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On the theory of torsional wave propagation in a solid elastic cylinder

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A new theory has been developed for the torsional wave propagation in a solid elastic cylinder. The method consists of an expansion of the displacement function in terms of suitable orthogonal polynomial. The boundary condition yields the frequency equation which reproduces the multimode system to a remarkable accuracy. Some numerical evaluations of the first few modes of the dispersion system and of the consecutive cut-off frequencies are also produced. The present work has been successfully utilized in solving an inhomogeneous elastic wave-guide problem.

1. Introduction

Extraction of information from the Pochhammer equation for the dispersion of torsional wave propagation in a solid elastic cylinder was accomplished by Owen (1950) and Davis (1956). The characteristic equation is simple in appearance but its transcendental nature hinders our inquisitions in a number of ways. Moreover in some cases this equation becomes too much cumbersome to deal with. For instance, inhomogeneous wave-guide problems have remained unsolved or atleast partly solved even for the simplest of cases. Unfortunately no theory is available in the literature which can replace the above one in dealing with a number of modes of this type of wave propagation in cylinders.

The method of expansion of displacement functions devised by Mindlin & his co-workers (1951, 1960) is taken help of and utilizing the proper boundary condition for the case under study new equations have been arrived at. The search for a suitable orthogonal polynomial which resembles actual displacement distribution profiles over the wave-guide cross section is however a labourious job and moreover is not often met with success. Jacobi polynomial has been found suitable for the present case by the author. The case of torsional wave propagation in an inhomogeneous solid cylinder would subsequently be solved with help of the above analysis.

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2. Theory

I. *Homogeneous cylinder case*

The theory developed here consists of an expansion of the displacement functions in terms of the Jacobi polynomial in the radial coordinate, the choice being limited by the need for a close representation of the actual displacements upto a fairly high modal number. The cylinder is assumed to be bounded by the surface at $r = a$ in the cylindrical coordinate system r, θ, z the axis of the cylinder being coincident with the Z direction of the coordinate system. In cosonance with the type of propagation under study the radial and the axial displacements are limited by the need for a close representation of the actual displacements upto a fairly high modal number. The cylinder is assumed to be bounded by the surface at $r = a$ in the cylinderical coordinate system r, θ, z by the axis of the cylinder being coindicent with the Z direction of the coordinate system. In consonance with the type of propagation under study the radial and the axial displacements are assumed to be zero. The circumferential component of the displacements is thus expressed assuming θ symmetry

$$u_\theta = \sum_u U_n(\alpha) u_n(z, t), \qquad \dots (1)$$

where $\alpha = r/a$

and $U_n(\alpha)$ is the Jacobi polynomial. The general term of the polynomial is expressed as

$$U_n(\alpha) = \alpha + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{(n+2)_k}{(k+1)!} \alpha^{2k+1}, \qquad \dots (2)$$

where $$\binom{n}{k} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}$$

$$(\beta)_k = \beta(\beta+1)\dots(\beta+k-1)$$

$$\beta_0 = 1$$

It has got the orthogonal property

$$4(n+1)^3 \int_0^1 U_m(\alpha) U_n(\alpha) \alpha d\alpha = 0, \; m \neq n$$

$$= 1, \; m = n \qquad \dots (3)$$

The stress equation of motion under assumed condition is

$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{2T_{r\theta}}{r} - P\frac{\partial^2 u_\theta}{\partial t^2} = 0 \qquad \dots (4)$$

The other equations of motion are automatically satisfied. The surviving stress equation of motion is written in the integrated form

$$\int_0^a \left(\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{2T_{r\theta}}{r} - P\frac{\partial^2 u_\theta}{\partial t^2} \right) r dr \delta u = 0. \tag{5}$$

Substituting eq. (1) in eq. (5) and utilizing the boundary condition $T_{r\theta} = 0$ at $r = a$ and finally equating the coefficients of δu_n both sides, the rod stress equation of motion is obtained as

$$\frac{\partial F_n}{\partial z} - \frac{P_n}{a} = \frac{P}{4(n+1)^3}\frac{\partial^2 u_n}{\partial t^2}, \tag{6}$$

where $$F_n = \int^1 T_{\theta z} U_n(\alpha) \alpha d\alpha$$

and $$P_n = \int_0^1 T_{r\theta}\left[\alpha \frac{\partial U_n(\alpha)}{\partial \alpha} - U_n(\alpha) \right] d\alpha. \tag{7}$$

The components of the rod strain can thus be expressed as

$$\epsilon_{\theta z} = \sum_n U_n \frac{\partial u_n}{\partial z}$$

$$\epsilon_{r\theta} = \frac{1}{a}\sum_n \left[\frac{\partial U_n(\alpha)}{\partial \alpha} u_n - \frac{U_n(\alpha) u_n}{\alpha} \right]. \tag{8}$$

The usual stress displacement relations then take the forms

$$T_{r\theta} = \frac{\mu}{a}\sum_n \left[\frac{\partial U_n(\alpha)}{\partial \alpha} u_n - \frac{U_n(\alpha) u_n}{\alpha} \right],$$

$$T_{\theta z} = \mu \sum_n U_n(\alpha)\frac{\partial u_n}{\partial z}, \tag{9}$$

