Second Order Sliding Mode Control for Nonlinear Affine Systems with Quantized Uncertainty [*]

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Abstract

This paper deals with the design of a Second-Order Sliding Mode (SOSM) control algorithm able to enhance the closed-loop performance depending on the current working conditions. The novelty of the proposed approach is the design of a nonsmooth switching line, based on the quantization of the uncertainties affecting the system. The quantized uncertainty levels allow one to define nested box sets in the auxiliary state space, i.e., the space of the sliding variable and its first time derivative, and select suitable control amplitudes for each set, in order to guarantee the convergence of the sliding variable to the sliding manifold in a finite time. The proposed algorithm is theoretically analyzed, proving the existence of an upperbound of the reaching time to the origin through the considered quantization levels.

Key words: Sliding mode control, nonlinear systems, uncertain dynamic systems, quantized signals, sliding surfaces

1 Introduction

Nowadays, Sliding Mode Control (SMC) is one of the most effective solution to control systems characterized by hard uncertainties [1, 2]. SMC is able to guarantee robustness against a wide class of disturbances, above all in case of matched uncertainties, i.e., uncertainties acting on the same channel of the control variable [2]. Yet, because of the discontinuous nature of the control law, the so-called chattering effect [3, 4] can be produced, i.e., high frequency oscillations of the controlled variable which can be disturbing for the actuators.

However, in the literature, several methods to perform chattering alleviation have been proposed, such as filtered sliding mode [5], boundary layer sliding mode [6] or fractional order sliding mode control [7]. Among these methodologies, the so-called Higher Order Sliding Mode (HOSM) control approaches, which involve not only the sliding variable, but also its time derivatives up to a certain order $r - 1$ [8–10],

consist in confining the discontinuity, necessary to steer the so-called sliding variable to zero, to a derivative of the control variable, so that the control signal actually fed into the plant is continuous. Because of the continuous nature of the control action, HOSM control approaches are appropriate to be applied even to electrical, electromechanical or mechanical systems [11, 12], as testified by [13–20].

In the classical formulation of SMC, the uncertain terms are assumed to be bounded with known bounds. It is also reasonable assuming that uncertainties can be linked to the system states because of state-dependent disturbances or different levels of confidence in the system model in different operating conditions [21]. This can imply a quantization of the uncertain terms such that different compact box sets can be defined in the state space. In the paper, the convergence to the origin of the auxiliary state space is proved, and an upperbound of the convergence time with respect to the worst realization of the uncertainties is analytically provided.

The present proposal provides a simple way to tune the amplitude of the discontinuous control action depending on the uncertainties quantization levels. Other interesting tuning mechanisms are presented in [21, 22]. Yet, they differ from the proposed approach since they are based on the a priori subdivision of the auxiliary state space into regions. Moreover, they rely on the use of the Suboptimal SOSM control law, while in this paper the SOSM control law with optimal reaching [10] is used inside each level, so that for

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each uncertainty quantization level a minimum time passage through the corresponding set is featured by the auxiliary state trajectory. In fact, the corresponding nonsmooth surface is the combination of different switching lines which result in being attractive with optimal reaching for the auxiliary state trajectories.

The paper is organized as follows. In Section 2 the problem is formulated, while in Section 3 the proposed strategy based on a nonsmooth switching line is presented. In Section 4 the stability analysis is discussed and an academic example is reported in Section 5. Some conclusions in Section 6 end the paper.

2 Problem Formulation

Consider a plant which can be described by the single-input system affine in the control variable

$$\dot{x}(t) = a(x(t)) + b(x(t))u(t) \tag{1}$$

where $x \in \Omega$ ($\Omega \subset \mathbb{R}^n$ bounded) is the state vector, the value of which at the initial time instant t_0 is $x(t_0) = x_0$, and $u \in \mathbb{R}$ is a scalar input subject to the saturation $[-\alpha, \alpha]$, while $a(x(t)) : \Omega \to \mathbb{R}^n$ and $b(x(t)) : \Omega \to \mathbb{R}^n$ are uncertain functions of class $\mathrm{C}^1(\Omega)$.

Define a suitable output function $\sigma(x) : \Omega \to \mathbb{R}$ of class $\mathrm{C}^2(\Omega)$. This function will play the role of "sliding variable" in the following, that is $\sigma(x)$ is the variable to steer to zero in a finite time in order to solve the control problem, according to classical sliding mode control theory [1]. The sliding variable $\sigma(x)$ has to be selected such that the following assumption holds.

Assumption 1 *If $u(t)$ in (1) is designed so that, in a finite time t_r (ideal reaching time), $\sigma(x(t_r)) = 0 \ \forall \, x_0 \in \Omega$ and $\sigma(x(t)) = 0 \ \forall \, t > t_r$, then $\forall \, t \geq t_r$ the origin is an asymptotically stable equilibrium point of (1) constrained to $\sigma(x(t)) = 0$.*

Note that Assumption 1 guarantees that the sliding mode control law to design is stabilizing.

