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ON SINGULAR CALOGERO-MOSER SPACES

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ABSTRACT. Using combinatorial properties of complex reflection groups we show that if the group W is different from the wreath product $\mathfrak{S}_n \wr \mathbb{Z}/m\mathbb{Z}$ and the binary tetrahedral group (labelled $G(m, 1, n)$ and G_4 respectively in the Shephard-Todd classification), then the generalised Calogero-Moser space $X_{\mathbf{c}}$ associated to the centre of the rational Cherednik algebra $H_{0,\mathbf{c}}(W)$ is singular for all values of the parameter \mathbf{c} . This result and a theorem of Ginzburg and Kaledin imply that there does not exist a symplectic resolution of the singular symplectic variety $\mathfrak{h} \times \mathfrak{h}^*/W$ when W is a complex reflection group different from $\mathfrak{S}_n \wr \mathbb{Z}/m\mathbb{Z}$ and the binary tetrahedral group (where \mathfrak{h} is the reflection representation associated to W). Conversely it has been shown by Etingof and Ginzburg that $X_{\mathbf{c}}$ is smooth for generic values of \mathbf{c} when $W \cong \mathfrak{S}_n \wr \mathbb{Z}/m\mathbb{Z}$. We show that this is also the case when W is the binary tetrahedral group. A theorem of Namikawa then implies the existence of a symplectic resolution in this case, completing the classification. Finally, we note that the above results together with work of Chlouveraki are consistent with a conjecture of Gordon and Martino on block partitions in the restricted rational Cherednik algebra.

1. INTRODUCTION

Let W be an irreducible complex reflection group and \mathfrak{h} its reflection representation. Etingof and Ginzburg [EG] associated to W a family of algebras, the *rational Cherednik algebras* $H_{t,\mathbf{c}}(W)$, depending on parameters t and \mathbf{c} . The definition is given in Section 2. When $t = 0$, these algebras have large centres and the geometry of the centre strongly influences the representation theory of the algebra. The affine variety $X_{\mathbf{c}}$ corresponding to the centre of the rational Cherednik algebra was called the generalised Calogero-Moser space at \mathbf{c} by Etingof and Ginzburg. They showed [EG, Corollary 1.14], that for generic values of the parameter \mathbf{c} , $X_{\mathbf{c}}$ is smooth when $W \cong G(m, 1, n)$. However, Gordon [Go, Proposition 7.3] showed that, for many Weyl groups W not of type A or $B(= C)$, $X_{\mathbf{c}}$ is a singular variety for all choices of the parameter \mathbf{c} . Using similar methods we extend this result to all irreducible complex reflection groups.

Theorem 1.1. *Let W be an irreducible complex reflection group, not isomorphic to $G(m, 1, n)$ or G_4 , and $X_{\mathbf{c}}$ the generalised Calogero-Moser space associated to W . Then $X_{\mathbf{c}}$ is a singular variety for all choices of the parameter \mathbf{c} . Conversely for $W \cong G_4$, $X_{\mathbf{c}}$ is a smooth variety for generic values of \mathbf{c} .*

This completes the classification of rational Cherednik algebras for which $X_{\mathbf{c}}$ is smooth for generic \mathbf{c} .

In [GK, Corollary 1.21], Ginzburg and Kaledin show that the existence of a symplectic resolution of the symplectic singularity $\mathfrak{h} \times \mathfrak{h}^*/W$ implies that $X_{\mathbf{c}}$ is smooth for generic \mathbf{c} . This result, together with Theorem 1.1 above implies the following corollary.

Corollary 1.2. *Let W be an irreducible complex reflection group with reflection representation \mathfrak{h} . Then there does not exist a symplectic resolution of $\mathfrak{h} \times \mathfrak{h}^*/W$ when $W \not\cong G(m, 1, n)$ or G_4 .*

It has been show by Wang [Wang, Proposition 1], that there exists a symplectic resolution of $\mathfrak{h} \times \mathfrak{h}^*/W$ when $W \cong G(m, 1, n)$, for all m and n . Similarly, since $X_{\mathbf{c}}$ is smooth for generic values of \mathbf{c} when $W \cong G_4$, a result of Namikawa [Na, Corollary 2.10], implies

Corollary 1.3. *There exists a symplectic resolution of the singular symplectic variety $\mathfrak{h} \times \mathfrak{h}^*/G_4$.*

In order to prove Theorem 1.1 we show that the restricted rational Cherednik algebra $\bar{H}_{0,\mathbf{c}}(W)$ has irreducible representations of dimension $< |W|$ for all values of \mathbf{c} when W is different from $G(m, 1, n)$ and G_4 . This implies that there exist blocks in $\bar{H}_{0,\mathbf{c}}(W)$ with nonisomorphic irreducible modules. Therefore the corresponding block partition of $Irr(W)$, as described in [GM], is trivial for generic values of \mathbf{c} if and only if W is $G(m, 1, n)$ or G_4 . A conjecture of Gordon and Martino [GM] then implies that the partitioning of $Irr(W)$ induced by the Rouquier families of the Hecke algebra $\mathcal{H}_{\mathbf{q}}(W)$ should also be trivial for generic choices of \mathbf{c} if and only if W is $G(m, 1, n)$ or G_4 . Work of Chlouveraki [Ch] on the cyclotomic Hecke algebras of exceptional complex reflection groups shows that this is indeed the case.

2. THE RATIONAL CHEREDNIK ALGEBRA AT $t = 0$

2.1. Definitions and notation. Let W be a complex reflection group, \mathfrak{h} its reflection representation over \mathbb{C} with $\dim(\mathfrak{h}) = n$, and \mathcal{S} the set of all complex reflections in W . Let $\omega : \mathfrak{h} \oplus \mathfrak{h}^* \rightarrow \mathbb{C}$ be the symplectic form on $\mathfrak{h} \oplus \mathfrak{h}^*$ given by $\omega((f_1, f_2), (g_1, g_2)) = f_2(g_1) - g_2(f_1)$ and $\mathbf{c} : \mathcal{S} \rightarrow \mathbb{C}$ a W -invariant function. For $s \in \mathcal{S}$, define $\omega_s : \mathfrak{h} \oplus \mathfrak{h}^* \rightarrow \mathbb{C}$ to be the restriction of ω on $\text{Im}(1 - s)$ and the zero form on $\text{Ker}(1 - s)$. The *rational Cherednik algebra* at parameter $t = 0$, as introduced by Etingof and Ginzburg [EG, page 250], is the quotient of the skew group algebra of the tensor algebra $T(\mathfrak{h} \oplus \mathfrak{h}^*)$ with W , $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes W$, by the ideal generated by the relations

$$(1) \quad [x, y] = \sum_{s \in \mathcal{S}} \mathbf{c}(s) \omega_s(x, y) s \quad \forall x, y \in \mathfrak{h} \oplus \mathfrak{h}^*$$

