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Very large amplitude cyclotron wave propagation in a cold collisionless magnetised plasma

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Abstract : The relativistic equation of motion of an electron in the presence of a very large amplitude electromagnetic wave and a uniform magnetic field is solved. Dispersion relation for the wave propagation is obtained and its dependance on the wave amplitude is studied. It is shown that in the electron cyclotron frequency range, the large amplitude waves have smaller guidance angle as compared to the waves of negligible amplitudes. The results are discussed, qualitatively, for the electromagnetic wave propagation in pulsar magnetosphere.

Keywords : Cyclotron waves, dispersion, wave guidance, pulsar magnetosphere.

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For small amplitude electromagnetic waves, dispersion characteristics and their guidance has been studied [1]. When amplitudes of waves become large, the nonlinear effects become important [2]. Among nonlinear effects, amplitude dependance of wave propagation is also one of the important topics to be studied.

Akhiezer and Polovin [3] and Akhiezer *et al* [4] have studied the nonlinear effects due to relativistic motions in a plasma. The solution of relativistic equation of motion suggests, among other aspects, the electron motions in the direction of wave propagation and transverse to it are coupled. Purely transverse or purely longitudinal waves are highly restricted solutions. Similar effects are seen by Kaw and Dawson [5] in the absence of background magnetic field, when waves of relativistic amplitudes propagate.

Here in this note, we analyse the nonlinear effects due to large amplitudes of waves by solving the relativistic equation of motion of an electron in the presence of a uniform magnetic field. Assuming the plasma to be cold, neglecting the temperature effects on the wave propagation, we obtain the dispersion relation. The ions are treated as to provide the charge neutralising background, which limits the largeness of amplitude of a wave. Effect of a wave amplitude on its propagation and its guidance is presented at the end.

Our new format of references has been adopted in this Note.

Basic equations

The plasma is assumed to be uniform, homogeneous and has a uniform magnetic field $\vec{B}_o$ along the z axis of the right hand co-ordinate system. The electron velocity $\vec{v}$ and density n are given by the relativistic equation of motion.

$$\frac{d\vec{P}}{dt} + (v.\nabla)\vec{P} = -e\vec{E} - \frac{e}{c}(\bar{v}\times\bar{B})$$

$$\vec{P} = m_o\bar{v}/(1 - v^2/c^2)^{1/2} \tag{1}$$

(where $\vec{P}$ is the relativistic momentum and m_o is the electron rest mass) and the continuity equation

$$\frac{\partial n}{\partial t} + \nabla.n\vec{v} = 0 \tag{2}$$

The Maxwell's equations for electric field $\vec{E}$ and magnetic field $\vec{B}$ of the wave are

$$\nabla\times\bar{E} = -\frac{1}{c}\frac{\partial B}{\partial t} \tag{3a}$$

$$\nabla\times\bar{B} = \frac{4\pi}{c}J + \frac{1}{c}\frac{\partial E}{\partial t} \tag{3b}$$

$$\nabla.\bar{E} = -4\pi e\,(n_o - n) \tag{3c}$$

$$\nabla\bar{B} = 0 \tag{3d}$$

here n_o is the equilibrium density.

For convenience, for wave like solutions, propagating along z axis given by the parameter $(z - ut)$, we use the following transformed variables

$$\xi = \frac{\omega_p}{c}(z - ut)\,;\ \beta = u/c\,;\ \bar{\rho} = \vec{P}/m_oc \tag{4}$$

where $\omega_p = (4\pi n_o e^2/m_o)^{1/2}$ is the electron plasma frequency and $\mu = c/u = 1/\beta$ is the refractive index. The subscripts x, y and z denote components along respective axes.

In terms of transformed variables (4) the continuity equation can be integrated, once, to give

$$n = n_o\beta(1+\rho^2)^{1/2}/[\beta(1+\rho^2)^{1/2} - \rho_z] \tag{5}$$

such that in the limit $\rho_z \to 0$, $n = n_o$ which is the requirement for existance of purely transverse electromagnetic waves. Since $n \geq 0$ we always find that $\rho_z < \beta(1+\rho^2)^{1/2}$.

