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A perturbation method for Maxwell's equation in a pumped medium : stable solutions

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The Maxwell's wave equation in a medium whose permittivity is undergoing a one-dimensional periodic space time variation by the action of a pump wave is solved by a perturbation technique based on the methods of Bogoliubov and Mitropolsky for non-linear oscillations. Expressions are obtained for the amplitudes of the various frequency components associated with the wave. The general dispersion relation is also obtained.

1. Introduction

The topic of propagation of electromagnetic waves in a medium whose permittivity is varied by the action of a pump wave is of contemporary interest for physicists and engineers. The notable contributions are found in the papers of Slater (1958), Tien (1958), Simon (1960), Cassedy & Oliner (1963), Kunz (1964), Holbery & Kunz (1966), and others. The effect of the pumping by an acoustic or electromagnetic wave is to produce a periodic variation of the permittivity of medium in space and time, determined, by the frequency and wave number of the pump wave. The complicated wave equation in such a medium has been solved by the above authors by numerous approximation techniques. The present paper deals with a perturbation method for the one-dimensional case and it has more general applicability than the others. This method can be suitably applied for several cases of wave propagation in nonlinear media.

In the ordinary case of a constant permittivity the wave equation is separable in the space and time parts. But when the permittivity is a function of space and time the wave equation is not separable. But if the wave equation is expressed in terms of a retarded time it will be separable in space and retarded time by the introduction of a suitable separation constant. The equation in terms of the retarded time will be one of the Mathieu type with periodic coefficients. The solution of the Mathieu equation has been discussed by McLachlan (1951) and has been used by Holbery & Kunz (1966). But the method will not be applicable to the equation in the present case. To solve the equation a perturbation method based on the methods of Bogoliubov & Mitropolsky (1961) is developed.

The solution of the wave equation falls under two heads. (a) The non-resonance case where the separation constant has any general value. In this case the wave amplitude is stable in space and time. (b) The resonance case where the separation constant has certain special relationship with the pump frequency and the propagation constant. In this case under certain conditions the solutions are unstable. This paper discusses the general non-resonance case.

2. Formulation of the Problem

Consider an infinite isotropic non-conducting non-dispersive medium of permittivity ϵ_1 which is subjected to the action of a pump wave of frequency Ω. (In this paper the frequency refers to the angular frequency), propagation constant K propagating along the x-direction. The effect of the wave is to modify the permittivity to a value ϵ given by

$$\epsilon(x, t) = \epsilon_1[1+h \cos(\Omega t-Kx), \qquad \ldots \quad (1)$$

where h is a factor much less than 1 called the modulation index. The pump wave is thus modulating the permittivity of the medium to the value given by eq. (1) and it does not have any other interaction with the propagating electromagnetic wave in the medium.

Assuming a linear relationship between the electric displacement vector $\boldsymbol{D}$ and the field vector $\boldsymbol{E}$ we obtain from Maxwell's electromagnetic equations in M.K.S. units.

$$\nabla \times \nabla \times \boldsymbol{E}+\mu_0 \frac{\partial^2}{\partial t^2} (\epsilon E) = 0, \qquad \ldots \quad (2)$$

where μ_0 is the magnetic permeability of the medium which is not affected by the pump wave. For a transverse electromagnetic wave propagating in the x-direction we have $E_z = E(x, t)$, $E_x = E_y = 0$. With ϵ given by eq. (1) we have from eq. (2)

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{C^2} \left[1+h \cos \Omega \left(t-\frac{x}{V}\right)\right] \frac{\partial^2 E}{\partial t^2}$$

$$+\frac{2h\Omega}{C^2} \sin \Omega \left(t-\frac{x}{V}\right)\frac{\partial E}{\partial t}+\frac{h\Omega^2}{C^2} \cos \Omega \left(t-\frac{x}{V}\right) = 0, \qquad \ldots \quad (3)$$

where $C = (\mu_0\epsilon_1)^{-\frac{1}{2}}$ is the velocity of propagation of the wave (called signal) in the unmodulated medium, $E = E(x, t)$ and $V = \Omega/K$ is the pump wave velocity.