Now consider the input-output map

$$\begin{cases} \dot{x}(t) = a(x(t)) + b(x(t))u(t) \\ y(t) = \sigma(x(t)) \\ x(t_0) = x_0 \ . \end{cases} \tag{2}$$

Assume that (2) is complete in Ω and has a uniform relative degree equal to 2. Moreover, assume that system (2) admits a global normal form in Ω, i.e., there exists a global diffeo-

morphism of the form $\Phi(x) : \Omega \to \Phi_\Omega \subset \mathbb{R}^n$,

$$\Phi(x) = \begin{pmatrix} \Psi(x) \\ \sigma(x) \\ a(x) \cdot \nabla\sigma(x) \end{pmatrix} = \begin{pmatrix} x_r \\ \xi \end{pmatrix}$$

$$\Psi : \Omega \to \mathbb{R}^{n-2}, \quad x_r \in \mathbb{R}^{n-2}, \quad \xi = \begin{pmatrix} \sigma(x) \\ \dot{\sigma}(x) \end{pmatrix} \in \mathbb{R}^2 \ ,$$

such that,

$$\begin{cases} \dot{x}_r = a_r(x_r, \xi) & \text{(3a)} \\ \dot{\xi}_1 = \xi_2 & \text{(3b)} \\ \dot{\xi}_2 = f(x_r, \xi) + g(x_r, \xi)u & \text{(3c)} \\ y = \xi_1 & \text{(3d)} \\ \xi(t_0) = \xi_0 & \text{(3e)} \end{cases}$$

with

$$a_r = \frac{\partial\Psi}{\partial x}(\Phi^{-1}(x_r, \xi))a(\Phi^{-1}(x_r, \xi))$$

$$f = a(\Phi^{-1}(x_r, \xi)) \cdot \nabla(a(\Phi^{-1}(x_r, \xi)) \cdot \nabla\sigma(\Phi^{-1}(x_r, \xi)))$$

$$g = b(\Phi^{-1}(x_r, \xi)) \cdot \nabla(a(\Phi^{-1}(x_r, \xi)) \cdot \nabla\sigma(\Phi^{-1}(x_r, \xi)))$$

where the obvious dependence on time is omitted. Note that, as a consequence of the uniform relative degree assumption, it yields

$$g(x_r, \xi) \neq 0, \quad \forall (x_r, \xi) \in \Phi_\Omega \ . \tag{4}$$

In the literature, see for instance [23], making reference to the previous system, subsystem (3b)-(3e) is called "auxiliary system". Since $a_r(\cdot)$, $f(\cdot)$, $g(\cdot)$ (the latter is assumed to be positive definite, for the sake of simplicity) are continuous functions and Φ_Ω is a bounded set, one has that

$$\exists \, F > 0 : \ |f(x_r, \xi)| \leq F \ \forall (x_r, \xi) \in \Phi_\Omega \tag{5}$$

$$\exists \, G_{\max} > 0 : \ g(x_r, \xi) \ \leq G_{\max} \ \forall (x_r, \xi) \in \Phi_\Omega \tag{6}$$

$$\exists \, G_{\min} > 0 : \ g(x_r, \xi) \ \geq G_{\min} \ \forall (x_r, \xi) \in \Phi_\Omega \ . \tag{7}$$

Note that, instead of (6) and (7), if $g(\cdot)$ was negative definite, one could analogously have the opposite inequalities. Moreover, the following assumption on the internal dynamics (3a) holds.

Assumption 2 *Given the auxiliary system* (3)*, the internal dynamics* (3a) *does not present finite time escape phenomena and the corresponding zero dynamics $a_r(x_r, 0)$ is globally asymptotically stable.*

Relying on (3)-(7) and assumptions 1 and 2, the control problem to solve is hereafter introduced.

Problem 1 *Design a feedback control law*

$$u(t) = \kappa(\sigma(x(t)), \dot{\sigma}(x(t))) \tag{8}$$

such that $\forall\, x_0 \in \Omega$, $\exists\, t_{\rm r} \geq 0\,:\, \sigma(x(t)) = \dot{\sigma}(x(t)) = 0$, $\forall\, t \geq t_{\rm r}$ *in spite of the uncertainties.*

The proposed control strategy has the merit to allow one to reformulate the control problem of stabilizing a nonlinear uncertain system, into a simpler control problem: that of stabilizing the auxiliary system (3b)-(3e) forced by a bounded input. In fact, it is sufficient to suitably select the sliding variable σ according to Assumption 1, to be able to determine (3b)-(3e), so that the explicit knowledge of Ψ is not actually necessary to solve the problem.

Remark 1 *Note that, if the sliding variable σ is steered to zero, this directly implies the asymptotic stability of the origin of the closed-loop system* (1) *since, by assumption, the zero dynamics* (3a) *of system* (1)*, transformed via the diffeomorphism $\Phi(x)$, is globally asymptotically stable.*

In the present work, in order to reduce the control effort of the input fed into the plant, relying on the 2-relative degree of system (2), a gain tuning mechanism is combined with the SOSM control strategy giving rise to a new control algorithm.

3 Nonsmooth Switching Line based SOSM Control

We are now in a position to introduce the proposed SOSM control algorithm based on a nonsmooth switching line.