The Maxwell's equations, in terms of variables (4) are found to be

$$\frac{dE_x}{d\xi} = \frac{4\pi n_o ec\beta^2\rho_x}{\omega_p(\beta^2-1)[\beta(1+\rho^2)^{1/2} - \rho_z]} \tag{6a}$$

$$\frac{dE_y}{d\xi} = \frac{4\pi n_o ec\beta^2\rho_y}{\omega_p(\beta^2-1)[\beta(1+\rho^2)^{1/2} - \rho_z]} \tag{6b}$$

$$\frac{dB_x}{d\xi} = \frac{4\pi n_o ec\beta\rho_x}{\omega_p(1-\beta^2)[\beta(1+\rho^2)^{1/2} - \rho_z]} \tag{6c}$$

$$\frac{dB_y}{d\xi} = \frac{4\pi n_0 ec\beta \rho_y}{\omega_p (\beta^2 - 1) [\beta (1 + \rho^2)^{1/2} - \rho_z]} \tag{6d}$$

To obtain the set of eqs. (6), we treate ($\beta - v_{z/c}$) as a constant. The components of equations of motion in same approximations, are

$$\frac{\partial^2 \rho_x}{\partial \xi^2} + \frac{\beta}{(\beta^2 - 1)} \frac{\rho_x}{[\beta (1 + \rho^2)^{1/2} - \rho_z]} - \frac{\omega_e (1 + \rho^2)^{1/2}}{\omega p [\beta (1 + \rho^2)^{1/2} - \rho_z]} \frac{\partial}{\partial \xi} \frac{\rho_y}{(1 + \rho^2)^{1/2}} = 0 \tag{7a}$$

$$\frac{\partial^2 \rho_y}{\partial \xi^2} + \frac{\beta}{(\beta^2 - 1)} \frac{\rho_y}{[\beta (1 + \rho^2)^{1/2} - \rho_z]} + \frac{\omega_e (1 + \rho^2)^{1/2}}{\omega p [\beta (1 + \rho^2)^{1/2} - \rho_z]} \frac{\partial}{\partial \xi} \frac{\rho_x}{(1 + \rho^2)^{1/2}} = 0 \tag{7b}$$

$$\frac{\partial^2 \rho_z}{\partial \xi^2} [\beta \rho_z - (1 + \rho^2)^{1/2}] + \frac{\rho_z}{[\beta (1 + \rho^2)^{1/2} - \rho_z]} = 0 \tag{7c}$$

In the equation set (7), the nonrelativistic electron gryofrequency is denoted by $\omega_e = eB_0/m_0 c$. These equations are identical to the equations of Akhiezer *et al* [4]. In the limit $\omega_e = 0$, these equations reduce to eqs. (3) of Kaw and Dawson [5].

The eqs. (7) are nonlinear and also show coupling between ρ_x, ρ_y and ρ_z. In fact general solutions are coupled longitudinal-cum-transverse modes, and therefore, it is hard to solve them analytically. For some special cases, the exact solutions can be obtained [3].

Dispersion relation :

Pure transverse wave solutions of eqs (7a-c) are possible when we put $\rho_z = 0$ which gives ρ^2 = constant. From the nature of the eqs. (7a-c), we substitute

$$\rho_x = \rho \cos (kz - \omega t) = \rho \cos \left(\frac{ck}{\omega_p} \xi\right) \tag{8}$$

$$\rho_y = \pm \rho \sin (kz - \omega t) = \pm \rho \sin \left(\frac{ck}{\omega_p} \xi\right)$$

and obtain the dispersion relation

$$\frac{c^2 k_+^2}{\omega^2} = 1 - \frac{\omega_n}{\omega \left[\omega (1 + \rho^2)^{1/2} \mp \omega_e \right]}$$

Using (8), the eqs. (6a-c), can be integrated once. Assuming the plane wave like solutions, the value of $E^2 = < E_x^2 + E_y^2 >$ (where < > denote the average) is found to be

$$\frac{e^2 E^2}{m_0^2 c^2} = \frac{\rho^2}{(1 + \rho^2)} \left[\omega (1 + \rho^2)^{1/2} \mp \omega_e \right] \tag{9}$$

From (9), we see that the relativistic factor is now defined in terms of $<E^2>$.

To obtain ρ^2 interms of the square of wave amplitudes, we use the approximations :
(a) when $\rho^2 >> 1$, we find

$$(1 + \rho^2)^{1/2} \quad \left[\frac{eE_0}{m_0 c} \pm \omega_e\right] / \omega \tag{10}$$

This relation is used by Papuashvili *et al* [6] to discuss the propagation of ultrarelativistic cyclotron waves in a cold collisionless plasma. (b) When $\rho^2 < 1$ by neglecting ρ^4 and higher order terms, we get from (9)

$$\rho^2 = \frac{e^2E^2/m_0^2c^2}{(\omega \pm \omega_e)^2} \quad \frac{1}{1 - \frac{e^2E^2/m_0^2c^2}{(\omega \pm \omega_e)^2}} \qquad (11)$$