Let $Z_{\mathbf{c}}$ denote the centre of $H_{0,\mathbf{c}}$ and $X_{\mathbf{c}} = \text{maxspec}(Z_{\mathbf{c}})$ the affine variety corresponding to $Z_{\mathbf{c}}$. The space $X_{\mathbf{c}}$ is called the *generalised Calogero-Moser space* associated to the complex reflection group W at parameter \mathbf{c} . By [EG, Proposition 4.5], we have an inclusion $A = \mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W \subset Z_{\mathbf{c}}$ and correspondingly a surjective morphism $\Upsilon_{\mathbf{c}} : X_{\mathbf{c}} \rightarrow \mathfrak{h}/W \times \mathfrak{h}^*/W$. This allows us to define the restricted rational Cherednik algebra $\bar{H}_{0,\mathbf{c}}(W)$ as

$$\bar{H}_{0,\mathbf{c}}(W) := \frac{H_{0,\mathbf{c}}(W)}{\langle A_+ \rangle}$$

where A_+ denotes the ideal in A of elements with zero constant term. From the defining relations (1) we see that putting \mathfrak{h}^* in degree 1, \mathfrak{h} in degree -1 and $\mathbb{C}W$ in degree 0 defines a \mathbb{Z} -grading on the rational Cherednik algebra $H_{t,\mathbf{c}}(W)$. The ideal $\langle A_+ \rangle$ is generated by elements that are homogeneous with respect

to this grading, therefore $\bar{H}_{0,\mathbf{c}}$ is also a \mathbb{Z} -graded algebra.

Let $Irr(W)$ be a complete set of non-isomorphic irreducible representation of W . We denote by $\mathbb{C}[\mathfrak{h}]^{coW}$ the coinvariant ring $\mathbb{C}[\mathfrak{h}]/C[\mathfrak{h}]_+^W$, where $C[\mathfrak{h}]_+^W$ is the ideal in $\mathbb{C}[\mathfrak{h}]$ generated by the elements in $\mathbb{C}[\mathfrak{h}]^W$ with zero constant term. We follow the notation introduced in [Go] and define

$$M(\lambda) := \bar{H}_{0,\mathbf{c}} \otimes_{\mathbb{C}[\mathfrak{h}]^{coW}} \lambda$$

to be the baby Verma $\bar{H}_{0,\mathbf{c}}$ -module associated to λ . This module is a graded $\bar{H}_{0,\mathbf{c}}$ -module with $M(\lambda)_i = 0$ for $i > 0$. By [Go, Proposition 4.3], $M(\lambda)$ has a simple head which we denote $L(\lambda)$.

We follow the notation of [Ste] with regards to complex reflection groups, and set $d = m/p$ when considering the groups $G(m, p, n)$. For an arbitrary finite dimensional \mathbb{Z} -graded vector space $M = \bigoplus_{i \in \mathbb{Z}} M_i$, the Poincaré polynomial of M will be denoted $P(M, t)$. We denote by $f_\lambda(t)$ the *fake polynomial* of the $\lambda \in Irr(W)$. This is defined as

$$f_\lambda(t) := \sum_{i \in \mathbb{Z}_{\geq 0}} (\mathbb{C}[\mathfrak{h}]_i^{coW} : \lambda) t^i$$

where $(\mathbb{C}[\mathfrak{h}]_i^{coW} : \lambda)$ is the multiplicity of λ in i^{th} degree of the coinvariant ring $\mathbb{C}[\mathfrak{h}]^{coW}$ (thought of here as a graded W -module).

We will also require the surjective map $\Theta : Irr(W) \rightarrow \Upsilon^{-1}(0)$, taking λ to the annihilator of $L(\lambda)$ in $Z_{\mathbf{c}}$, as defined in [Go, paragraph 5.4]. This map has the property that a fiber $\Theta^{-1}(\mathfrak{m})$ is a singleton set if and only if \mathfrak{m} is a smooth closed point in $X_{\mathbf{c}}$ ([Go, Theorem 5.6]).

2.2. General results. Let $\{s_1, \dots, s_k\}$ be a conjugacy class consisting of complex reflections in W and ζ the eigenvalue of s_1 (and hence all s_i) not equal to 1 when thinking of W as a subgroup of $GL(\mathfrak{h})$. For $1 \leq i \leq k$, let ω_{s_i} be the restricted symplectic form on $\mathfrak{h} \oplus \mathfrak{h}^*$ as defined above. Let $\pi_{s_i} : \mathfrak{h} \oplus \mathfrak{h}^* \rightarrow \text{Im}(1 - s_i)$ be the projection map along $\text{Ker}(1 - s_i)$, so that $\omega_{s_i} = \omega \circ \pi_{s_i}$, and define $\Omega = \sum_{i=1}^k \omega_{s_i}$.

Lemma 2.1. *Let W , ω and Ω be as above. Then*

$$\Omega = \frac{k}{n} (1 - \zeta)^{-1} (1 - \zeta^{-1})^{-1} (2 - \zeta - \zeta^{-1}) \omega.$$

Proof. Since each ω_{s_i} is alternating and \mathbb{C} -linear, $\Omega \in \bigwedge^2(\mathfrak{h} \oplus \mathfrak{h}^*)$. Let $x \in \mathfrak{h} \oplus \mathfrak{h}^*$. Then x decomposes uniquely as $x_1 + x_2$, with $x_1 \in \text{Im}(1 - s_i)$ and $x_2 \in \text{Ker}(1 - s_i)$. By definition, there exists $y \in \mathfrak{h} \oplus \mathfrak{h}^*$ such that $(1 - s_i)y = x_1$. Then $(1 - g s_i g^{-1})(gy) = g(1 - s_i)g^{-1}gy = g(1 - s_i)y = gx_1$ implying that $gx_1 \in \text{Im}(1 - g s_i g^{-1})$. Also $(1 - s_i)x_2 = 0$ implies that $(1 - g s_i g^{-1})gx_2 = 0$ hence gx decomposes as $gx_1 + gx_2$ with $gx_1 \in \text{Im}(1 - g s_i g^{-1})$ and $gx_2 \in \text{Ker}(1 - g s_i g^{-1})$. Therefore $\pi_{g s_i g^{-1}} = g \pi_{s_i} g^{-1}$ and $\omega_{s_i}(g^{-1}x, g^{-1}y) = \omega_{g s_i g^{-1}}(x, y)$. Hence $\Omega \in (\bigwedge^2(\mathfrak{h}^* \oplus \mathfrak{h}))^W$. By [EG, Lemma 2.23] $\dim(\bigwedge^2(\mathfrak{h}^* \oplus \mathfrak{h}))^W = 1$, therefore there exists $\lambda \in \mathbb{C}$ such that $\Omega = \lambda \omega$. Consider $\Omega'(x, y) = \Omega((x, 0), (0, y))$, where $x \in \mathfrak{h}$ and $y \in \mathfrak{h}^*$. Recall that ζ is the eigenvalue of s_i not equal to 1, then $\pi_{s_i}(x) = (1 - \zeta)^{-1}(1 - s_i)x$ and