3.1 Design of the Switching Line

Making reference to the SOSM algorithm with optimal reaching presented in [10], let $\alpha_{\rm r}$ be the reduced control amplitude, which is the minimum amplitude of $\ddot{\sigma}$ in presence of the maximum realization of the uncertainty terms when $u = \pm\alpha$ is applied, i.e.,

$$\alpha_{\rm r} = G_{\min}\alpha - F > 0 \,, \tag{9}$$

such that the so-called *switching line* is defined as

$$\widetilde{\mathcal{S}} := \left\{ (\sigma, \dot{\sigma}) \in \mathbb{R}^2 \,:\, \sigma = -\frac{\dot{\sigma}|\dot{\sigma}|}{2\alpha_{\rm r}} \right\} \,. \tag{10}$$

Now, in this paper a new nonsmooth switching line is defined. The idea is based on the fact that the uncertain terms f and g, depending on ξ, are quantized relaying on a partition of the auxiliary system state space into m stripes $\mathcal{B}_i$, $i = 1, \ldots, m$, with

$$\mathcal{B}_i := \left\{ (\sigma, \dot{\sigma}) \in \mathbb{R}^2 \,:\, \underline{\sigma}_i \leq \sigma \leq \overline{\sigma}_i \right\} \tag{11}$$

where $\underline{\sigma}_i < 0$ and $\overline{\sigma}_i > 0$ are constants, with $\underline{\sigma}_i < \underline{\sigma}_{i+1} < 0$ and $\overline{\sigma}_i > \overline{\sigma}_{i+1} > 0$, $i = 1, 2, \ldots, m-1$. Then, instead of

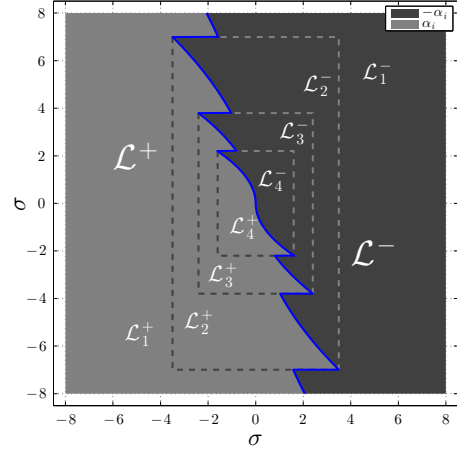
Figure 1. Switching line $\mathcal{S}$ in case of the proposed SOSM algorithm, and the quantization levels

considering unique functions f and g, we can consider the instances of f and g in the stripes, hereafter denoted as

$$f_{\mathcal{B}_i}(x_{\rm r}, \xi) = \{f(x_{\rm r}, \xi) : \xi \in \mathcal{B}_i\} \tag{12}$$
$$g_{\mathcal{B}_i}(x_{\rm r}, \xi) = \{g(x_{\rm r}, \xi) : \xi \in \mathcal{B}_i\} \,,$$

with $i = 1, \ldots, m$.

Let $\partial\mathcal{B}_i$ be the boundaries of the sets $\mathcal{B}_i$, and consider the switching line included in the ith level as

$$\mathcal{S}_i = \mathcal{S}_i^+ \cup \mathcal{S}_i^- := \left\{ (\sigma, \dot{\sigma}) \in \mathbb{R}^2 \,:\, \sigma = -\frac{\dot{\sigma}|\dot{\sigma}|}{2\alpha_{{\rm r},i}} \right\} \tag{13}$$

where $\mathcal{S}_i^+$ and $\mathcal{S}_i^-$ are the sets of points belonging to $\mathcal{S}_i$ with $\dot{\sigma} > 0$ and $\dot{\sigma} < 0$, respectively.

For each level i, in order to define the corresponding switching line, one has

$$\alpha_{{\rm r},i} = \overline{\mathcal{G}}_{\min,i}\alpha_i - \overline{\mathcal{F}}_i > 0 \tag{14}$$

where $\overline{\mathcal{F}}_i$ and $\overline{\mathcal{G}}_{\min,i}$ are positive constants such that

$$|f_{\mathcal{B}_i}(x_{\rm r}, \xi)| \leq \overline{\mathcal{F}}_i \qquad i = 1, \ldots, m. \tag{15}$$
$$g_{\mathcal{B}_i}(x_{\rm r}, \xi) \geq \overline{\mathcal{G}}_{\min,i} > 0$$

Assumption 3 *The bounds $\overline{\mathcal{F}}_i$, $\overline{\mathcal{G}}_{\min,i}$ are known, with $\overline{\mathcal{F}}_i \geq \overline{\mathcal{F}}_{i+1}$, and $\overline{\mathcal{G}}_{\min,i} \leq \overline{\mathcal{G}}_{\min,i+1}$, $i = 1, \ldots, m-1$.*

Then, we select $\alpha_{\mathrm{r},i} > \alpha_{\mathrm{r},i+1}$, and

$$\alpha_i > \frac{\overline{\mathcal{F}}_i}{\underline{\mathcal{G}}_{\min,i}} , \qquad (16)$$

such that $\alpha_i > \alpha_{i+1}$, $i = 1, \ldots, m-1$.