Wave eq. (3) is to be solved for the electric field. But since it is not separable in the space and time part we can introduce a transformation of variables so that the resulting differential equation is separable in the new variables. This transformation can be introduced by setting

$$X = x, \quad \tau = t - \frac{x}{V} \qquad \ldots \quad (4)$$

where τ can be considered as a retarded time. Using eq. (4) and with

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} \text{ and } \frac{\partial}{\partial x} = \frac{\partial}{\partial X} - \frac{1}{V}\frac{\partial}{\partial \tau},$$

we can write eq. (2) in the form

$$\frac{\partial^2 E}{\partial X^2} + \left(\frac{1}{V^2} - \mu_0 \epsilon\right)\frac{\partial^2 E}{\partial \tau^2} - \frac{2}{V}\frac{\partial^2 E}{\partial X \partial \tau} - 2\mu_0 \frac{\partial E}{\partial \tau}\frac{d\epsilon}{d\tau} - \mu_0 E \frac{d^2 \epsilon}{d\tau^2} = 0, \qquad \ldots \quad (5)$$

where

$$E = E(x, \tau) \text{ and } \epsilon = \epsilon(\tau) = \epsilon_1(1 + \cos \Omega\tau). \qquad \ldots \quad (6)$$

We can investigate the solution of eq. (5) having the form

$$E(X, \tau) = T(\tau) \exp i\beta X, \qquad \ldots \quad (7)$$

where β is a separation constant which can be real or complex. Using eq. (7) in eq. (5) we get an ordinary differential equation for T in the form

$$\left(\frac{1}{V^2} - \mu_0 \epsilon\right)\frac{d^2 T}{d\tau^2} - 2\left(\frac{i\beta}{V} + \mu_0 \epsilon'\right)\frac{dT}{d\tau} - (\beta^2 + \mu_0 \epsilon'')T = 0, \qquad \ldots \quad (8)$$

where

$$T = T(\tau),\ \epsilon' = \frac{d\epsilon}{d\tau} \text{ and } \epsilon'' = \frac{d^2\epsilon}{d\tau^2}.$$

Eq. (8) can be transformed into a differential equation where the first derivative is removed by a substitution

$$T(\tau) = G(\tau) \exp \int \eta(\tau) d\tau. \qquad \ldots \quad (9)$$

The function $\eta(\tau)$ can be chosen such that when eq. (9) is substituted in eq. (8) the coefficient of $dG/d\tau$ in the resulting differential equation for G is zero. With this condition applied we get

$$\eta(\tau) = \frac{\dfrac{i\beta}{V} + \mu_0 \epsilon'}{\dfrac{1}{V^2} - \mu_0 \epsilon} \qquad \ldots \quad (10)$$

and with ϵ given by eq. (6) we get

$$\exp \int \eta(\tau) d\tau = \frac{T_0}{(1 + \alpha h \cos \Omega\tau)} \exp i\alpha_1 \left[\frac{2}{(1 - h^2\alpha^2)^{\frac{1}{2}}} \tan^{-1}\left\{\left(\frac{1 + h\alpha}{1 - h\alpha}\right)^{\frac{1}{2}} \tan \frac{\Omega\tau}{2}\right\}\right] \qquad \ldots \quad (11)$$

where T_0 is an arbitrary constant of integration.

$$\alpha = \frac{1}{\frac{C^2}{V^2}-1}, \quad \alpha_1 = \frac{C^2}{V}\frac{\beta}{\Omega}\alpha.$$

Using eqs. (9) and (10) in eq. (8), we get

$$\left(\frac{1}{V^2}-\mu_0\eta\right)^2 \frac{d^2G}{d\tau^2}+\left(\beta^2\mu_0\epsilon-\frac{i\beta}{V}\mu_0\epsilon'\right)G = 0 \qquad \text{.. (12)}$$

The two linearly independent solutions G_1 and G_2 of eq. (12) can be substituted in eq. (9) and the electric field E in the modulated medium can be obtained from eq. (7).

3 Perturbation Method

The solution of eq. (12) in closed form is not easy and as in similar problems a suitable perturbation technique is to be used. Here a technique based on the methods of Bogliubov & Mitropolsky (1961) for non-linear oscillations is developed.