$\pi_{s_i}(y) = (1 - \zeta^{-1})^{-1}(1 - s_i)y$. Expanding $\Omega'(x, y)$

$$\begin{aligned}\Omega'(x, y) &= \sum_{i=1}^k \omega((1 - \zeta)^{-1}(1 - s_i)x, (1 - \zeta^{-1})^{-1}(1 - s_i)y) \\ &= (1 - \zeta)^{-1}(1 - \zeta^{-1})^{-1} \sum_{i=1}^k [\omega(x, y) - \omega(s_i x, y) - \omega(x, s_i y) + \omega(s_i x, s_i y)] \\ &= (1 - \zeta)^{-1}(1 - \zeta^{-1})^{-1} \omega(x, \left(\sum_{i=1}^k 2 - s_i - s_i^{-1}\right)y)\end{aligned}$$

Define $\phi = \left(\sum_{i=1}^k 2 - s_i - s_i^{-1}\right) : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$, a W -homomorphism. The trace of ϕ is $2nk - (n-1)k - k\zeta - (n-1)k - k\zeta^{-1} = k(2 - \zeta - \zeta^{-1})$. Since \mathfrak{h}^* is irreducible, Schur's lemma says that $\phi(y) = \frac{k}{n}(2 - \zeta - \zeta^{-1})y$ and hence $\lambda = \frac{k}{n}(1 - \zeta)^{-1}(1 - \zeta^{-1})^{-1}(2 - \zeta - \zeta^{-1})$. \square

We also require the notion of a generalised baby Verma module, which are baby Verma modules above points other than the origin in $\mathfrak{h}/W \times \mathfrak{h}^*/W$.

Definition 2.2. Let $(p, q) \in \mathfrak{h}/W \times \mathfrak{h}^*$, W_q the stabiliser subgroup of q in W and E an irreducible W_q -module. Then we define the *generalised baby Verma* module

$$\Delta_{\mathbf{c}}(E; p, q) := H_{0, \mathbf{c}}(W) \otimes_{\mathbb{C}[\mathfrak{h}]^W \otimes_{\mathbb{C}[\mathfrak{h}^*]} \rtimes W_q} E$$

where the action of $\mathbb{C}[\mathfrak{h}]^W \otimes_{\mathbb{C}[\mathfrak{h}^*]} \rtimes W_q$ on E is given by $(f \otimes g \otimes w) \cdot e = f(p)g(q)w \cdot e$ for all $f \in \mathbb{C}[\mathfrak{h}]^W$, $g \in \mathbb{C}[\mathfrak{h}^*]$, $w \in W_q$.

Since $\mathbb{C}[\mathfrak{h}]^W \otimes_{\mathbb{C}[\mathfrak{h}^*]} \rtimes W_q \subseteq Z_{\mathbf{c}}$, Schur's lemma implies that, for every irreducible $H_{0, \mathbf{c}}$ -module L , there exists $(p, r) \in \mathfrak{h}/W \times \mathfrak{h}^*/W$ such that $(f \otimes g) \cdot l = f(p)g(r)l$, for all $l \in L$, $f, g \in \mathbb{C}[\mathfrak{h}]^W \otimes_{\mathbb{C}[\mathfrak{h}^*]} \rtimes W_q$. Choosing a point q in the orbit represented by r we write $(p, r) = (p, Wq)$ and say that the irreducible $H_{0, \mathbf{c}}$ -module L lies above (p, Wq) .

Lemma 2.3. Let L be an irreducible $H_{0, \mathbf{c}}$ -module lying above (p, Wq) . Then there exist $E \in \text{Irr}(W_q)$ and a surjective $H_{0, \mathbf{c}}$ -homomorphism $\phi : \Delta_{\mathbf{c}}(E; p, q) \twoheadrightarrow L$.

Proof. The action on L of the commutative ring $\mathbb{C}[\mathfrak{h}^*]$ gives a decomposition $L = \bigoplus_{q' \in \mathfrak{h}^*} L_{q'}^{\text{gen}}$ of L into generalised eigenspaces. That is, for each $l \in L_{q'}^{\text{gen}}$ and $f \in \mathbb{C}[\mathfrak{h}^*]$, there exists an $N \in \mathbb{N}$ such that $(f - f(q'))^N \cdot l = 0$ (since L is finite dimensional, we can choose N to be independent of f and l).

Choose q' such that $L_{q'}^{\text{gen}} \neq 0$, so that $(f - f(q'))^N$ acts as zero on $L_{q'}^{\text{gen}}$ for all $f \in \mathbb{C}[\mathfrak{h}^*]^W$. As L lies over (p, Wq) we see that $(f - f(q))$ also acts nilpotently on $L_{q'}^{\text{gen}}$ and $f(q) = f(q')$. Since W is a finite group, each orbit in \mathfrak{h}^* is closed, therefore $q' \in Wq$ and we can find $w \in W$ such that $w \cdot q = q'$. Now let $0 \neq L_{q'} \subseteq L_{q'}^{\text{gen}}$ be the space of elements l in $L_{q'}^{\text{gen}}$ such that $(f - f(q')) \cdot l = 0$, for all $f \in \mathbb{C}[\mathfrak{h}^*]$. Then $w^{-1}(L_{q'}) \neq 0$ and $f \cdot (w^{-1}l) = w^{-1} \cdot ({}^w f)(q')l = f(q)w^{-1} \cdot l$ implies that $w^{-1}(L_{q'}) \subseteq L_q$. Thus L_q is a nonzero W_q -submodule of L because $f \cdot (v \cdot l) = v \cdot f(q)l = f(q)(v \cdot l)$ for all $f \in \mathbb{C}[\mathfrak{h}]$, $v \in W_q$ and $l \in L_q$. Choose an irreducible W_q -submodule E of L_q . The inclusion $E \hookrightarrow L$ induces a $H_{0, \mathbf{c}}$ -homomorphism $\phi : \Delta_{\mathbf{c}}(E; p, q) \rightarrow L$. The fact that L is irreducible implies that this is a surjection. \square