Remark 2 *The inequalities $\alpha_i > \alpha_{i+1}$, $i = 1, \ldots, m-1$ imply that the effort of the control input fed into the plant is reduced when the auxiliary state trajectory moves towards the inner levels.*

Define now the box sets as

$$\mathcal{Z}_i := \mathcal{B}_i \cap \left\{ (\sigma, \dot{\sigma}) \in \mathbb{R}^2 \, : \, \underline{\dot{\sigma}}_i \leq \dot{\sigma} \leq \overline{\dot{\sigma}}_i \right\} \qquad (17)$$

where $\underline{\dot{\sigma}}_i$ and $\overline{\dot{\sigma}}_i$ are obtained as the ordinates of the points given by the intersection between the boundaries of the set $\mathcal{B}_i$ and the switching line of the ith level, i.e., $\left\{ (\overline{\sigma}_i, \underline{\dot{\sigma}}_i) \, (\underline{\sigma}_i, \overline{\dot{\sigma}}_i) \right\} = \mathcal{S}_i \bigcap \partial \mathcal{B}_i$. Finally, the proposed *non-smooth switching line*, as illustrated in Figure 1, is defined as

$$\mathcal{S} := \left(\bigcup_{i=1}^{m-1} \mathcal{S}_i \cap (\mathcal{Z}_i \setminus \mathcal{Z}_{i+1}) \right) \cup (\mathcal{S}_m \cap \mathcal{Z}_m) \qquad (18)$$

while the quantization levels are

$$\mathcal{L}_i = \mathcal{L}_i^+ \cup \mathcal{L}_i^- := \mathcal{Z}_i \setminus \mathcal{Z}_{i+1} \qquad (19)$$

with $i = 1, \ldots, m-1$, $\mathcal{L}_m = \mathcal{Z}_m$, and $\mathcal{L}_i^+$ and $\mathcal{L}_i^-$ being the regions on the left and on the right of the switching line (18), respectively (see Figure 1), i.e.,

$$\begin{aligned} \mathcal{L}_i^+ &:= \left\{ (\sigma, \dot{\sigma}) \in \mathcal{L}_i \, : \, \sigma < -\tfrac{\dot{\sigma}|\dot{\sigma}|}{2\alpha_{\mathrm{r},i}} \right\} \cup \mathcal{S}_i^+ \\ \mathcal{L}_i^- &:= \left\{ (\sigma, \dot{\sigma}) \in \mathcal{L}_i \, : \, \sigma > -\tfrac{\dot{\sigma}|\dot{\sigma}|}{2\alpha_{\mathrm{r},i}} \right\} \cup \mathcal{S}_i^- . \end{aligned} \qquad (20)$$

3.2 The Proposed Control Law

Consider system (3), with the auxiliary state space partitioned as in (19). Assume also that, for $(\sigma, \dot{\sigma}) \in \mathcal{L}_i$, $g(\cdot)$ and $f(\cdot)$ satisfy constraints (15). The control parameters α_i are chosen so as to satisfy the constraint (16). Then, the control law is defined as

$$u(t) = \alpha_i \operatorname{sgn}_{\mathcal{L}_i^\pm} \qquad (21)$$

where

$$\operatorname{sgn}_{\mathcal{L}_i^\pm} = \begin{cases} +1 & \text{if } (\sigma, \dot{\sigma}) \in \mathcal{L}_i^+ \\ -1 & \text{if } (\sigma, \dot{\sigma}) \in \mathcal{L}_i^- \end{cases} \qquad (22)$$

with $i = 1, \ldots, m$. Note that, differently from [21], the control law (21) does not depend on the box sets, but only on the portion of the switching line of the current state-space level. This implies that no sliding mode is enforced on the boundary of the nested box regions.

4 Stability Analysis

With reference to HOSM control algorithms with optimal reaching (see [10]), in this section, a characterization of the proposed algorithm, which consists of control laws that switch over a branch of parabolas arcs, is provided. Specifically, in the next Lemma the existence of m invariant regions $\mathcal{L}_{\mathrm{I},i}$ is demonstrated.

Lemma 1 *Consider the state-space partitioned into the m regions defined in (19). Assume that the bounds (15) hold. Then, all the quantization regions $\mathcal{L}_i$ of the auxiliary state-space contain invariant sets $\mathcal{L}_{\mathrm{I},i} \subset \mathcal{L}_i$ of the form*

$$\mathcal{L}_{\mathrm{I},i} := \mathcal{L}_i \setminus \{ \mathcal{L}_{0,i} \cup \mathcal{L}_{1,i} \} \qquad (23)$$

where

$$\mathcal{L}_{0,i} := \left\{ (\sigma, \dot{\sigma}) \, : \, \sigma > -\frac{\dot{\sigma}|\dot{\sigma}|}{2\alpha_{\mathrm{r},i}} + \overline{\sigma}_i, \, \dot{\sigma} > 0 \right\} \qquad (24)$$

$$\mathcal{L}_{1,i} := \left\{ (\sigma, \dot{\sigma}) \, : \, \sigma < -\frac{\dot{\sigma}|\dot{\sigma}|}{2\alpha_{\mathrm{r},i}} + \underline{\sigma}_i, \, \dot{\sigma} < 0 \right\} . \qquad (25)$$

Moreover, the sets $\mathcal{L}_{\mathrm{I},i}$ are the maximum obtainable domains of attraction for the given switching sets.

