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On relativistic wave equation for particles of integer spins

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Abstract : The representation given by Mathews to describe particles of integer spins is modified in such a way that the Hamiltonian does not contain the matrix C_0 which is a null matrix. An extreme relativistic limit of the modified Hamiltonian is also obtained

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1. Introduction

Following the work of Weaver *et al* [1] regarding the relativistic wave equation for particles of arbitrary spin, Mathews [2] obtained two Hamiltonians of the form,

$$H_1 = \sum_{\mu=\frac{1}{2}}^{S} E\, C_\mu \tanh 2\mu\theta + \beta \sum_{\mu=\frac{1}{2}}^{S} E\, B_\mu \operatorname{sech} 2\mu\theta \tag{1.1}$$

$$H_2 = \sum_{\mu=0}^{S} E\, C_\mu \coth 2\mu\theta + \beta \sum_{\mu=0}^{S} E\, C_\mu \operatorname{cosech} 2\mu\theta \tag{1.2}$$

$$\tanh \theta = p/E.$$

Of the above two, former is quantizable for half-integer spins and the latter is quantizable for inter spins. The integer Spin Hamiltonian is rather peculiar [3].

The matrix C_0 occurring in H_2 is a null matrix.

This can be seen from the relation,

$$C_\mu = \Lambda_\mu - \Lambda_{-\mu} \tag{1.3}$$

$$B_\mu = \Lambda_\mu + \Lambda_{-\mu} \tag{1.4}$$

where Λ_μ is a certain projection operator possessing the property,

$$\Lambda_\mu \Lambda_\upsilon = \Lambda_\mu\, \delta\mu\upsilon. \tag{1.5}$$

By virtue of the above relation, we get

$$C_\mu C_\upsilon = (\Lambda_\mu \Lambda_{-\mu})(\Lambda_\upsilon - \Lambda_{-\upsilon}) = B_\mu \delta\mu\upsilon, \tag{1.5a}$$

$$C_\mu B_\upsilon = (\Lambda_\mu - \Lambda_{\ \mu})(\Lambda_\upsilon + \Lambda_{\ \upsilon}) = C_\mu \delta\mu\upsilon, \tag{1.5b}$$

$$B_\mu B_\upsilon = (\Lambda_\mu + \Lambda_{-\mu})(\Lambda_\upsilon + \Lambda_{-\upsilon}) = B_\mu \delta\mu\upsilon. \tag{1.5c}$$

For $\mu = 0$, we have from eqs. (1.3-1.5)

$$C_0 = 0$$

The occurrence of C_0 in the RHS of eq. (1.2), makes the first term in the expansion i.e. E C_0 CothO indeterminate since $C_0 = 0$ and CothO $\rightarrow \infty$.

In this paper, an attempt has been made to modify the Hamiltonian H_2 and other relevant operators in such a way that they do not contain the matrix C_0 and consequently the above mentioned indeterminate term. This elimination of C_0 gives us a new representation which we call the modified Mathews representation (MMR). The MMR has one distinct advantage. The extreme relativistic representation which one obtains from MMR is free from the null matrix C_0. Earlier work regarding the extreme relativistic representation [4] contained the matrix C_0 and its presence was considered to be an unhappy feature of the work.

The broad outline of the paper is as follows :

The operator (S) linking the Hamiltonian H_2 and the Hamiltonian $H_c = \beta E$ is available [5]. (Here C stands for Canonical representation). Taking this operator S and eliminating the matrix C_0 from this operator, a new operator R is constructed. By using the operator R, a Hamiltonian H_2' is obtained from H_c by a similarity transformation. The Hamiltonian H_2' thus obtained, remains free from the matrix C_0. Relevant expressions for the position, spin and boost operators are obtained. Finally, the extreme relativistic limit of the Hamiltonian H_2' is obtained.

2. Operator S linking H_2 and H_c

The Operator S linking the Hamiltonian H_2 and H_c is of the form

$$S = \sum_{\mu=0}^{S} \left[\delta^{(+)}{}_\mu (I + \beta) B_\mu + \sigma_3 \delta^{(-)}\mu (I + \beta) C_\mu \right] \tag{2.1}$$

$$S^{-1} = \sum_{\mu=0}^{S} \left[\beta^{(+)}{}_\mu (I + \beta) B_\mu + \sigma_3 \beta^{(\ )}{}_\mu (I + \beta) C_\mu \right], \text{ where} \tag{2.2}$$

$$\delta^{(+)}{}_\mu = (m_0/8E)^{1/2} m^{\ \mu}{}_0 (E + p)^{\ \mu} \left[(E + p)^{2\mu} \pm m^{2\mu}{}_0\right] \tag{2.3}$$

$$\beta^{(\pm)}{}_\mu = (E/2m_0)^{1/2} m^\mu{}_0 (E + p)^\mu \left[(E + p)^{2\mu} \pm m^{2\mu}{}_0\right]^{-1} \tag{2.4}$$

