Indian J. Phys. **47**, 385–389 (1973)

Notes on the covariance of Maxwell's equations

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(*Received* 6 *July* 1972)

It is shown that the Maxwell equations are covariant with respect to the space-time transformation,

$$x' = (x - c^2t/w)\gamma, \quad t' = (t - x/w)\gamma, \quad y' = y, \quad z' = z, \quad \gamma = (1 - c^2/w^2)^{-\frac{1}{2}}$$

$|w| > c$. A few consequences of this are discussed

1. Introduction

The covariance property of the Maxwell equation with respect to the full Lorentz group is quite familiar. But it is also well known that the group of automorphism of Maxwell's equation is larger than this as it contains the conformal group (Cunningham 1910, Bateman 1910). This comprises of transformations which are non-linear in space-time Sen (1936) arrived at a very important result that the linearty of space-time transformations is a necessary consequence of the principle of equivalence and continuity. Recently Dutta *et al* (1970) have pointed to the interesting fact that the linearity, is almost a mathematical consequence of the principle of relativity. On the other hand, long back, Frank (1911) showed that if one restricted oneself to the only linear group of space-time transformation, then the group, with respect to which Maxwell's equations were covariant, was the Lorentz group together with the ordinary affine group. This consists of linear transformations ($\boldsymbol{r}$, ct) to ($\boldsymbol{r}'$, ct') such that

$$\boldsymbol{r}'.\boldsymbol{r}' - c^2t'^2 = \pm k(\boldsymbol{r}.\boldsymbol{r} - c^2t^2); \qquad \ldots \quad (1)$$

where k is a positive constant. With the positive sign, the constant leads to the admission of the group of transformations, which is associated with the change of scale for length or time or for both This is the common invariance group of all physical phenomena. But, with the negative sign is associated, in general, linear space-time transformations with complex coefficients which are not physically meaningful. In this context it may not be irrelevant to mention that recently there has been attempts to introduce complex space-time transformations to investigate the possibility of faster-than-light particles. However, it has been shown by the author (Sen Gupta 1966) that velocities greater than that of light may be introduced in the linear space-time transformations which keep the relation (1) invariant with positive sign on the right hand side. Hence, the Maxwell equations should also be covariant under such transformations. The object of this short paper is to show this explicitly and discuss some of its physical consequences. Nevertheless, it is worth mentioning that the negative sign in (1) may still be admitted. In this case the space-time transformations are effected by multiplying them with $\pm i$, so that their ratios, which are physically important quantities, are always real. The consequences of these transformations on general tensors are multiplications by i, $-i$, 1 or -1 consistent with the rank of the tensor. But these do not lead to any new physical content of the theory.

In recent years, there has been a good deal of discussion about particles and systems moving with velocity greater than that of light (in empty space). This upper bound of the velocity follows in a natural manner from the usual expression of the Lorentz transformation, which is one of the invariance group of Maxwell's equations. It is worthwhile to investigate whether it is possible to express the Lorentz transformations by incorporating velocities greater than that of light. At this stage, it should be emphasized that in deriving the usual expression for the Lorentz transformation, in addition to the relation (1), it is assumed that in the limit when the relative velocity is small, the transformations should lead to those of Galilei. This may be imperative when one is not only interested in the Maxwell equations but also the equations of motion in classical mechanics. As a matter of fact, in a previous investigation the author (Sen Gupta 1966) showed that linear transformations of space-time, restricting only to the relation (1), might be expressed with velocities greater than that of light. In the following section, after a brief discussion on these transformations, we shall show the covariance of Maxwell's equations with respect to these transformations. The last section is a discussion on some of its consequences.

2. The Covariance of Maxwell's Equation

a) The space-time transformations

We shall confine our discussions to the accelerating part of the transformations, leaving out the spatial rotation; since it does not contribute anything new in the context of our present investigation. Let the space-time transformation be be written in the form

$$x' = (x-c^2t/w)\gamma,\ y' = y,\ z' = z,\ t' = (t-x/w)\gamma;\ \gamma = (1-c^2/w^2)(\ldots)^{-\frac{1}{2}} \quad \ldots \quad (2)$$

$|w| > c$. It is easy to see that

$$x'^2+y'^2+z'^2-c^2t'^2 = x^2+y^2+z^2-c^2t^2. \quad \ldots \quad (3)$$

The coefficients are real only when $w^2 > c^2$. They also form a group. It was for the first time introduced by the author (Sen Gupta 1966). Two successive transformations as given by (2) with parameters w_1, w_2 lead to a transformation with the parameter w' given by

$$w' = \frac{c^2+w_1w_2}{w_1+w_2} \tag{4}$$

On writing (2) in the form

$$x' = -\frac{(t-xw/c^2)c^2}{(w^2-c^2)^{\frac{1}{2}}}, \quad y' = y, \quad z' = x, \quad t' = -\frac{x-wt}{(w^2-c^2)^{\frac{1}{2}}},$$

one notes, but for the coefficients and the negative sign, the roles of space and time in this transformation are interchanged with those of the Lorentz transformation, which in turn, is responsible for introducing the velocity greater than that of light. Naturally, the limiting transformation obtained from (2) by taking the limit $(w/c) \to \infty$ is

$$x' = x, \quad y' = y, \quad z' = z, \quad t' = t-x/w. \tag{5}$$

It is quite different from the Galilei transformation. The most general element of this limiting group obtained by incorporating a spatial rotation $\mathcal{R}$ is

$$\boldsymbol{r}' = \mathcal{R}\boldsymbol{r}, \quad t' = t-(\boldsymbol{n}\cdot\boldsymbol{r})/w, \tag{6}$$

where $\boldsymbol{n}$ is the unit vector along the direction of velocity. The structure of this group has been studied by Levy-Leblond (1965) and the author (Sen Gupta 1966, 1968*a*).