The solution falls under two heads, (a) the so-called non-resonance case where the separation constant β is not in the neighbourhood of the quantity

$$N\frac{\Omega}{2C}\left(\frac{C^2}{V^2}-1\right),$$

N being an integer, (b) the resonance case when β is in the neighbhourhood of

$$\frac{N\Omega}{2V}\left(\frac{C^2}{V^2}-1\right).$$

The non resonance case is dealt in this article

Substituting for ϵ and ϵ' from eq. (1) and assuming $\alpha h << 1$, we can write eq. (12) in the form

$$\frac{d^2G}{d\tau^2}+\nu^2G = -\nu^2[h(\alpha_3\cos\Omega\tau+i\alpha_2\sin\Omega\tau)$$

$$+h^2(\alpha_4\cos^2\Omega\tau+i\alpha_2\alpha\sin 2\Omega\tau)+\ldots]G. \qquad \text{.. (13)}$$

where

$$\nu^2 = \frac{(\beta C)^2}{\left(\frac{C^2}{V^2}-1\right)}, \quad \alpha_3 = 1+2\alpha, \quad \alpha_4 = 2\alpha+3\alpha^2. \qquad \text{... (14)}$$

In most cases of practical interest h is small and hence the expansion used to obtain eq. (13) is valid.

Eq. (14) can be interpreted in general as the oscillation of a system with natural frequency ν subjected to a small periodic perturbation represented by the terms on the right hand side. It is to be noted that when $v \to \infty$, $\alpha_2 = 0$ and for small h neglecting terms in h^2 and above, eq. (13) is reduced to the Mathieu's equation which has been intensively studied and applied to several problems. In the present case the solution of eq. (13) can be studied to the second order in h so that terms in h and h^2 can be retained.

The general solution of eq. (13) will consist of a wave with a fundamental frequency ν and harmonic components of frequencies $n\Omega+m\nu$ (the integers m and n varying from $-\infty$ to $+\infty$) and relative amplitudes depending on h, Ω and ν. But when n and m are such that one of the harmonic frequencies is equal to the natural frequency ν, i.e.,

$$n\Omega+m\nu = \nu$$

or ... (15)

$$\nu = \frac{p}{q}\Omega,$$

where p and q are integers, the amplitude of the particular harmonic will be comparable with that of the fundamental and we say there is resonance. However, it will be observed later that resonance will not occur for all values of p and q and it occurs for ν in the neighbourhood of $N\Omega/2$, where N is an integer When substituted for ν from eq. (14) we get the resonance condition as

$$\beta \approx \frac{N\Omega}{2C}\left(\frac{C^2}{V^2}-1\right). \qquad \ldots (16)$$

In the non-resonance case the solution of eq. (13) has to be sought in the form

$$G = f\cos\psi+hu_1(f, \psi, \Omega\tau)+h^2u_2(f, \psi, \Omega\tau)+\ldots, \qquad \ldots (17)$$

Where the function u_1, u_2 etc. are periodic in both the angular variables ψ and $\Omega\tau$ with a period 2π. The amplitude f and the total phase ψ are determined by the following defferential equations,

$$\frac{df}{d\tau} = hR_1(f)+h^2R_2(f)+\ldots \qquad \ldots (18)$$

$$\frac{d\psi}{d\tau} = \nu+hS_1(f)+h^2S_2(f)+\ldots, \qquad \ldots (19)$$

when $h = 0$, we note that f is a constant and $\psi = \nu\tau$ so that the unperturbed solution is

$$G = f\cos\nu\tau. \qquad \ldots (20)$$

The quantities R_1, R_2, S_1, S_2, ... etc. on the right hand sides of eqs. (18) and (19) must depend only on the amplitude f since in the non-resonance case the phase of the natural oscillation has no dependence on the phase of the perturbation. But in the resonance case the situation will be different.

Substituting eqs. (17), (18) and (19) in eq. (13) and equating the coefficients of like powers of h on both sides of the resulting equation we get

$$\nu^2 \frac{\partial^2 U_1}{\partial \psi^2} + 2\nu \frac{\partial^2 U_1}{\partial \psi \partial \tau} + \frac{\partial^2 U_1}{\partial \tau^2} + \nu^2 U_1 = a_0 + 2f\nu S_1 \cos\psi + 2\nu R_1 \sin\psi, \quad \ldots (21)$$

$$\nu^2 \frac{\partial^2 U_2}{\partial \psi^2} + 2\nu \frac{\partial^2 U_1}{\partial \psi \partial \tau} + \frac{\partial^2 U_1}{\partial \tau^2} + \nu^2 U_2 = a_1 + 2f\nu S_2 \cos\psi + 2\nu R_2 \sin\psi, \quad \ldots (22)$$

where

$$a_0 = -\nu^2 f \cos\psi(\alpha_3 \cos\Omega\tau + i\alpha_2 \sin\Omega\tau) \quad \ldots (23)$$