$$\sigma_3\beta + \beta\sigma_3 = II; \ \sigma^2{}_3 = \beta^2 = I. \tag{2.4a}$$

Eliminating the matrix C_0 from the above expression, let us define operators R and R^{-1} such that

$$R = \sum_{\mu=1}^{S} \left[\delta^{(+)}{}_{\mu} (I + \beta) B_{\mu} + \sigma_3 \delta^{(-)}{}_{\mu} (I + \beta) C_{\mu} \right] + k_1 B_c \tag{2.5}$$

$$R^{-1} = \sum_{\mu=1}^{S} \left[\beta^{(+)}{}_{\mu} (I + \beta) B_{\mu} + \sigma_3 \beta^{(-)}{}_{\mu} (I + \beta) C_{\mu} \right] + k_2 B_o . \tag{2.6}$$

The consistance of eq. (2.5) with eq. (2.6) demands that $k_1 k_2 = 1$. Of the many possibilities, we choose the simplest one, that is, $k_1 = k_2 = 1$. Accordingly we get,

$$R = \sum_{\mu=1}^{S} \left[\delta^{(+)}{}_{\mu} (I + \beta) B_{\mu} + \sigma_3 \delta^{(-)}{}_{\mu} (I + \beta) C_{\mu} \right] + B_o \tag{2.7}$$

$$R^{-1} = \sum_{\mu=1}^{S} \left[\beta^{(+)}{}_{\mu} (I + \beta) B_{\mu} + \sigma_3 \beta^{(-)}{}_{\mu} (I + \beta) C_{\mu} \right] + B_o . \tag{2.8}$$

Let us define a wave-function ψ such that

$$\psi' = R \psi_c \tag{2.9}$$

where the wave function satisfies the Schrödinger equation

$$i \frac{\partial \psi_c}{\partial t} = \beta E \psi . \tag{2.10}$$

If G_c represents a poincare group generator relevant for the representation provided by ψ_c, the corresponding generator G' for the wave function ψ' is given by the similarity transformation,

$$G' = R G_c R^{-1} . \tag{2.11}$$

Replacing G_c by $H_c = \beta E$, we obtain the Hamiltonian H'_2 by the relation

$$H'_2 = R \beta E R^{-1} . \tag{2.12}$$

By virtue of eq. (2.7) and (2.8) we get

$$H'_2 = \sum_{\mu=1}^{S} E C_{\mu} \coth 2\mu\theta + \beta \sum_{\mu=1}^{S} E C_{\mu} \operatorname{cosech} 2\mu\theta + \sigma_3 E B_o . \tag{2.13}$$

As desired, The Hamiltonian H'_2 does not contain the null matrix C_o. It may be noted here that in eq. (2.13) the summation runs from $\mu = 1$ and not from $\mu = 0$ as in eq. (1.2)

3. Lorentz invariant scalar product and observable in MMR

In the ψ_c representation, the Lorentz invariant scalar product is of the form

$$(\psi_c \cdot \psi_c) = \int \psi^{+}{}_c \, \sigma_3 \, \psi_c \, d^3x . \tag{3.1}$$

Transforming the above equation to ψ' representation we have

$$(\psi', \psi') = \int \psi'^{+} M' \psi' d^3x$$

where

$$M' = (R^{-1})^{+} \sigma_3 R^{-1}$$

$$= (1/2m_o) [\beta H'_2 + H'_2 \beta] + \sigma_3 B_o. \quad (3.2)$$

The expectation value of an operator O in the state is defined as

$$<O> = \int \psi'^{+} M' O \psi' d^3x \quad (3.3)$$

for O to be real, we should have

$$M'O = O^{+} M'^{+}. \quad (3.4)$$

The conventional position operator x and the spin operator S do not satisfy the above requirement. However, by exploiting the eq. (2.11), suitable well behaving expressions for position and spin observables can be obtained. Replacing G_c by x and S in the eq. (2.11) we get

$$x = R \chi R^{-1}. \quad (3.5)$$

$$S' = R S R^{-1}. \quad (3.6)$$

Since x and S are observables in the ψ_c representation x' and S' are observables in the ψ' represention. By replacing R and R^{-1} by their equivalent expressions, the right hand sides of the eq. (3.5) and (3.6) can be evaluated. For spin 1, they are of the form,

$$x' = \chi + \{m_o + 2E - (2m_o E)^{1/2}\}/(8m_o E)^{1/2} (I + \beta) B_1 \sigma_3 \tau/p$$
$$+ (m_o/8E)^{1/2} (I + \beta) \sigma_3 \tau/p - B_o \sigma_3 \tau/p$$
$$- i (I - \beta) C_1 \lambda/2p + i (\sigma_3/p^2) (m_o E/8)^{1/2} (I + \beta) B_o \lambda$$
$$+ \sigma_3 (I + \beta) C_1 \sigma_3 \tau/(2m_o E)^{1/2} - i p B_1/p^2 + i\beta B_1 p(m_o^2 + E^2) / 2p^2E^2. \quad (3.7)$$