3. SINGULAR GENERALISED CALOGERO-MOSER SPACES

3.1. The main result.

Theorem 3.1. *For all W not isomorphic to $G(m, 1, n)$ or G_4 and for all parameters \mathbf{c} , the variety $X_{\mathbf{c}}$ is singular.*

By [EG, Proposition 3.8] the statement of Theorem 3.1 is equivalent to the statement: *for W not isomorphic to $G(m, 1, n)$ or G_4 and for all parameters \mathbf{c} there exists an irreducible $H_{0,\mathbf{c}}(W)$ -module L with $\dim L < |W|$.* Therefore Theorem 3.1 follows from

Proposition 3.2. *For each W not isomorphic to $G(m, 1, n)$ or G_4 , there exists an irreducible W -module λ such that for all parameters \mathbf{c} , the irreducible $\bar{H}_{0,\mathbf{c}}(W)$ -module $L(\lambda)$ has dimension $< |W|$.*

The proof of Proposition 3.2 will occupy the remainder of Section 3. The irreducible complex reflection groups were classified by Shephard and Todd [ST] and either belong to an infinite family labelled $G(m, p, n)$, where $m, p, n \in \mathbb{N}$ and $p \mid m$, or to one of 34 exceptional groups G_4, \dots, G_{37} .

Lemma 3.3. *Let W be a complex reflection group. Let $\lambda \in \text{Irr}(W)$ be the unique representation corresponding to a smooth point of $\Upsilon^{-1}(0)$ in $X_{\mathbf{c}}$ i.e. $\Theta(\lambda)$ is smooth in $X_{\mathbf{c}}$. Then the Poincaré polynomial of $L(\lambda)$ as a graded vector space is given by*

$$(2) \quad P(L(\lambda), t) = \frac{\dim(\lambda)t^{b_{\lambda^*}} P(\mathbb{C}[\mathfrak{h}^*]^{coW}, t)}{f_{\lambda^*}(t)}$$

where λ^* is the irreducible W -module dual to λ , and b_{λ} the trailing degree of the fake polynomial $f_{\lambda}(t)$.

Proof. By [Go, Lemma 4.4, paragraphs (5.2) and (5.4)], the graded composition factors of $M(\lambda)$ are all of the form $L(\lambda)[i]$, for some $i \geq 0$. Therefore we can find a multiset $\{i_1, \dots, i_k\}$ such that as a graded W -module

$$M(\lambda) \cong L(\lambda)[i_1] \oplus L(\lambda)[i_2] \oplus \dots \oplus L(\lambda)[i_k].$$

Since $\Theta(\lambda)$ is a smooth point in $X_{\mathbf{c}}$, [EG, Theorem 1.7] says that $L(\lambda) \cong \mathbb{C}W$ as a W -module. Hence it contains a unique copy of the trivial representation T . Assume this copy occurs in degree a in $L(\lambda)$. Then it will occur in degree $a - i_j$ in $L(\lambda)[i_j]$. As a graded W -module, $M(\lambda) \cong \mathbb{C}[\mathfrak{h}^*]^{coW} \otimes \lambda$. The fact that $[\tau \otimes \lambda : T] = \delta_{\tau\lambda^*}$ implies that the graded multiplicity of T in $M(\lambda)$ equals the graded multiplicity of λ^* in $\mathbb{C}[\mathfrak{h}^*]^{coW}$. The graded multiplicity of λ^* in $\mathbb{C}[\mathfrak{h}^*]^{coW}$ is $f_{\lambda^*}(t)$. Hence $P(M(\lambda), t) = t^{-a} f_{\lambda^*}(t) P(L(\lambda), t)$. The lowest nonzero term of $P(L(\lambda), t)$ occurs in degree zero implying that $a = b_{\lambda^*}$. The formula follows by noting that $P(M(\lambda), t)$ is $\dim(\lambda)P(\mathbb{C}[\mathfrak{h}^*]^{coW}, t)$. \square

Since $L(\lambda)$ is a finite dimensional module, the above lemma shows that the right hand side of equation (2) is a polynomial in $\mathbb{Z}[t, t^{-1}]$ with integer coefficients. Moreover, [Go, Lemma 4.4] shows that it is actually in $\mathbb{Z}[t]$ and that the degree 0 coefficient is $\dim \lambda$.

3.2. The infinite series. We show that for $p \neq 1$ and $W = G(m, p, n) \neq G(2, 2, 3)$ we can choose an irreducible representation λ of $G(m, p, n)$ such that Lemma 3.3 does not hold. Thus $L(\lambda)$ will have dimension $< |G(m, p, n)|$, proving Proposition 3.2 in this case. The group $G(2, 2, 3)$ is the Weyl group corresponding to the Dynkin diagram $D_3 = A_3$ and hence $G(2, 2, 3) \cong S_4$. By [EG, Corollary 16.2], $X_{\mathbf{c}}$ is smooth for generic and hence all non-zero \mathbf{c} in this case.

We give a description of the parameterization of irreducible $G(m, p, n)$ -modules. The reader should consult [Ste, pages 379-381] for details. An m -multipartition $\underline{\lambda}$ of n is an ordered m -tuple of partitions $(\lambda^0, \dots, \lambda^{m-1})$ such that $|\lambda^0| + \dots + |\lambda^{m-1}| = n$. Let $\mathcal{P}(m)$ denote the set of all m -multipartitions of n . The cyclic group $\mathbb{Z}/p\mathbb{Z} = \langle g \rangle$ acts on $\mathcal{P}(m)$: g moves each entry of $\underline{\lambda}$ d places to the right i.e.

$$g \cdot (\lambda^0, \dots, \lambda^{m-1}) = (\lambda^{m-d}, \lambda^{m-d+1}, \dots, \lambda^{m-1}, \lambda^0, \dots, \lambda^{m-1}),$$

(recall from subsection 2.1 that $d = m/p$). For $\underline{\lambda} \in \mathcal{P}(m)$, we denote the orbit $\mathbb{Z}/p\mathbb{Z} \cdot \underline{\lambda}$ by $\{\underline{\lambda}\}$ and $\text{Stab}_{\mathbb{Z}/p\mathbb{Z}}(\underline{\lambda})$ is the stabiliser subgroup of $\mathbb{Z}/p\mathbb{Z}$ with respect to $\underline{\lambda}$. Then the irreducible representations of $G(m, p, n)$ are labelled by distinct pairs $(\{\underline{\lambda}\}, \epsilon)$, where $\epsilon \in \text{Stab}_{\mathbb{Z}/p\mathbb{Z}}(\underline{\lambda})$.