$$\begin{aligned} a_1 = & -\nu^2 U_1(\alpha_3 \cos\Omega\tau + i\alpha_2 \sin\Omega\tau) \\ & -\nu^2 f \cos\psi(\alpha_4 \cos^2\Omega\tau + i\alpha\alpha_2 \sin 2\Omega\tau) \\ & + \left(R_1 \frac{\partial R_1}{\partial f} - fS_1^2\right) \cos\psi - \left(fR_1 \frac{\partial S_1}{\partial f} + 2R_1 S_1\right) \sin\psi \\ & + 2\nu R_1 \frac{\partial^2 U_1}{\partial f \partial \psi} + 2\nu S_1 \frac{\partial^2 U_1}{\psi^2} + 2R_1 \frac{\partial^2 U_1}{\partial f \partial \tau} + 2S_1 \frac{\partial^2 U_1}{\partial \psi \partial \tau}. \end{aligned} \quad \ldots (24)$$

The functions a_0, a_1 are periodic in ψ and $\Omega\tau$ and moreover depend on f. From the relations given by eqs. (21) and (22). U_1 U_2, R_1, R_2, S_1, S_2 are to be determined. The first step is to evaluate U_1, R_1 and S_1. The function a_0 can be expanded into a double Fourier series given by

$$a_0(f, \psi, \Omega\tau) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{nm}^{(0)}(f) \exp i(n\Omega\tau + m\psi), \quad \ldots (25)$$

where $a_{nm}^{(0)}$ are the Fourier coefficients. Multiplying both sides of eq. (25) by $\exp i(n\theta + m\psi)$ (where $\theta = \Omega\tau$) and integrating with respect to θ and ψ over a complete cycle we get

$$a_{nm}^{(0)}(f) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} a_0(f, \psi, \theta) \exp -i(n\theta + m\psi) d\theta d\psi. \quad \ldots (26)$$

Similarly U_1 can be expanded in a double Fourier series with the Fourier coefficients $U_{nm}^{(1)}$ given by

$$U_1(f, \psi, \Omega\tau) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U^{(1)}_{n}(f) \exp i(n\Omega\tau + m\psi) \quad \ldots (27)$$

Substituting eqs. (25) and (27) in eq. (21) we get

$$\sum_n \sum_m [\nu^2-(n\Omega+m\nu)^2]U^{(1)}{}_{nm}(f)\exp i(n\Omega\tau+m\psi)$$

$$= 2f\nu S_1 \cos\psi+2\nu R_1 \sin\psi+\sum_n \sum_m a_{nm}{}^{(0)}(f)\exp i(n\Omega\tau+m\psi) \quad \ldots (28)$$

Since we are dealing with the non-resonance case it is necessary to determine such values of $U_{nm}{}^{(1)}$, R_1, S_1 that U_1 will not contain the resonance term wth frequency $n\Omega+m\nu = \nu$. For this, those $U_{nm}{}^{(1)}(f)$ for which $\nu^2-(n\Omega+m\nu)^2 = 0$ should be zero. This condition is satisfied for $n = 0$, $m = \pm 1$. Hence $U_{01}{}^{(1)} = U_{0-1}{}^{(1)} = 0$. With these conditions applied, and equating the coefficients of equal harmonics in eq. (28) we get

$$U_{nm}{}^{(1)}(f) = \frac{a_{nw}{}^{(0)}(f)}{\nu^2-(n\Omega+m\nu)^2} \text{ (with } n \neq 0, m \neq +1) \quad \ldots (29)$$

$$2\nu R_1 = -i[a_{01}{}^{(0)}+a_{0-1}{}^{(0)}] \quad \ldots (30)$$

$$2f\nu S_1 = -[a_{01}{}^{(0)}+a_{0-1}{}^{(0)}] \quad \ldots (31)$$

Using eqs. (26), (30) and (31) we get

$$R_1 = \frac{1}{4\pi^2\nu}\int_0^{2\pi}\int_0^{2\pi} a_0(f,\psi,\theta)\sin\psi\, d\theta d\psi \quad \ldots (32)$$

$$S_1 = \frac{1}{4\pi^2 f\nu}\int_0^{2\pi}\int_0^{2\pi} a_0(f,\psi,\theta)\cos\psi\, d\theta d\psi. \quad \ldots (33)$$