PROOF. The proof of the Lemma follows from [24], where it is proved that, assuming $\sigma(t_0) = \sigma_0 \in \mathcal{L}_{0,i}$, in presence of the maximum realization of the uncertainties, the system will move on a parabolic arc, the equation of which is the following

$$\sigma = \frac{\dot{\sigma}|\dot{\sigma}|}{2\alpha_{\mathrm{r},i}} + \overline{\sigma}_i + \varepsilon \qquad (26)$$

with $\varepsilon > 0$ (the case with $\sigma_0 \in \mathcal{L}_{1,i}$ is specular). Then, it is easy to see that starting inside $\mathcal{L}_{0,i}$ or $\mathcal{L}_{1,i}$, this arc intersects the σ-axis outside $\mathcal{L}_i$, and one can conclude that $\mathcal{L}_{\mathrm{I},i}$ are the maximum regions of attraction. $\square$

In the next Theorem, the finite-time stability property of the controlled auxiliary system is proved by exploiting the bang-bang principle [25]. Specifically, an explicit expression for the convergence time of the auxiliary trajectory to the origin of the auxiliary state space is found. Note that, proving this result directly implies that the origin of the state space of the closed-loop system (1) is an asymptotically stable equilibrium point since, by assumption, the zero dynamics of system (2) is asymptotically stable.

Theorem 1 *Given system (1), with the input-output map (2), controlled via the control law (21), such that for the*
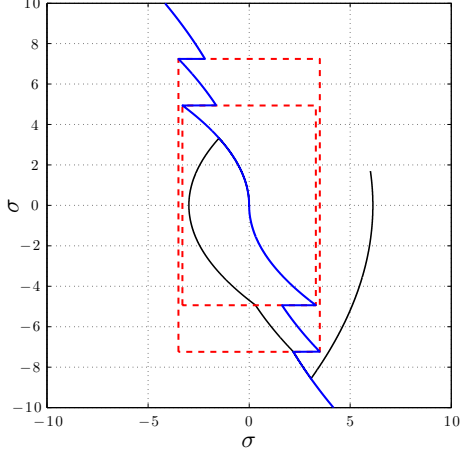
Figure 2. Performance of a perturbed double integrator controlled via the proposed nonsmooth switching line based SOSM algorithm. The bounds of the uncertainties are $\overline{\mathcal{F}}_1 = 8$, $\overline{\mathcal{F}}_2 = 3.5$, $\overline{\mathcal{F}}_3 = 3.3$, $\overline{\mathcal{G}}_{\min,1} = \overline{\mathcal{G}}_{\min,2} = \overline{\mathcal{G}}_{\min,3} = 1$, while the initial conditions are $\xi_0 = [\xi_1(0), \xi_2(0)] = [6, 1.7]^T$

worst possible realization of the disturbance terms (i.e., $f(x_r(t), \sigma(t), t) \equiv -\overline{\mathcal{F}}_i \cdot \mathrm{sgn}(u(t))$ and $g(x_r, \sigma(t), t) \equiv \overline{\mathcal{G}}_{\min,i}$) Lemma 1 holds and the trajectory $(\sigma, \dot{\sigma}) \in \mathcal{L}_{I,i}$, $\forall i = 1, \ldots, m$ and $\forall t \geq t_0$, then, the sliding variable σ and its first time derivative $\dot{\sigma}$, i.e., ξ_1 and ξ_2 in the auxiliary system (3)*, are steered to zero in a finite time t_r.*

PROOF. The proof of the convergence of the system trajectory to the origin of the auxiliary state space $\{\sigma, \dot{\sigma}\}$ does not depend on the number of box sets, but directly follows from the results presented in [10, Theorem 2] and [26]. More specifically, when the system trajectory reaches the inner level $\mathcal{L}_m$, the control law (21) coincides with the second order sliding mode control law with optimal reaching introduced in [10], so that the finite time convergence to the origin of the controlled system is guaranteed as shown in [10, 26].

Consider the auxiliary system (3) with the worst-case realization of the uncertain terms and controlled by applying the control input (21), then the system dynamics can be written as the following double integrator plant, i.e.,

$$\begin{aligned} \dot{\xi}_1(t) &= \xi_2(t) \\ \dot{\xi}_2(t) &= u_r(t) \end{aligned} \tag{27}$$

in which $u_r = (\alpha_{r,i}/\alpha_i)u$ implicitly takes into account the effect of the disturbance terms in the generic region $\mathcal{L}_i$.

Let $(0, 0) = (\sigma(t_r), \dot{\sigma}(t_r)) = (\xi_1(t_r), \xi_2(t_r))$, where t_r is the reaching time to the origin, given a certain initial condition, by using the control law (21). From system (27), integrating backward from τ_i to t, τ_i being the initial time instant within the ith region, since it obeys the Newton's

laws and the control law assumes constant value inside each region $\mathcal{L}_i$, it is possible to get

$$\xi_2(t) = \xi_2(\tau_i) + u_r(t - \tau_i) \tag{28}$$

$$\xi_1(t) = \xi_1(\tau_i) + \xi_2(\tau_i)(t - \tau_i) + \frac{1}{2}u_r(t^2 + \tau_i^2) - u_r\tau_i t \tag{29}$$

which represent the velocity and the corresponding position associated to the sliding variable, respectively.