Let $(t)_{(n)} = (1-t) \dots (1-t^{n-1})(1-t^n)$ and for λ a partition of n , denote by $n(\lambda) = \sum_{i=1}^k (i-1)\lambda_i$ the partition statistic. The young diagram D_λ of a partition λ is the finite subset of $\mathbb{N} \times \mathbb{N}$, justified to the south west (in the French style), representing λ . For $(i, j) \in D_\lambda$, we denote by $h(i, j)$ the hook length at (i, j) . The hook polynomial is defined to be

$$H_\lambda(t) = \prod_{(i,j) \in D_\lambda} (1 - t^{h(i,j)}).$$

[Ste, Corollary 6.4] states that the fake polynomial of the irreducible representation labelled by $(\{\underline{\lambda}\}, \epsilon)$ is

$$(3) \quad f_{\{\underline{\lambda}\}}(t) = \frac{1 - t^{dn}}{1 - t^{mn}} R_{\{\underline{\lambda}\}}(t) I_{\underline{\lambda}}(t^m),$$

where

$$R_{\{\underline{\lambda}\}}(t) = \sum_{\underline{\mu} \in \{\underline{\lambda}\}} t^{r(\underline{\mu})} \quad \text{with} \quad r(\underline{\mu}) = \sum_{i=0}^{m-1} i|\mu^i|, \quad \text{and} \quad I_{\underline{\lambda}}(t) = (t)_{(n)} \prod_{i=1}^m \frac{t^{n(\lambda^i)}}{H_{\lambda^i}(t)}.$$

Note that the formula only depends on the orbit and not on the choice of stabiliser.

We wish to calculate the rational function (2) for a well chosen representation $(\{\underline{\mu}\}, \epsilon)$ of the irreducible representations of $G(m, p, n)$. By [Hu, Theorem 3.15], the Poincaré polynomial of the coinvariant ring of W is given by

$$P(\mathbb{C}[\mathfrak{h}^*]^{coW}, t) = \prod_{i=1}^n \frac{1 - t^{d_i}}{1 - t}$$

where d_1, \dots, d_n are the degrees of a set of fundamental homogeneous invariant polynomials of W (d_1, \dots, d_n are independent, up to reordering, of the polynomials choosen). By [ST, page 291], $d_1, \dots, d_n = m, 2m, \dots, (n-1)m, dn$ when $W = G(m, p, n)$.

Hence, if the dual representation of $(\{\underline{\mu}\}, \epsilon)$ is $(\{\underline{\lambda}\}, \eta)$, equation (2) becomes

$$\begin{aligned}
P(L(\{\underline{\mu}\}, \epsilon), t) &= \\
&= \frac{\dim(\{\underline{\mu}\}, \epsilon) t^{b_{\{\underline{\lambda}\}}} (1-t^m)(1-t^{2m}) \dots (1-t^{(n-1)m})(1-t^{dn}) \prod_{i=0}^{m-1} H_{\lambda^i}(t^m)(1-t^{mn})}{(1-t)^n (1-t^{dn}) R_{\{\underline{\lambda}\}}(t) (t^m)_{(n)} \prod_{i=0}^{m-1} t^{n(\lambda^i)m}} \\
(4) \quad &= \frac{\dim(\{\underline{\mu}\}, \epsilon) t^{b_{\{\underline{\lambda}\}}} \prod_{i=0}^{m-1} H_{\lambda^i}(t^m)}{(1-t)^n R_{\{\underline{\lambda}\}}(t) \prod_{i=0}^{m-1} t^{n(\lambda^i)m}}
\end{aligned}$$

Let $k \in \mathbb{N}$ such that $t^k \mid R_{\{\underline{\lambda}\}}(t)$ but $t^{k+1} \nmid R_{\{\underline{\lambda}\}}(t)$ in $\mathbb{Z}[t]$ and write $R_{\{\underline{\lambda}\}}(t) = t^k \tilde{R}_{\{\underline{\lambda}\}}(t)$. Then rearrange equation (3) as

$$(5) \quad f_{\{\underline{\lambda}\}}(t) = \left(t^k \prod_{i=0}^{m-1} t^{n(\lambda^i)m} \right) \tilde{R}_{\{\underline{\lambda}\}}(t) \left(\frac{1-t^{dn}}{1-t^{mn}} (t^m)_{(n)} \prod_{i=1}^m \frac{1}{H_{\lambda^i}(t^m)} \right)$$

Since each $H_{\lambda^i}(t^m)$ is a product of factors of the form $(1-t^l)$, the product in the right most bracket consists entirely of factors of the form $(1-t^l)$. Therefore

$$t^{b_{\{\underline{\lambda}\}}} = t^k \prod_{i=0}^{m-1} t^{n(\lambda^i)m}$$

and equation (4) becomes

$$(6) \quad P(L(\{\underline{\mu}\}, \epsilon), t) = \frac{\dim(\{\underline{\mu}\}, \epsilon) \prod_{i=1}^m H_{\lambda^i}(t^m)}{(1-t)^n \tilde{R}_{\{\underline{\lambda}\}}(t)}.$$

To contradict Lemma 3.3 and hence prove Proposition 3.2 we have

Lemma 3.4. *Let $p \neq 1$ and $W = G(m, p, n)$ with $W \neq G(2, 2, 3)$. Then there exists $(\{\underline{\mu}\}, \epsilon) \in \text{Irr}(W)$ such that the right hand side of equation (6) is not an element of $\mathbb{C}[t]$.*

Proof. We consider the cases $n = 2, 3$ and $n > 3$ separately. For $n > 3$ choose $(\{\underline{\mu}\}, \epsilon)$ such that its dual representation is $\underline{\lambda} = (\lambda^0, \emptyset, \dots, \emptyset)$, where $\lambda^0 = (2, 2, 1, 1, \dots, 1)$. Then

$$\tilde{R}(t) = R(t) = 1 + t^{dn} + t^{2dn} + \dots + t^{(p-1)dn} = \frac{1-t^{mn}}{1-t^{dn}}$$

and for this particular m -multipartition we have

$$\prod_i H_{\lambda^i}(t^m) = H_{\lambda^0}(t^m) = (1 - t^{2m})(1 - t^m)(1 - t^{(n-1)m})(1 - t^{(n-2)m}) \prod_{i=1}^{n-4} (1 - t^{im})$$