Knowing R_1 and S_1, f and ψ can be evaluated to the first order in h and U_1 determined as

$$U_1 = \sum_{n\neq 0}\sum_{m\neq\pm 1} \frac{1}{4\pi^2}\frac{\exp i(n\Omega\tau+m\psi)}{[\nu^2-(n\Omega+m\nu)^2]}\int_0^{2\pi}\int_0^{2\pi} a_0 \exp\ i(n\theta+m\psi)d\theta d\psi \quad \ldots (34)$$

The above procedure can be continued to determined R_2, S_2 and U_2. With R_1, S_1 and U_1 determined from eqs. (32), (33) and (34) $a_1(f, \psi, \Omega\tau)$ can be evaluated from eq. (24). Expanding a_1 and u_2 in double Fourier series, substituting in eq. (22) and proceeding in the same way as before, we get

$$R_2 = -\frac{1}{4\pi^2\nu}\int_0^{2\pi}\int_0^{2\pi} a_1 \sin\psi\, d\theta d\psi \quad \ldots (35)$$

$$S_2 = \frac{1}{4\pi^2 f\nu}\int_0^{2\pi}\int_0^{2\pi} a_1 \cos\psi\, d\theta d\psi \quad \ldots (36)$$

$$U_2 = \frac{1}{4\pi^2}\sum_{n\neq 0}\sum_{m\neq\pm 1}\frac{\exp i(n\Omega\tau+m\psi)}{\nu^2-(n\Omega+m\nu)^2}\int_0^{2\pi}\int_0^{2\pi} a_1 \exp -i(n\theta+m\psi)d\theta d\psi. \quad \ldots (37)$$

On evaluating the appropriate integrals it is found that

$$R_1 = 0, \quad S_1 = 0, \quad R_2 = 0, \quad S_2 = \nu\left[\frac{(\alpha_3^2-\alpha_2^2)^2}{4(\Omega^2-4\nu^2)} + \frac{\alpha_4}{4}\right] \qquad \ldots (38)$$

$$U_1 = f[C_1 \exp i(\psi+\theta) + C_2 \exp -i(\psi+\theta) + C_3 \exp i(\psi-\theta) + C_4 \exp -i(\psi-\theta)] \qquad \ldots (39)$$

$$U_2 = f[D_1 \exp i(\psi+2\theta) + D_2 \exp -i(\psi+2\theta) + D_3 \exp i(\psi-2\theta) + D_4 \exp -i(\psi-2\theta)], \qquad \ldots (40)$$

where

$$\psi = \nu\tau + h^2 S_2 \tau \qquad \ldots (41)$$

$$\left.\begin{aligned} C_1 &= \frac{\nu^2}{4}\,\frac{(\alpha_3+\alpha_2)}{\nu^2-(\nu+\Omega)^2} & C_2 &= -\frac{\nu^2}{4}\,\frac{(\alpha_3-\alpha_2)}{\nu^2-(\nu+\Omega)^2} \\ C_3 &= \frac{\nu^2}{4}\,\frac{(\alpha_3-\alpha_2)}{\nu^2-(\nu-\Omega)^2} & C_4 &= -\frac{\nu^2}{4}\,\frac{(\alpha_3+\alpha_2)}{\nu^2-(\nu-\Omega)^2} \end{aligned}\right\} \qquad \ldots (42)$$

$$\left.\begin{aligned} D_1 &= \frac{\nu^2}{8}\,\frac{[\nu C_1(\alpha_3+\alpha_2)-\alpha_4-2\alpha_2\alpha]}{\nu^2-(\nu+2\Omega)^2} \\ D_2 &= \frac{\nu^2}{8}\,\frac{[C_2(\alpha_3-\alpha_2)-\alpha_4+2\alpha_2\alpha]}{\nu^2-(\nu+2\Omega)^2} \\ D_3 &= \frac{\nu^2}{8}\,\frac{[C_3(\alpha_3-\alpha_2)-\alpha_4+2\alpha_2\alpha]}{\nu^2-(\nu-2\Omega)^2} \\ D_4 &= \frac{\nu^2}{8}\,\frac{[C_4(\alpha_3+\alpha_2)-\alpha_4-2\alpha_2\alpha]}{\nu^2-(\nu-2\Omega)^2} \end{aligned}\right\} \qquad (43)$$