Consider to start from the external region $\mathcal{L}_1$ and to reach $\mathcal{S}_1^-$. For the sake of brevity only the cases starting on the right of the nonsmooth switching line will be considered, that is the case in which the initial sign of the control law is negative (the opposite case is specular). Three different steps can be distinguished.

Step 1, Case 1 $((\sigma, \dot{\sigma}) \in \mathcal{L}_1^- \setminus \mathcal{S}_1^-)$. The initial control $u_r = -\alpha_{r,1}$ is applied to drive the state along the parabola passing through $(\xi_1(t_0), \xi_2(t_0))$ to the switching line at which time $\tau_{s,1}$ the control is switched to $u_r = \alpha_{r,1}$. Note that for $\xi_2 < 0$ the switching line has the form $\xi_1 = \xi_2^2/(2\alpha_{r,1})$. From equation (28), squaring and dividing for $2\alpha_{r,1}$, one has

$$\frac{\xi_2^2(t)}{2\alpha_{r,1}} = \frac{\xi_2^2(\tau_1)}{2\alpha_{r,1}} + \frac{\alpha_{r,1}}{2}(t^2 + \tau_i^2 - 2\tau_1 t) - \xi_2(\tau_1)(t - \tau_1). \tag{30}$$

Instead from equation (29) one has

$$\xi_1(t) = \xi_1(\tau_1) + \xi_2(\tau_1)(t - \tau_1) - \frac{\alpha_{r,1}}{2}(t^2 + \tau_i^2) + \alpha_{r,1}\tau_1 t. \tag{31}$$

Subtracting (30) to (31), one has

$$\frac{\xi_2^2(\tau_1)}{2\alpha_{r,1}} = \xi_1(\tau_1) + 2\xi_2(\tau_1)(t - \tau_1) - \alpha_{r,1}(t^2 + \tau_i^2) + 2\alpha_{r,1}\tau_1 t. \tag{32}$$

From equation (32), one obtains

$$\begin{aligned} &\alpha_{r,1}t^2 - 2(\xi_2(\tau_1) + \alpha_{r,1}\tau_1)t + \\ &+ \left(2\xi_2(\tau_1)\tau_1 + \alpha_{r,1}\tau_i^2 - \xi_1(\tau_1) + \frac{\xi_2^2(\tau_1)}{2\alpha_{r,1}}\right) = 0 \,. \end{aligned} \tag{33}$$

Solving (33) and considering the positive root to make $\tau_{s,1}$ positive for all $\xi_2(\tau_1)$, the switching time $\tau_{s,1}$ is

$$\tau_{s,1} = \frac{\xi_2(\tau_1)}{\alpha_{r,1}} + \tau_1 + \sqrt{\frac{\xi_1(\tau_1)}{\alpha_{r,1}} + \frac{\xi_2^2(\tau_1)}{2\alpha_{r,1}^2}} \,. \tag{34}$$

From equation (28), one has that

$$\xi_2(\tau_{s,1}) = \xi_2(\tau_1) - \alpha_{r,1}(\tau_{s,1} - \tau_1) \,. \tag{35}$$

Assume now to reach the border of the region $\mathcal{L}_2$, that is

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$\underline{\dot{\sigma}}_2$, with $u_\mathrm{r} = \alpha_{\mathrm{r},1}$ so as to have

$$\xi_2(T_1) = \xi_2(\tau_{\mathrm{s},1}) + \alpha_{\mathrm{r},1}(T_1 - \tau_{\mathrm{s},1}) = \underline{\dot{\sigma}}_2 \;. \qquad (36)$$

Finally, with $\tau_1 = t_0$, the reaching time is

$$T_1 = \frac{\underline{\dot{\sigma}}_2}{\alpha_{\mathrm{r},1}} + \frac{\dot{\sigma}(t_0)}{\alpha_{\mathrm{r},1}} + t_0 + 2\sqrt{\frac{\sigma(t_0)}{\alpha_{\mathrm{r},1}} + \frac{\dot{\sigma}^2(t_0)}{2\alpha_{\mathrm{r},1}^2}} \;. \qquad (37)$$

Step 1, Case 2 ($(\sigma, \dot{\sigma}) \in \mathcal{S}_1^-$). From equation (28), one has that, by applying $u_\mathrm{r} = \alpha_{\mathrm{r},1}$,

$$\xi_2(T_1) = \xi_2(\tau_1) + \alpha_{\mathrm{r},1}(T_1 - \tau_1) \;. \qquad (38)$$

Assume now to reach the border of the region $\mathcal{L}_2$, that is $\underline{\dot{\sigma}}_2$, so as to have

$$\xi_2(T_1) = \xi_2(\tau_1) + \alpha_{\mathrm{r},1}(T_1 - \tau_1) = \underline{\dot{\sigma}}_2 \;. \qquad (39)$$

Then, with $\tau_1 = t_0$, the reaching time is

$$T_1 = \frac{\underline{\dot{\sigma}}_2}{\alpha_{\mathrm{r},1}} - \frac{\dot{\sigma}(t_0)}{\alpha_{\mathrm{r},1}} + t_0 \;. \qquad (40)$$