Equation (6) becomes

$$(7) \quad P(L(\{\underline{\mu}\}, \epsilon), t) = \frac{\dim(\{\underline{\mu}\}, \epsilon)(1 - t^{2m})(1 - t^m)(1 - t^{(n-1)m})(1 - t^{(n-2)m})(t^m)_{n-4}(1 - t^{dn})}{(1 - t^{mn})(1 - t)^n}.$$

The numerator of (7) factorises over \mathbb{C} as a product of factors $(1 - \omega t)$, where ω is a primitive k^{th} root of unity with $1 \leq k < mn$, whereas the denominator contains at least one factor of the form $(1 - \sigma t)$, where σ is a primitive $(mn)^{\text{th}}$ root of unity. Therefore, since $\mathbb{C}[t]$ is an Euclidean domain, the right hand side of (7) cannot not lie in $\mathbb{C}[t]$.

For $n = 2$ and $m \geq n$, take $\underline{\lambda} = ((1), (1), \emptyset \dots \emptyset)$. Then

$$\prod_i H_{\lambda^i}(t^m) = (1 - t^m)^2 \quad R(t) = \frac{t(1 - t^{2m})}{1 - t^{2d}}, \quad \text{and} \quad \tilde{R}(t) = \frac{1 - t^{2m}}{1 - t^{2d}}.$$

Substituting into (6)

$$P(L(\{\underline{\mu}\}, \epsilon), t) = \frac{\dim(\{\underline{\mu}\}, \epsilon)(1 - t^m)^2(1 - t^{2d})}{(1 - t^{2m})(1 - t)^2}.$$

By the same reasoning as above, since $2m > 2d, m$, this rational function is not a polynomial.

Similarly, for $n = 3$ and $m \geq n$, take $\underline{\lambda} = ((1), (1), (1), \emptyset \dots \emptyset)$. Then

$$\prod_i H_{\lambda^i}(t^m) = (1 - t^m)^3 \quad R(t) = \frac{t^3(1 - t^{3m})}{1 - t^{3d}}, \quad \text{and} \quad \tilde{R}(t) = \frac{1 - t^{3m}}{1 - t^{3d}}.$$

Substituting into (6)

$$P(L(\{\underline{\mu}\}, \epsilon), t) = \frac{\dim(\{\underline{\mu}\}, \epsilon)(1 - t^m)^3(1 - t^{3d})}{(1 - t^{3m})(1 - t)^3}.$$

Once again, this rational function is not a polynomial because $3m > 3d, m$. □

3.3. The Exceptional Groups. Using the computer algebra program [GAP] together with the package [CHE] we calculate for each exceptional complex reflection group W (excluding G_4), the number of irreducible representations λ for which the polynomial $t^{-b_{\lambda^*}} f_{\lambda^*}(t)$ does not divide $P(\mathbb{C}[\mathfrak{h}]^{coW}, t)$ in $\mathbb{C}[t]$. Table (3.3) gives the results of these calculations. For each of these λ , Lemma 3.3 does not hold and hence $\dim L(\lambda) < |W|$ for all values of \mathbf{c} . Since there is always at least one such λ for every exceptional group, Proposition 3.2 is proved for the exceptional groups.

The code used to produce the data in Table (3.3) is available on the author's website [Be]. For every exceptional group, the fake polynomials of the irreducible characters are listed there. The remainder of

TABLE 1. Number of irreducibles that fail Lemma 3.3

Group	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
# failures	3	6	13	2	16	15	43	1	4	9	18	15	55	70	164	18	42	12	4
Group	24	25	26	27	28	29	30	31	32	33	34	35	36	37					
# failures	8	3	10	26	5	24	24	40	33	30	148	9	30	75					

$P(\mathbb{C}[\mathfrak{h}]^{coW}, t)$ on division by $t^{-b^*} f_{\lambda^*}(t)$ is also listed. In addition, this information is available for many of the groups $G(m, p, n)$ of rank ≤ 5 .

4. THE EXCEPTIONAL GROUP G_4

The group G_4 , as labelled in [ST], is the binary tetrahedral group. It can be realised as a finite subgroup of the group of units in the quaternions

$$G_4 = \{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$$

and has order 24. It is generated by the elements $s_1 = \frac{1}{2}(-1 + i + j - k)$ and $s_2 = \frac{1}{2}(-1 + i - j + k)$ and has presentation $G_4 = \langle s_1, s_2 | s_1^3 = s_2^3 = (s_1 s_2)^6 = 1 \rangle$. It has seven conjugacy classes which we label $Cl_1 = \{1\}$, Cl_2 , Cl_3 , Cl_4 , Cl_5 , Cl_6 , and Cl_7 . The character table is

Class	1	2	3	4	5	6	7
Size	1	1	4	4	6	4	4
Order	1	1	3	3	4	6	6
T	1	1	1	1	1	1	1
V_1	1	1	ω^2	ω	1	ω^2	ω
V_2	1	1	ω	ω^2	1	ω	ω^2
W	2	-2	-1	-1	0	1	1
\mathfrak{h}	2	-2	$-\omega^2$	$-\omega$	0	ω^2	ω
\mathfrak{h}^*	2	-2	$-\omega$	$-\omega^2$	0	ω	ω^2
U	3	3	0	0	-1	0	0

where ω is a primitive cube root of unity. Note that the reflection representation \mathfrak{h} has dimension 2, therefore G_4 is a rank 2 complex reflection group.

The group G_4 has two classes which consist of complex reflections and we label these reflections as

$$\begin{aligned} Cl_3 &= \{s_1, s_2, s_3, s_4\} \\ &= \left\{ \frac{1}{2}(-1 + i + j - k), \frac{1}{2}(-1 + i - j + k), \frac{1}{2}(-1 - i + j + k), \frac{1}{2}(-1 - i - j - k) \right\} \end{aligned}$$

and

$$\begin{aligned} Cl_4 &= \{t_1, t_2, t_3, t_4\} \\ &= \left\{ \frac{1}{2}(-1 - i - j + k), \frac{1}{2}(-1 + i - j - k), \frac{1}{2}(-1 - i + j - k), \frac{1}{2}(-1 + i + j + k) \right\} \end{aligned}$$

Unlike all other exceptional irreducible complex reflection groups we have

Theorem 4.1. *For generic values of \mathbf{c} , the generalised Calogero-Moser space $X_{\mathbf{c}}$ associated to G_4 is a smooth variety.*

Proof. The theorem is proved by showing that each irreducible $H_{0,\mathbf{c}}$ -module is isomorphic to the regular representation of G_4 . By [EG, Proposition 3.8], this is equivalent to the statement of the theorem. Let $E = T \oplus V_1 \oplus V_2 \oplus 3U$ and $F = \mathfrak{h} \oplus \mathfrak{h}^* \oplus W$ be two G_4 -modules.