If the solution is assumed as

$$G = f\sin\psi + hU_1' + h^2U_2' + \ldots \qquad \ldots (44)$$

we get by similar procedure

$$U_1' = if[-C_1 \exp i(\psi+\theta) + C_2 \exp -i(\psi+\theta) - C_3 \exp i(\psi-\theta) + C_4 \exp -i(\psi-\theta)] \qquad \ldots (45)$$

$$U_2' = if[-D_1 \exp i(\psi+2\theta) + D_2 \exp -i(\psi+2\theta) - D_3 \exp i(\psi-2\theta) + D_4 \exp -i(\psi-2\theta)]. \qquad \ldots (46)$$

From eqs. (17), (39), (40), (44), (45) and (46) we can write the two solutions for G as

$$G = f \exp i\psi[1 + h(2C_1 \exp i\theta + 2C_3 \exp -i\theta) + h^2(2D_1 \exp 2i\theta + 2D_3 \exp -2i\theta) + \ldots] \qquad \ldots (47)$$

and

$$G = f \exp -i\psi[1 + h(2C_2 \exp -i\theta + 2C_4 \exp i\theta) + h^2(2D_2 \exp -2i\theta + 2D_4 \exp 2i\theta) + \ldots]. \qquad \ldots (84)$$

In the above discussions, the cases $\nu = \Omega/2$ and $\nu = \Omega$ (corresponding to $n = 0$, $m = \pm 1$ etc.) are obviously resonance conditions as evident from the coefficients given by eqs. (42) and (43). Since S_2 is real, the solutions do not change their amplitude exponentially with increasing τ and hence can be said to represent stable solutions. These solutions are valid for all $\nu \neq N\Omega/2$ ($N = 1, 2, \ldots$). Hence stable solutions occur for all $\nu \neq N\Omega/2$ or

$$\beta \neq \frac{N\Omega}{2C}\left(\frac{C_2}{V^2}-1\right).$$

4. Electric Field in the Medium

The expression for the electric field $E(x, t)$ can be obtained from eq. (7) by the use of eqs. (9), (10), (47) and (48) with eq. (47) we get the solution as

$$E(x,t) = L \exp i(\omega_1 t - k_1 x)\left[1 + L_1 \exp i\Omega\left(t-\frac{x}{V}\right) + L_2 \exp - i\Omega\left(t-\frac{x}{V}\right)\right.$$
$$\left. + L_3 \exp 2i\Omega\left(t-\frac{x}{V}\right) + L_4 \exp -2i\Omega\left(t-\frac{x}{V}\right) + \ldots\right], \quad \ldots (49)$$

where L can be called the fundamental amplitude depending mainly upon f the relative amplitudes L_1 and L_2 of the harmonics are proportional to h while L_3 and L_4 are proportional to h^2,

$$\omega_1 = \frac{C\beta}{\frac{C}{V}-1} + h^2\left(\Omega\frac{\alpha_1\alpha^2}{2} + S_2\right) \quad \ldots (50)$$

$$k_1 = \frac{\omega_1}{C} + h^2\left(\Omega\frac{\alpha_1\alpha^2}{2} + S_2\right)\left(\frac{1}{V} - \frac{1}{C}\right). \quad (51)$$

Eq. (49) represents a forward wave in the direction of pump wave having a fundamental frequency ω_1 and wave vector k_1.

The phase velocity ω_1/k_1 is not C, but depend upont he modulation index h and the pump wave frequency Ω and veocity V. Thus the medium is turned dispersive by the effect of the pump wave. The associated harmonics have frequencies $\omega_1 + \Omega$, $\omega_1 \pm 2\Omega$ etc. and their velocities are different from those of fundamental.

When eq. (48) is used, the electric field

$$E(x,t) = L' \exp i(\omega_2 t + k_2 x)\left[1 + L_1' \exp i\Omega\left(t-\frac{x}{V}\right) + L_2' \exp i\Omega\left(t-\frac{x}{V}\right)\right.$$
$$\left. + L_3' \exp 2i\Omega\left(t-\frac{x}{V}\right) + L_4' \exp -2i\Omega\left(t-\frac{x}{V}\right)\right], \quad \ldots (52)$$

where L' is the fundamental amplitude L'_1, L'_2, L'_3, L'_4 the relative amplitudes of the harmonics as in eq. (49). Here

$$\omega_2 = \frac{C\beta}{\frac{C}{V}+1} + h^2\left(\frac{\Omega\,\alpha_1\alpha^2}{2} + S_2\right) \quad \text{.. (53)}$$

$$k_2 = \frac{\omega_2}{C} - h^2\left(\frac{\Omega\,\alpha_1\alpha^2}{2} + S_2\right)\left(\frac{1}{V} - \frac{1}{C}\right). \quad \text{.. (54)}$$

Eq. (52) thus represents a backward wave in a direction opposite to the pump wave with a frequency different from that of the forward wave. The wave vector is also different from that of the forward wave and so is the velocity. The frequencies ω_1 and ω_2 depend on the particular choice of β. For a given value of β there will be two dominant frequencies excited in the medium and they travel in opposite directions with different velocities.