Step 2 ($(\sigma, \dot{\sigma}) \in \mathcal{L}_2^+$). From equation (28), one has that, by applying $u_\mathrm{r} = \alpha_{\mathrm{r},2}$,

$$\xi_2(T_2) = \xi_2(\tau_2) + \alpha_{\mathrm{r},2}(T_2 - \tau_2) \;. \qquad (41)$$

Assume now to reach the border of the region $\mathcal{L}_3$, that is $\underline{\dot{\sigma}}_3$, so as to have

$$\xi_2(T_2) = \xi_2(\tau_2) + \alpha_{\mathrm{r},2}(T_2 - \tau_2) = \underline{\dot{\sigma}}_3 \;. \qquad (42)$$

Consider that $\tau_2 = T_1$ so that $\xi_2(T_1) = \underline{\dot{\sigma}}_2$, then the reaching time is

$$T_2 = \frac{\underline{\dot{\sigma}}_3}{\alpha_{\mathrm{r},2}} - \frac{\underline{\dot{\sigma}}_2}{\alpha_{\mathrm{r},2}} + T_1 \;. \qquad (43)$$

Then, for all the regions $\mathcal{L}_i$ $i = 2, \ldots, m-1$, it is possible to write

$$T_i = \frac{\underline{\dot{\sigma}}_{i+1}}{\alpha_{\mathrm{r},i}} - \frac{\underline{\dot{\sigma}}_i}{\alpha_{\mathrm{r},i}} + \sum_{j=1}^{i-1} T_j \;. \qquad (44)$$

Step 3 ($(\sigma, \dot{\sigma}) \in \mathcal{L}_m^+ \setminus \mathcal{S}_m^+$). Since the target is the origin, the net time is

$$T_m = -\frac{\dot{\sigma}(T_{m-1})}{\alpha_{\mathrm{r},1}} + T_{m-1} + 2\sqrt{-\frac{\sigma(T_{m-1})}{\alpha_{\mathrm{r},m}} + \frac{\dot{\sigma}^2(T_{m-1})}{2\alpha_{\mathrm{r},m}^2}} \;. \qquad (45)$$

Note that $\dot{\sigma}(T_{m-1}) = \overline{\dot{\sigma}}_m$, so that one has

$$T_m = -\frac{\overline{\dot{\sigma}}_m}{\alpha_{\mathrm{r},m}} + T_{m-1} + 2\sqrt{-\frac{\sigma(T_{m-1})}{\alpha_{\mathrm{r},m}} + \frac{\overline{\dot{\sigma}}_m^2}{2\alpha_{\mathrm{r},m}^2}} \;. \qquad (46)$$

Finally, one can conclude that the convergence to the origin, in the worst case of uncertainty, occurs in a finite time T_m. In general, this implies a finite convergence time $t_\mathrm{r} \leq T_m$, which concludes the proof. $\qquad \square$

Remark 3 *Note that, the finite time t_r is given by summing the time intervals needed to pass from the external region to the inner one. Given T_i, $i = 1, \ldots, m-1$ as the minimum time inside each quantization region, the whole convergence time is not the minimum one but an upperbound in case of the maximum realization of the uncertainty. Figure 2 shows the performance of a perturbed chain of integrators in the case of the worst realization of the uncertainties. The state trajectory belongs to each region at most in one switch, and the convergence time can be explicitly estimated according to Theorem 1, resulting in $4.403\,\mathrm{s}$ compared to the actual one equal to $4.404\,\mathrm{s}$.*

5 Illustrative Example

In this section, in order to assess the properties of the proposed nonsmooth SOSM control strategy, an illustrative example is briefly discussed. Consider the nonlinear uncertain system,

$$\begin{cases} \dot{x}_1(t) = x_2(t) + x_3(t) \\ \dot{x}_2(t) = x_3(t) \\ \dot{x}_3(t) = 0.5\cos(x_2(t)) + u(t) \\ y(t) \;= x_1(t) \end{cases} \qquad (47)$$

where the initial condition is $x(0) = [6\ 0.1\ -1.8]^T$. Then, the system is stabilized by choosing the sliding variable $\sigma(x(t))$ as the controlled variable $y(t)$. Note that, (47) has a uniform relative degree equal to 2, and it admits a global normal form, i.e., there exists the global diffeomorphism

$$\Phi(x(t)) = \begin{pmatrix} x_2(t) \\ x_1(t) \\ x_2(t) + x_3(t) \end{pmatrix} = \begin{pmatrix} x_\mathrm{r}(t) \\ \xi_1(t) \\ \xi_2(t) \end{pmatrix}$$

such that

$$\begin{cases} \dot{x}_\mathrm{r}(t) = x_3(t) \\ \dot{\xi}_1(t) = \xi_2(t) \\ \dot{\xi}_2(t) = x_3(t) + 0.5\cos(x_2(t)) + u(t) \\ y(t) \;= \xi_1(t) \\ \xi(t_0) = \xi_0 \end{cases} \qquad (48)$$