Claim 1. Let L be a finite dimensional $H_{0,\mathbf{c}}$ -module, then $L \cong aE \oplus bF$, for some $a, b \in \mathbb{Z}_{\geq 0}$.

To prove Claim 1 we use an argument similar to that of [EG, Proposition 16.5]. Let $\rho : H_{0,\mathbf{c}} \rightarrow \text{End}_{\mathbb{C}}(L)$ realise the action of $H_{0,\mathbf{c}}$ on L . Then, for all $x, y \in \mathfrak{h} \oplus \mathfrak{h}^*$, we have the commutation relation

$$(8) \quad [\rho(x), \rho(y)] = c_1 \sum_{i=1}^4 \omega_{s_i}(x, y) \rho(s_i) + c_2 \sum_{j=1}^4 \omega_{t_j}(x, y) \rho(t_j)$$

By Lemma 2.1, $\sum_{i=1}^4 \omega_{s_i} = \sum_{j=1}^4 \omega_{t_j} = 2\omega$. Taking traces on both sides of equation (8)

$$(9) \quad 0 = c_1 2\omega(x, y) \text{Tr}_L(s_1) + c_2 2\omega(x, y) \text{Tr}_L(t_1) \quad \forall x, y \in \mathfrak{h} \oplus \mathfrak{h}^*$$

Since c_1 and c_2 are generic i.e. take values in a dense open subset of \mathbb{C}^2 , and equation (9) is linear, we have $0 = 2\omega(x, y) \text{Tr}_L(s_1) = 2\omega(x, y) \text{Tr}_L(t_1)$. The fact that ω is nondegenerate implies that Tr_L is zero on Cl_3 and Cl_4 .

Using the fact that s_1 is a complex reflection and $\dim \mathfrak{h}^* = 2$, we can choose a nonzero $x_1 \in \mathfrak{h}^*$ such that $s_1(x_1) = x_1$. Then $s_1[x_1, y] = [x_1, s_1 y]$ for all $y \in \mathfrak{h}$. Since $s_1(x_1) = x_1$, $x_1 \in \text{Ker}(1 - s_1)$ and hence $\omega_{s_1}(x_1, y) = 0$ for all $y \in \mathfrak{h}$. Similarly, $s_1 t_1 = 1$ implies that $x_1 \in \text{Fix}(t_1)$ and hence $\omega_{t_1}(x_1, y) = 0$. Therefore, multiplying both sides of equation (8) on the left by $\rho(s_1)$ and taking traces

$$0 = c_1 \sum_{i=2}^4 \omega_{s_i}(x_1, y) \text{Tr}_L(s_1 s_i) + c_2 \sum_{j=2}^4 \omega_{t_j}(x_1, y) \text{Tr}_L(s_1 t_j)$$

Again, using the fact that c_1, c_2 are generic, we get

$$0 = \sum_{i=2}^4 \omega_{s_i}(x_1, y) \text{Tr}_L(s_1 s_i) = \sum_{j=2}^4 \omega_{t_j}(x_1, y) \text{Tr}_L(s_1 t_j)$$

Since $s_1 s_2, s_1 s_3$ and $s_1 s_4$ all belong to Cl_7 and $s_1 t_2, s_1 t_3, s_1 t_4$ all belong to Cl_5 we have

$$0 = \sum_{i=2}^4 \omega_{s_i}(x_1, y) \text{Tr}_L(s_1 s_i) = 2\omega(x_1, y) \text{Tr}_L(s_1 s_2)$$

$$0 = \sum_{j=2}^4 \omega_{t_j}(x_1, y) \text{Tr}_L(s_1 t_j) = 2\omega(x_1, y) \text{Tr}_L(s_1 t_2)$$

Therefore Tr_L is zero on Cl_7 and Cl_5 .

We can also multiply both sides of equation (8) on the left by $\rho(t_1)$ instead of $\rho(s_1)$. Noting that $t_1^2 \in Cl_3$, $t_1 t_2, t_1 t_3, t_1 t_4 \in Cl_6$ and repeating the above argument shows that Tr_L is also zero on Cl_6 .

Therefore any element of G_4 that has nonzero trace on L must belong to Cl_1 or Cl_2 . Hence the character associated to L must take values $(n, m, 0, 0, 0, 0, 0)$, for some $n \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}$, on the classes Cl_1, Cl_2, \dots, Cl_7 . Taking inner products shows that

$$L \cong \frac{1}{|G_4|}(n+m)E \oplus \frac{2}{|G_4|}(n-m)F$$

Setting $a = \frac{1}{|G_4|}(n+m)$ and $b = \frac{2}{|G_4|}(n-m)$ proves Claim 1.

Claim 2. Let L be an irreducible representation of $H_{0,\mathbf{c}}$, with \mathbf{c} generic. Then L must be isomorphic to $E \oplus F$ or $\mathbb{C}G_4$ as a G_4 -module.

If L is irreducible then $\dim L \leq 24$. Therefore Claim 1 implies that $L \cong E, 2E, nF, 1 \leq n \leq 4, E \oplus F$ or $\mathbb{C}G_4$. Assume that L is isomorphic to E as a G_4 -module. The action of \mathfrak{h}^* on L defines a G_4 -equivariant linear map $\phi: \mathfrak{h}^* \rightarrow \text{End}_{\mathbb{C}}(E)$. The G_4 -module $\text{End}_{\mathbb{C}}(E)$ decomposes as

$$\begin{aligned} \text{End}_{\mathbb{C}}(E) &\cong (T \otimes T) \oplus 2(T \otimes V_1) \oplus 2(T \otimes V_2) \oplus 6(T \otimes U) \oplus (V_1 \otimes V_1) \oplus 2(V_1 \otimes V_2) \oplus \\ &6(V_1 \otimes U) \oplus (V_2 \otimes V_2) \oplus 6(V_2 \otimes U) \oplus 9(U \otimes U) \cong 12T \oplus 12V_1 \oplus 12V_2 \oplus 36U \end{aligned}$$