(*b*) *The transformation of Maxwell's Equation*

The Maxwell equations may be written as

$$\left.\begin{aligned} &\nabla\times\boldsymbol{H}-\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t} = \boldsymbol{j}, && \nabla\cdot\boldsymbol{E} = \rho \\ &\nabla\times\boldsymbol{E}+\frac{1}{c}\frac{\partial\boldsymbol{H}}{\partial t} = 0, && \nabla\cdot\boldsymbol{H} = 0. \end{aligned}\right\} \quad \ldots \tag{7}$$

Let us introduce the space-time transformation

$$r'_{\parallel} = (r_{\parallel}-c^2t/w)\gamma, \; t' = (t-r_{\parallel}/w)\gamma, \; \boldsymbol{r}'_{\perp} = \boldsymbol{r}_{\perp}. \quad \ldots \tag{8}$$

For any vector $\boldsymbol{q}$, $q_{\parallel} = (\boldsymbol{n}\cdot\boldsymbol{q})$, $\boldsymbol{n}$ the unit vector along the direction of velocity and $\boldsymbol{q}_{\perp} = \boldsymbol{q}-\boldsymbol{n}q_{\parallel}$. It can be easily checked that the Maxwell equation (7) can, be written in a covariant form by introducing the following transformation of the field quantities $\boldsymbol{E},\boldsymbol{H}$;

$$\left.\begin{aligned} &E'_{\parallel} = E_{\parallel}, && \boldsymbol{E}'_{\perp} = \{\boldsymbol{E}_{\perp}+(c/w)\boldsymbol{n}\times\boldsymbol{H}\}\gamma \\ &H'_{\parallel} = H_{\parallel}, && \boldsymbol{H}'_{\perp} = \{\boldsymbol{H}_{\perp}-(c/w)\boldsymbol{n}\times\boldsymbol{E}\}\gamma \end{aligned}\right\} \tag{9}$$

and the charge-current

$$j'_{\parallel} = (j_{\parallel} - c\rho/w)\gamma, \quad \rho' = (\rho - cj_{\parallel}/w)\gamma \quad \boldsymbol{j}'_{\perp} = \boldsymbol{j}_{\perp}. \tag{10}$$

The above transformation shows explicitly that the Maxwell equations may also be written in a covariant form by incorporating a velocity greater than that of light It is obvious from (9) that as usual $\boldsymbol{E}.\boldsymbol{E}-\boldsymbol{H}.\boldsymbol{H}$ and $\boldsymbol{E}.\boldsymbol{H}$ are invariants The usual expression for the Poynting theorem is also valid in the transformed system, but the physical interpretations of the relevant quantities, namely $\boldsymbol{E}.\boldsymbol{E}+\boldsymbol{H}.\boldsymbol{H}$ and $\boldsymbol{E}\times\boldsymbol{H}$ should not be carried over without due alterations. On the contrary they need critical examination in the context of the physical process pertinent to the problem

3. Discussion

It needs to be mentioned that the transformations (2) may be obtained from the usual expression for the Lorentz transformation on replacing v/c, $(v < c)$ of the latter by c/w Both the ratios being less than unity, it leads to $w > c$. An interesting consequence of this transformations on the charge-current is worth-mentioning. In order to show this, let us take the simple case of charge-current due to a point charge, moving with uniform velocity $\boldsymbol{n}u$, thus

$$\rho = \rho_0\delta(\boldsymbol{r}-\boldsymbol{n}ut), \qquad \boldsymbol{j} = \boldsymbol{n}\rho u/c \tag{11}$$

$(\rho_0 = \text{constant})$ and the transformed charge-current

$$\rho' = (1-u/w)\rho\gamma, \quad j'_{\parallel} = \{(u/c)-(c/w)\}\gamma\rho, \quad \boldsymbol{j}'_{\perp} = \boldsymbol{j}_{\perp}. \tag{12}$$

For particles moving with a velocity $|u| > c$, the above equations show that with (1) $w = u$, $\rho' = 0$ and $j_{\parallel}$ is only a function of time; and (2) $w > u$, ρ' the charge changes sign, but the current changes direction only if $u < c^2/w$. This possibility of reducing the charge current only to current depending only on time was utilized by the author in studying possible nature of the electromagnetic field produced by faster-than-light charged particles (Sen Gupta 1971*a*). In the investigations of the Cerenkov radiation, (Iwanenko & Sokolow 1953, Sen Gupta 1965, 1971*a,b*), the change in the sign of the electric field, due to a particle moving in a homogeneous medium, when the velocity increases from the phase velocity of the electromagnetic waves in the medium, is quite well-known. It may be effectively looked upon as an apparent change in sign of the charge As a matter of fact with the help of a transformation similar to that in (2) the author (Sen Gupta 1968*b*) was able to reduce the problem of the Cerenkov radiation to that of an antenna It is also worthwhile to mention that Sommerfeld (1904*a, b*) and Schott (1912) in their investigations on the electromagnetic field due to charges moving with velocity greater that of light also noted this change of sign of the electric field.

Finally, the transformation (2) may be combined with the usual Lorentz transformation ($v < c$), for the velocities in the same direction, the resulting velocity w' is given by

$$w' = \frac{v+w}{1+vw/c^2} \geqslant c^2 \quad (\text{for} \quad w > c, \quad v < c).$$

In our exposition we have confined ourselves to the transformation properties of the Maxwell equations only. As noted in the introduction the classical equations of motion are not covariant with respect to these transformations, as the classical laws of motion are covariant only with the Galilei transformations, in the non-relativistic limit

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