Thus the rod stresses are expressed as

$$F_n = \mu \int_0^1 U_n^2(\alpha)\frac{\partial u_n}{\partial z}\alpha d\alpha,$$

$$P_n = \frac{\mu}{a}\int_0^1 \left[\frac{\partial U_n(\alpha)}{\partial \alpha} - \frac{U_n(\alpha)}{\alpha} \right] u_n \alpha d\alpha. \tag{10}$$

The first few terms of the above series are given below

$$F_0 = \frac{\mu}{4}\frac{\partial u_0}{\partial z} \qquad P_0 = 0$$

$$F_1 = \frac{\mu}{32}\frac{\partial u_1}{\partial z} \qquad P_1 = \frac{3}{2}\frac{\mu u_1}{a}$$

$$F_2 = \frac{\mu}{108}\frac{\partial u_2}{\partial z} \qquad P_2 = \frac{16}{9}\frac{\mu u_2}{a}$$

The general equation of motion thus can be written as

$$\frac{\partial^2 u_n}{\partial z^2} - M_n^2 u_n = \frac{1}{c_2^2}\frac{\partial^2 u_n}{\partial t^2} \qquad \dots \quad (11)$$

where M_n $(n = 0, 1, 2 \dots)$ is a set of constants dependent on the mode number and on the radius of cross-section of the cylinder. First few members of the set are given below

$$\begin{aligned} M_0 &= 0 \\ M_1 &= 4.3^{1/2}/a \\ M_2 &= 8.3^{1/2}/a \end{aligned} \qquad \dots \quad (12)$$

Assuming a solution of the form $\exp(i\omega t - ik_0 z)$ the required dispersion relation is obtained as,

$$\frac{\omega^2}{c_2^2 k_0^2} = \frac{M_n^2}{k_0^2} + 1. \qquad \dots \quad (13)$$

The first mode predicted by the above theory gives dispersionless propagation with a velocity c_2 where $c_2 = (\mu/p)^{\frac{1}{2}}$. This is in exact agreement with the results obtained from the previous analysis. All the higher modes are dispersive and the agreement with the Owen's results is again close. The first three modes of the dispersion family predicted by the above theory are shown in figure 1.

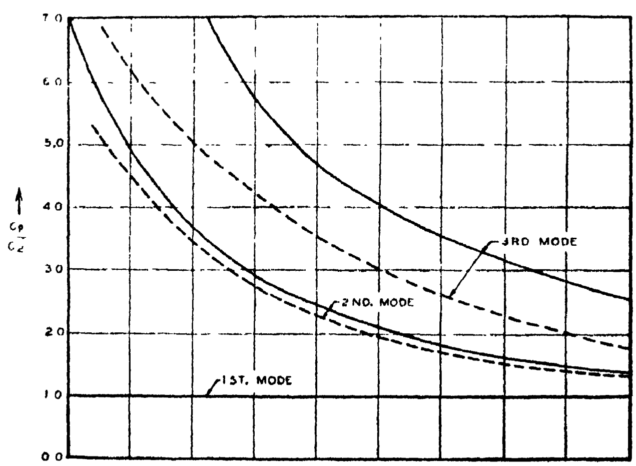

Fig. 1. Phase velocity dispersion curves for torsional waves in cyiinder.—author's curve, — — — Owen's curve.

For the sake of comparison Owen's curves are also drawn alongside. With a Mindlin-Herrmann type constant adjustment process, however, the present curves can be made exactly identical in all respects to the curves obtained from the Bessel function distribution.

2. *Inhomogeneous cylinder case*

Let the elastic inhomogeneity of the material of the cylinder be wholly axial and be assumed after Dutta (1956) as

$$\mu = \mu_0 + \mu_1 z \qquad (14)$$

It can be shown easily that with the usual assumptions for torsional waves for the present study the form of inhomogeneity of the other elastic Lame constant does not affect the results of the analysis, two equations of motion being automatically satisfied in an identical manner. The remaining equation of motion as deduced earlier and expressed in eq. (6) will remain valid for the inhomogeneous case but the values of the rod stresses F_n and P_n are to be calculated afresh taking care of eq. (14). On assuming the displacement of the form

$$u_n(z,t) = u_1(z)\exp(i\omega t), \qquad \ldots \quad (15)$$

and a little more calculation gives the displacement equation of motion as,

$$\zeta \frac{d^2u_n}{d\zeta^2} + \frac{du_n}{d\zeta} + u_n\left[\frac{\omega^2}{\chi^2} - K_n\zeta\right] = 0 \qquad \ldots \quad (16)$$

where

$$\zeta = \mu_0 + \mu_1 z$$

$$\chi^2 = \mu_1^2/\rho$$

and

$$K_n = M_n^2/\mu_1^2$$

For the first branch $K_0 = 0$, and with the substitution $\eta = 2\omega\sqrt{\zeta}\chi$ eq. (16) can be brought to the form

$$\frac{d^2u_0}{d\eta^2} + \frac{1}{\eta}\frac{du_0}{d\eta} + u_0 = 0 \qquad \ldots \quad (17)$$

The above equation is the well known Bessel differential of zeroth order. Hence the solution is written as

$$u_0 = AJ_0(\eta) + BY_0(\eta). \qquad \ldots \quad (18)$$

where $J_0(\eta)$ and $Y_0(\eta)$ are the Bessel functions of zero order and of first and second kind respectively and A and B are two constants to be evaluated from the boundary conditions.