$$S' = S - C_1 i \sigma_3 (S \times p) /p - B_o S + (m_o/8E)^{1/2} (I + \beta) B_o S$$
$$+ i [\{2p^2 m_o E\}/p(8m_o E)^{1/2}] (I - \beta) B_1 (S \times p) /p$$
$$- (m_o E/8p^2)^{1/2} i (I - \beta) (S \times p) /p$$
$$+ (E/2m_o)^{1/2} (I + \beta) C_1 i \sigma_3 (S \times p) /p. \quad (3.8)$$

Here, $\tau = (\lambda \times p) /p$ and $\lambda = \sigma_3 S$.

4. Boost operator

A representation is said to be completely defined only when we have determined expressions for all the operators of Poincare group in the representation. The Poincare group operators are : Hamiltonian (H), momentum (p), the angular momentum (J) and the boost operator (k).

For the modified Mathews representation, the Hamiltonion is given in the equation (2.13). The angular momentum J is given as $J = (\alpha \times p) + S$,

where S in the Spin operator.

To complete the description of the representation, the expression for the operator k is to be determined. The boost operator k links the wave functions ϕ' and ψ' in two different inertial frames. If ϕ' is the wave function at a space-time point in one frame, the wave function ψ' in another Lorentz frame is given as

$$\psi' = (I + i\, d\mathbf{v}.\mathbf{k}')\, \phi'. \tag{4.1}$$

Here, $d\mathbf{v}$ is the infinitesimal relative velocity between the two frames. An expression for the Boost operator can be obtained from the similarity transformation

$$k' = R\, k_c R^{-1}. \tag{4.2}$$

where

$$k_c = t\, p - \frac{1}{2}\,(x\, H_c + H_c\, x) + \beta\,(S \times p)/(E + m_o)$$

is the boost operator in the ψ_c representation. For the Spin 1 case, K' is of the form,

$$k' = tp - \frac{1}{2}\,(x' H'_2 + H'_2 x') + H'_2(S \times p)/E\,(E + m_o).$$

5. Extreme relativistic representation

A representation suitable for describing Dirac particles with extreme relativistic energies was first studied by Mendlowitz [6] and later by Cini and Touschek [7]. The Cini-Touschek representation given for Spin - $\frac{1}{2}$ was generalised by one of us [4] for arbitrary Spin. For any Spin-S; the extreme relativistic Hamiltonian is of the form

$$H_E = \sum_{\mu = 0 \text{ or } 1/2} E\, C_\mu. \tag{5.1}$$

For integer spin case, the Hamiltonian H_E contains the null matrix C_o and thus exhibits a drawback.

In this section, we obtain an extreme relativistic Hamiltonian H'_E by projecting the MMR to the extreme relativistic limit. For the integer Spin case, this Hamiltonian H'_E does not contain the matrix C_o and hence is a better candidate than the one obtained earlier by one of us.

To project the Hamiltonian H'_2 to the extreme relativistic limit, we recall the similarity transformation (2.12)

$$H'_2 = R\, \beta\, E\, R^{-1}. \tag{5.2}$$

In the extreme relativistic limit, the ratio $m_o/p \to 0$. Injecting this high momentum approximation into the operators R and R^{-1} we have

$$R\,(m_o/p \to 0) \to T = \sum_{\mu=1}^{s} \delta_\mu \,\{(I+\beta)\,B_\mu + \sigma_3\,(I+\beta)\,C_\mu\} + B_o \quad (5.3)$$

$$R^{1}\,(m_o/p \to 0) \to T^{-1} = \sum_{\mu=1}^{s} \beta_\mu \,\{(I+\beta)\,B_\mu + \sigma_3\,(I+\beta)\,C_\mu\} + B_o \quad (5.4)$$

$$\delta_\mu = (m_o/8p)^{1/2}\,(2p/m_o)^{\mu};\ \beta_\mu = (p/2m_o)^{1/2}\,(m_o/2p)^{\mu}.$$

The extreme relativistic Hamiltonian can now be obtained as,

$$H'_E = T\beta E T^{-1} = \sum_{\mu=1}^{s} E\,\{C_\mu\} + \sigma_3\,B_o\,E. \quad (5.5)$$

Other operators (G'_E) relevant for this extreme relativistic representation can be obtained from the similarity transformation.

6. Conclusion

The representation given by Mathews to describe particles of integer spins has been modified, in such a way that the Hamiltonian does not contain the null matrix C_0. By projecting the modified Hamiltonian to the extreme relativistic limit, we have obtained an extreme relativistic representation. Now it remains to be seen whether or not the modified Hamiltonian (H'_2) is quantizable. Work in this direction is in progress and will be reported in a future publication.

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