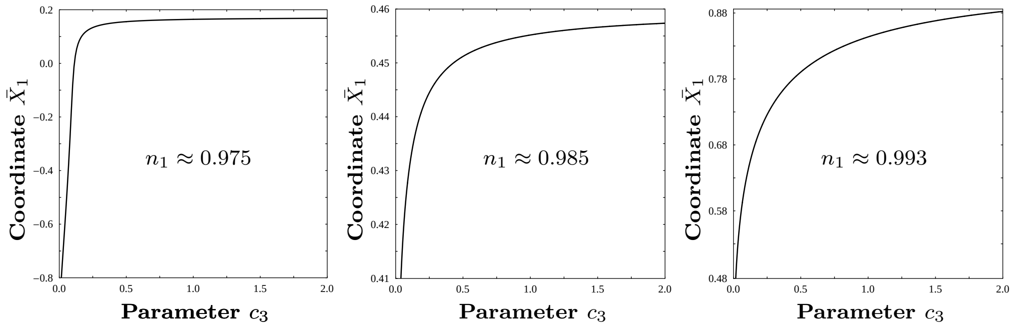

Figure 4. Plots of the dimensionless coordinate $\bar{X}_1$ at which ellipticity is lost as a function of the dimensionless material parameter c_3 for three fixed values of n_1 close to 1.

and hence, for $0 < c_3 < 21.75$ loss of ellipticity is initiated from $\lambda_1 \approx 0.724$ ($\bar{X}_1 \approx -0.394$) when $c_3 \to 0$, and, for $c_3 \geq 21.75$, from $\lambda_1 = \frac{\sqrt{3}}{3}$ ($\bar{X}_1 \approx 0.333$) independently of the value of c_3.

At this point, it should be emphasized that the necessary and sufficient conditions for strong ellipticity to hold given in (105)–(107), and hence their consequences for the specific class of the strain-energy functions (78), are purely local. However, for the considered geometry and deformation, if the symmetry is to be maintained this would suggest that ellipticity should be lost simultaneously at each point on a surface $r =$ constant. This would imply that $\mathbf{n} = \mathbf{e}_r$. If this is the case then strongly elliptic modes of bending deformation with the required symmetry are sustainable if and only if the simple requirements

$$\hat{\hat{W}}_{11} > 0, \quad \bar{\bar{W}}_1 + \bar{\bar{W}}_4 > 0 \tag{113}$$

hold jointly. These conditions actually hold for the particular special model considered since then we have $n_2 = 0$ in the left-hand side of (111) and $c_3 > 0$. Accordingly, for the material model examined above we conclude that ellipticity failure is possible only if the deformation becomes non-symmetric. It is worth noting, however, that if $\mathbf{n}$ is taken to be not strictly radial but very close to the direction of $\mathbf{e}_r$, failure of ellipticity can occur. This point is illustrated in Figure 4 where we plot the solutions $\bar{X}_1$ of (111) against the parameter c_3 for three fixed values of n_1 close to 1.

6 Incompressible materials

For the incompressible theory and with reference to the work of Jiang and Ogden [8, 9], general forms of strain-energy functions in respect of bending deformation (under plane strain) for transversely isotropic elastic materials may be identified. For this purpose, we define the strain-energy functions $\bar{w}(I_1, I_4) = \bar{\bar{W}}(I_1, 1, I_4)$ and $\hat{w}(\lambda_1) = \hat{\hat{W}}(\lambda_1, \lambda_2)$, where $\bar{\bar{W}}$ can, for example, be one of the functions discussed in Section 4.2 or any other function satisfying the required conditions.

Here, the equilibrium equation imposes no restriction on the strain-energy but simply serves to determine the Lagrange multiplier, p say, involved in the expression

$$\boldsymbol{\sigma} = 2\bar{w}_1\mathbf{B} + 2\bar{w}_4\mathbf{m} \otimes \mathbf{m} - p\mathbf{I}$$

for the (plane strain) Cauchy stress, where $\mathbf{I}$ is the (two-dimensional) identity tensor.

As a result the specialization (62) discussed in the Section 4.2.1 is now admissible and may be written as

$$\bar{w} = \bar{w}_{\text{iso}}(I_1) + \bar{w}_{\text{fib}}(I_4).$$

This is one possibility within a very wide class of incompressible transversely materials that may be examined under the considered bending deformation.

Acknowledgements. The work of F. Kassianidis is supported by the UK Engineering and Physical Sciences Research Council and by the University of Glasgow.

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