For all the higher branches $K_n \neq 0$ and with the substitutions $u_1(z) = \dfrac{\phi_1(z)}{4K_n\zeta^{\frac{1}{2}}}$ and $\zeta = -\xi/2K_n^{\frac{1}{2}}$ the following equation can be obtained from the eq. (16)

$$\frac{d^2\phi_1}{d\xi^2} + \phi_1\left[\frac{1}{4\xi^2} + \frac{\omega^2}{2\chi^2 K_n^{\frac{1}{2}}\xi} - \frac{1}{4}\right] = 0. \qquad \ldots \quad (19)$$

The solution of the above equation can be obtained in terms of Whittekar functions (Whittekar & Watson 1962) as

$$\phi_1(\xi) = A_1 W_{k_0}(\xi) + B_1 W_{-k_0}(-\xi) \qquad \dots \quad (20)$$

where $K = \omega^2/2\chi^2 K_n^{\frac{1}{2}}$

and A_1 and B_1 are constants.

Thus the expression for $u_n(\zeta)$ can be written as

$$u_n(\zeta) = [A_1 W_{k_0}(2K_n^{\frac{1}{2}}\zeta) + B_1 W_{k_0}(-2K_n^{\frac{1}{2}}\zeta)]/4K_n\zeta^{\frac{1}{2}}. \qquad \dots \quad (21)$$

Now the frequency equation of the system under different boundary conditions at the ends. For a free-free bar the end conditions are

$$\left.\frac{du_n}{dz}\right|_{z=0} = 0$$

$$\left.\frac{du_n}{dz}\right|_{z=l} = 0 \qquad \dots \quad (22)$$

The frequency equation of the system is thus easily obtained as,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0 \qquad \dots \quad (23)$$

where

$$a_{11} = \frac{W_{k_0}(2K_n^{\frac{1}{2}}\mu_0)}{4K_n\mu_0^{1/2}} - \frac{\mu_1 W_{k_0}(2K_n^{\frac{1}{2}}\mu_0)}{8K_n\mu_0^{3/2}}$$

$$a_{12} = \frac{W'_{-k_0}(-2K_n^{\frac{1}{2}}\mu_0)}{4K_n\mu_0^{\frac{1}{2}}} - \frac{\mu_1 W_{-k_0}(-2K_n^{\frac{1}{2}}\mu_0)}{8K_n\mu_0^{3/2}},$$

$$a_{22} = \frac{W'_{-k_0}\{-2K_n^{\frac{1}{2}}(\mu_0+\mu_1 l)\}}{4K_n(\mu_0+\mu_1 l)^{\frac{1}{2}}} - \frac{\mu_1 W_{-k_0}\{-2K_n^{\frac{1}{2}}(\mu_0+\mu_1 l)\}}{8K_n(\mu_0+\mu_1 l)^{3/2}}$$

$$a_{21} = \frac{W'_{k_0}\{2K_n^{\frac{1}{2}}(\mu_0+\mu_1 l)\}}{4K_n(\mu_0+\mu_1 l)^{\frac{1}{2}}} - \frac{\mu_1 W_{k_0}\{2K_n^{\frac{1}{2}}(\mu_0+\mu_1 l)\}}{8K_n(\mu_0+\mu_1 l)^{3/2}}$$

where dashes denote differentiation with respect to z.

5. Discussions

The present theory gives some more insight into the role of the separate stress components in the processes of dispersion of torsional waves in homogeneous isotropic cylinder. The most important achievement of the proposed theory lies in the complete removal of the transcendental nature from the frequency equation of torsional waves in homogeneous cylinders. Another triumphant feature of the theory for the homogeneous cylinder lies in the complete identity of the eq. (11) with a corresponding equation deduced by Jones (1959) though the method of approach was completely different from the present one.

The equation for the homogeneous cylinder also predicts separate cut-off frequencies for the higher order modes like the existing theory does. The values of these cut-off frequencies are given by the general expression

$$\omega_{c_n} = \beta_n$$

where the values of β_n are $\sqrt{48}\, c_2/a$ and $\sqrt{198}c_2/a$ for $n = 1$ and $n = 2$ corresponding to the two lower order modes just above the lowest order mode.

For the inhomogeneous cylinder problem Bessel functions are found not sufficient to describe the displacement functions. The general frequency equation involving Whittekar functions can, however, be shown to be breaking down to equation consisting of only Bessel functions for the lowest order mode.

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