But for a given frequency excited in the medium the values of β will be different for the forward and backward waves. If this frequency is ω, for the forward wave β is given by the relation

$$\omega = \frac{C\beta}{\frac{C}{V}-1} + h^2\left(\frac{\Omega\,\alpha_1\alpha^2}{2} + S_2\right) \quad \text{.. (55)}$$

and the wave is represented as

$$E(x,t) = A \exp i\lambda_a x[\exp i(\omega t - k_0 x) + a_1 \exp i\{(\omega+\Omega)t - (k_0+K)x\} + a_{-1} \exp i\{(\omega-\Omega)t - (k_0-K)x\} + \ldots], \quad \text{... (56)}$$

where A is an arbitary constant

$$\left.\begin{aligned}
\lambda_a &= -\frac{\frac{h^2}{4}\frac{\omega}{C}\left[\left(\frac{C}{V}+1\right)+\frac{\omega^2}{\Omega^2}\left(\frac{C}{V}-3\right)\right]}{\left(1-\frac{C}{V}\right)\left[\left(\frac{C}{V}+1\right)^2-4\frac{\omega^2}{\Omega^2}\right]}\\
k_0 &= \frac{\omega}{C}\\
a_1 &= -\frac{h}{2}\,\frac{(\omega+\Omega)^2}{\Omega\left(1-\frac{C}{V}\right)\left[\Omega\left(\frac{C}{V}+1\right)+2\omega\right]}\\
a_{-1} &= -\frac{h}{2}\,\frac{(\omega-\Omega)^2}{\Omega\left(1-\frac{C}{V}\right)\left[\Omega\left(\frac{C}{V}+1\right)-2\omega\right]}
\end{aligned}\right\} \quad \text{... (57)}$$

For the backward wave β is given by the relation

$$\omega = \frac{C\beta}{\frac{C}{V}+1} + h^2\left(\frac{\Omega\,\alpha_1\alpha^2}{2} + S_2\right) \qquad \ldots (58)$$

The wave is given by

$$E(x,t) = B\exp i\lambda_b x[\exp i(\omega t + k_0 x) + b_1 \exp i\{(\omega+\Omega)t + (k_0 - K)x\} + b_{-1}\exp i\{(\omega-\Omega)t + (k_0+K)x\} + \ldots], \qquad \ldots (59)$$

where B is an arbitrary constant and

$$\left.\begin{aligned} \lambda_b &= -\frac{h^2}{4}\frac{\omega}{C}\frac{\left[\left(1-\frac{C}{V}-\frac{\omega^2}{\Omega^2}\left(\frac{C}{V}+3\right)\right)\right]}{\left(1+\frac{C}{V}\right)\left[\left(1-\frac{C}{V}\right)^2 - 4\frac{\omega^2}{\Omega^2}\right]} \\ b_1 &= -\frac{h}{2}\frac{(\omega+\Omega)^2}{\Omega\left(1+\frac{C}{V}\right)\left[\Omega\left(1-\frac{C}{V}\right)+2\omega\right.} \\ b_{-1} &= -\frac{h}{2}\frac{(\omega-\Omega)^2}{\Omega\left(1+\frac{C}{V}\right)\left[\Omega\left(1-\frac{C}{V}\right)-2\omega\right.} \end{aligned}\right\} \qquad \ldots (60)$$

Eqs. (56) and (60) show that in a permittivity modulated medium the electric field exists as a superposition of waves of frequencies ω, $\omega \pm \Omega$ etc. with different amplitudes. The different frequency components have different velocities and the medium is turned dispersive. Besides the velocity of each frequency component will be different for the waves travelling along the direction of the pump wave and opposite. For a real ω, the propagation constant k_0 is also real. The amplitudes of the waves do not grow exponentially in space and time and hence the waves are stable in the medium.

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