$\xi_0 = [6\ -1.7]^T$ being the initial condition. Relying on systems (47)-(48) it is possible to set the bounds in (5)-(7) equal to $F = 8$, $G_\mathrm{min} = G_\mathrm{max} = 1$. To perform the simulation tests, the Euler solver is used with a numerical integration
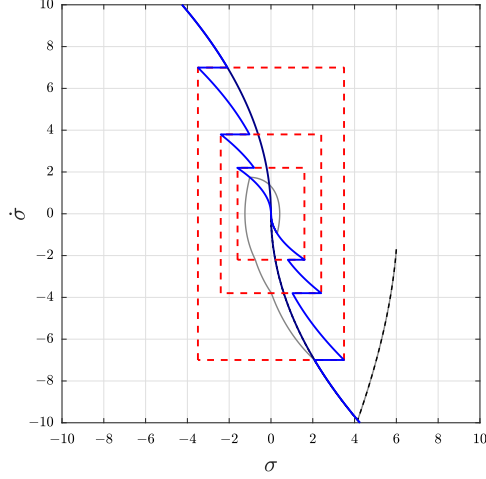
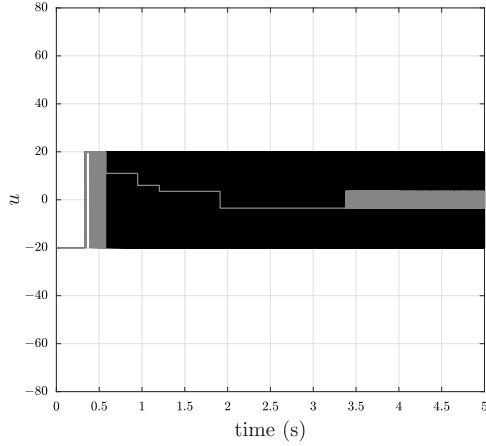Figure 3. Auxiliary state-space trajectory

Figure 4. Time evolution of the control law u when the new algorithm is applied (gray) and when the control algorithm in [10] is used (black)
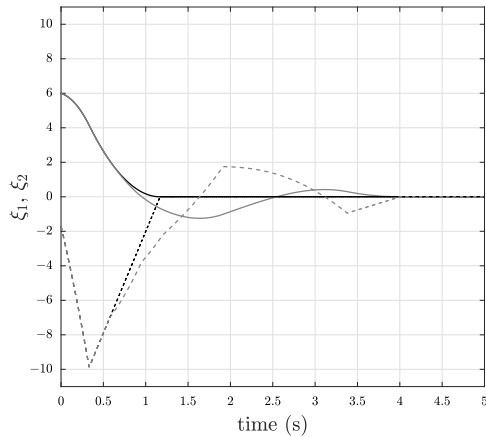

Figure 5. Time evolution of the state ξ_1 (solid line) and ξ_2 (dashed line) when the new algorithm is applied (gray) and when the control algorithm in [10] is used (black)

step equal to 0.0001 s. Note that, this choice of σ implies that the solved problem is an output feedback control problem (see, [2,27,28] as examples of sliding mode control solutions for this problem). According to the procedure described in Section 3, we assume to known that the uncertainty is quantized in 4 levels, so that $m = 4$ with $\overline{\mathcal{F}}_1 = 8$, $\overline{\mathcal{F}}_2 = 4$, $\overline{\mathcal{F}}_3 = 3$, $\overline{\mathcal{F}}_4 = 2$, $\underline{\mathcal{G}}_{\text{min},1} = \underline{\mathcal{G}}_{\text{min},2} = \underline{\mathcal{G}}_{\text{min},3} = \underline{\mathcal{G}}_{\text{min},4} = 1$, and the corresponding nonsmooth switching line (see Figure 3) is derived as in (18), with $\alpha_1 = 20$, $\alpha_2 = 11$, $\alpha_3 = 6$ and $\alpha_4 = 3.5$, respectively.

It is useful to compare the proposed algorithm with the already published control law with optimal reaching in [10]. To this end, we consider three indexes: i) the root mean square (RMS) value of the sliding variable, σ_{RMS}; ii) the control effort E_c; iii) the reaching time t_r.

The trajectories of the auxiliary system, equal to (48), are reported in Figure 3, while Figures 4 and 5 show the time evolution of the control signal and the response curves of the auxiliary system states in case of the new algorithm and in case of the control law proposed in [10], respectively. More specifically, in Figures 4 the main advantage of the proposed approach with respect the other one clearly appears, in terms of control effort which is progressively reduced through the quantized levels.

One can note that, while the RMS value of the sliding variable is similar (3.4844 and 3.2197), the control effort is strongly reduced by using the new control law (8.1424 with respect to 20). Of course, this implies that the convergence finite time is longer than that of the control algorithm in [10] (3.989 s with respect to 1.145 s). Hence, although the reaching time is increased, the sliding mode is ensured even progressively reducing the gain through the levels (see Figure 4), which is beneficial in many practical mechanical and electromechanical cases.

6 Conclusions

A new second order sliding mode control algorithm based on a nonsmooth switching line is proposed in the paper. Starting from a quantization of the uncertainty affecting the system, it is possible to define nested sets which allow to determine a nonsmooth line of attraction. A piece-wise tuning of the control amplitude can therefore be easily realized, providing an extremely compact control law, which resembles the classical first order sliding mode control law. The convergence of the auxiliary system state trajectory is analyzed in the paper and an upperbound of the convergence time in case of the worst realization of the uncertainty is provided.

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