This shows that \mathfrak{h}^* is not a summand of $\text{End}_{\mathbb{C}}(E)$. Thus ϕ must be the zero map. Similarly, the action of \mathfrak{h} must also be zero on E . This implies that the right hand side of equation (8) must also act as zero on E . In particular, it must act as zero on $T \subset E$. This means that

$$0 = c_1 \sum_{i=1}^4 \omega_{s_i}(x, y) + c_2 \sum_{j=1}^4 \omega_{t_j}(x, y) = 2(c_1 + c_2)\omega(x, y)$$

This is a contradiction because c_1, c_2 are generic and ω is nondegenerate. Hence L cannot be isomorphic to E . Repeating the above argument for F we have

$$\begin{aligned} \text{End}_{\mathbb{C}}(F) &\cong (\mathfrak{h} \otimes \mathfrak{h}) \oplus 2(\mathfrak{h} \otimes \mathfrak{h}^*) \oplus 2(\mathfrak{h} \otimes W) \oplus \\ &(\mathfrak{h}^* \otimes \mathfrak{h}^*) \oplus 2(\mathfrak{h}^* \otimes W) \oplus (W \otimes W) \cong 3T \oplus 3V_1 \oplus 3V_2 \oplus 9U \end{aligned}$$

Therefore \mathfrak{h}^* and \mathfrak{h} must act as zero on F . If we consider the right hand side of equation (8), this time restricted to $W \subset F$ then we have

$$0 = c_1 \sum_{i=1}^4 \omega_{s_i}(x, y) \rho|_W(s_i) + c_2 \sum_{j=1}^4 \omega_{t_j}(x, y) \rho|_W(t_j)$$

Taking the trace of this equation gives $0 = -2(c_1 + c_2)\omega(x, y)$, which is a contradiction because c_1, c_2 are generic and ω is nondegenerate. Therefore $L \not\cong F$. The same reasoning shows that L cannot be isomorphic to $2E$ or $nF, 2 \leq n \leq 4$ either. This proves Claim 2.

Claim 3. Let L be an irreducible $H_{0,c}$ -module, then L cannot be isomorphic to $E \oplus F$ as a G_4 -module.

By Lemma 2.3, there exists a generalised Verma module $\Delta_{\mathbf{c}}(M; p, q)$ and a surjective homomorphism $\phi : \Delta_{\mathbf{c}}(M; p, q) \rightarrow L$. As a G_4 -module we have

$$\Delta_{\mathbf{c}}(M; p, q) = H_{0,c}(W) \otimes_{\mathbb{C}[\mathfrak{h}]^W \otimes_{\mathbb{C}[\mathfrak{h}^*] \rtimes W_q} M \cong \mathbb{C}G_4 \otimes \text{Ind}_{(G_4)_q}^{G_4} M \cong k\mathbb{C}G_4$$

where $(G_4)_q$ is the stabiliser of $q \in \mathfrak{h}^*$ and $k = [G_4 : (G_4)_q] \dim M$. The generalised Verma module $\Delta_{\mathbf{c}}(M; p, q)$ has a finite composition series. Each factor of this series must have dimension ≤ 24 . Therefore, by Claim 2, each factor is isomorphic to either $\mathbb{C}G_4$ or $E \oplus F$ as a G_4 -module. Hence there exist $m, n \in \mathbb{N}$ such that $k\mathbb{C}G_4 \cong m\mathbb{C}G_4 \oplus n(E \oplus F)$ with $n \geq 1$. But then $n(E \oplus F) \cong (k - m)\mathbb{C}G_4$, which is a contradiction. This completes the proof of Claim 3 and the theorem. \square

We can now apply a result of Namikawa [Na, Corollary 2.10], which we state for the convenience of the reader.

Theorem 4.2 (Namikawa). *Let (X, ω) be an affine symplectic variety equipped with a \mathbb{C}^* -action such that*

- *the weights of \mathbb{C}^* on X are all positive and there exists a unique fixed point $0 \in X$,*
- *the symplectic form ω has positive weight $l > 0$.*

Then the following are equivalent

- (1) *X has a crepant resolution*
- (2) *X has a smoothing by a Poisson deformation.*

Corollary 4.3. *Let X be the symplectic singularity $\mathfrak{h} \times \mathfrak{h}^*/G_4$. There exists a symplectic resolution $\pi : Z \rightarrow X$ of X .*

First we recall some definitions from [Ko, page 236], the reader should consult that article for details. A variety will mean a quasi-projective variety over \mathbb{C} . Let X, Y be normal varieties with K_X \mathbb{Q} -Cartier and $f : Y \rightarrow X$ a birational morphism. We can write

$$K_Y \equiv f^*(K_X) + A$$

If E is a prime exceptional divisor on Y then the *discrepancy* of E with respect to X (denoted $a(E, X)$) is defined to be the coefficient of E in A . If $f' : Y' \rightarrow X$ is another birational morphism and $E' \subset Y'$ the birational transform of E on Y' then $a(E, X) = a(E', X)$. Therefore $a(E, X)$ depends only on E and not on Y . The variety X is called *canonical* if $a(E, X) \geq 0$ for all E .

Proof. The affine variety X is four dimensional and normal. By [Wa, Watanabe's Theorem] X has Gorenstein singularities and hence the canonical divisor K_X is trivial (and hence Cartier). The affine variety $V = \mathfrak{h} \times \mathfrak{h}^*$ is smooth and therefore V is canonical. Since G_4 is a finite group, the quotient map $\pi : V \rightarrow X$ is a finite dominant morphism and $\pi^*K_X = \pi^*\mathcal{O}_X = \mathcal{O}_V = K_V$. Therefore we can apply [Ko, Proposition 3.16] which says that X is canonical. Therefore the pair (X, \emptyset) is a Kawamata log terminal pair (as defined in [AHK]) and we can apply [AHK, Lemma 2.1] to conclude that there exists an effective \mathbb{Q} -factorial terminal pair (Y, B) together with a birational morphism $f : Y \rightarrow X$ such that

$$K_Y + B \equiv f^*(K_X)$$

However as noted above we can write $K_Y \equiv f^*(K_X) + A$ with $A = -B$. Since X is canonical $a(E, X) \geq 0$ for all exceptional prime divisors E on Y . Hence A is an effective divisor. But B is also effective therefore $A = B = 0$ and we deduce that $f : Y \rightarrow X$ is a crepant morphism. As noted in [EG, Section 4.14], $\{X_c\}_{c \in \mathbb{C}^2}$ is a Poisson deformation of X . Therefore Theorem 4.1 says that X has a smoothing by a Poisson deformation. Now we can apply Namikawa's result [Na, Theorem 2.4] and conclude that there exists a symplectic resolution $\pi : Z \rightarrow X$. \square

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