Substituting the value given by (10), we get the dispersion relation

$$\mu_\pm^2 = 1 - \frac{\omega_p^2}{\omega\left[\frac{eE}{m_0 c}\right]}$$

The ρ^2 from (11) results in the dispersion relation

$$\mu_\pm^2 = 1 - \frac{\omega_p^2}{\omega\left[\omega\left(1 + \frac{e^2E^2/m_0^2c^2}{(\omega \pm \omega_e)^2} - \left[\frac{e^2E^2/m_0^2c^2}{(\omega \pm \omega_e)^2}\right]\right) \pm \omega_e\right]}$$

which in the limit $e^2E^2/m_0^2c^2\,(\omega \pm \omega_e)^2 < 1$ reduces to

$$\mu_\pm^2 = 1 - \frac{\omega_p^2}{\omega\left[\omega\left(1 + \frac{e^2E^2/m_0^2c^2}{(\omega \pm \omega_e)^2}\right) \pm \omega_e\right]} \qquad (12)$$

On the basis of eq. (12), we discuss the guidance of electron cyclotron waves.

The guidance of waves :

The electromagnetic cyclotron waves of frequency ω and wave number $\vec{k}$ (ω_{pi}, $\omega_i < \omega < \omega_e$, ω_{pe} where ω_{pi}, ω_{pe} are plasma frequencies and ω_i, ω_e are cyclotron frequencies) are known to be guided along magnetic field lines [7, 8]. In the case of ultra-relativistic electron plasma, it is known that two types of circularly polarized waves with different refraction coefficient can exist and conditions for such wave excitation have been discussed [6].

For waves propagating at an angle θ with respect to the magnetic field $\vec{B}_0$ the angle, α, made by the group velocity direction with the magnetic field is given by

$$\tan\alpha = \frac{1}{k}(\partial\omega/\partial\theta)_k / (\partial\omega/\partial k)_\theta$$

This angle, α_r, for finite amplitude waves, from the relation (12), with the negative sign, gives

$$\tan\alpha_r = \frac{c^2k^2\omega_e \sin\theta - \omega^2\omega_e \sin\theta}{2c^2k^2\,[\omega(1+\rho^2)^{1/2} - \omega_e\cos\theta]}$$

In the limit $\rho \to 0$ i.e. $E \to 0$, we get α_0 suggesting that the group velocity direction makes smaller angle than the phase velocity direction. The ratio

$$\frac{\tan \alpha_r}{\tan \alpha_0} = 1 - \frac{\omega p^2}{2(\omega - \omega_e \cos \theta)} \qquad \text{when } \rho^2 < 1$$

is always less than unity for $\rho \neq 0$. This shows that waves of finite amplitudes make small angle with respect to magnetic field, thereby they are more efficiently guided along the magnetic field lines.

Another interesting aspect off-angle propagation is that the electron cyclotron waves have resonance at $(\omega - k_{\parallel} v_{\parallel} + N\omega_e) = 0$, (where $n = 0, \pm 1, \pm 2 \ldots$). Such waves will have Landau resonance for $N = 0$ along with the cyclotron resonance for $N = -1$. This phenomenon, in the case of finite amplitude waves will be interesting because the resonant velocity $v_{\parallel}$ is related to $v_{\perp}$. Depending upon the electron distribution function, Landau resonance leads to damping of waves whereas the cyclotron resonance can lead to damping or growth of the wave [9].

Pulses from pulsars are interpreted interms of plasma collective effects, instability and nonlinear processes etc. The synchrotron emissions by relativistic particles in the presence of strong magnetic fields and the antenna theory of radio emissions are the two processes which seem appropriate [10]. These radiations propagate along the magnetic field lines. When observed intensities extrapolated backward, the waves have very large amplitudes. Therefore, the qualitative analysis presented here is appropriate.

Morever, the fast rotation rate and strong magnetic fields generate non-neutral plasma in the magnetosphere of the pulsar. The charges are redistributed. The electrons and negatively charged particles occupy the polar regions, positive charges equatorial region and there is a vacuum separating these two [11]. Wave propagation studies presented here are appropriate to one component plasma. The radiation passing through the magnetised plasma, in the restricted frequency region, is guided along the magnetic field lines by the large amplitudes of the waves. This also supports the basic idea of electromagnetic wave propagation along the magnetic field lines.

From the analysis we see that one of the non-linear effects introduced by a large amplitude is the dependance of the dispersion relation on wave amplitudes. There is a guidance of waves along the magnetic field lines. The results are encouraging and needs further detailed studies.

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