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Magnetohydrodynamic laminar flow along a vertical wall

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The problem of magnetohydrodynamic laminar flow of an electrically conducting fluid along a vertical wall under the influence of gravity and in presence of transverse magnetic field is studied. The effect of transverse magnetic field on the boundary layer thickness and on the film thickness from a vertical wall is considered when surface tension is neglected

Symbols Used

x = coordinate axis along direction of motion

y = coordinate axis normal to wall

u = x-component of local velocity

v = y-component of local velocity

P, ρ, μ, σ = pressure, density, viscosity and electrical conductivity of the fluid, respectively

g = acceleration due to gravity

ν = Kinematic viscosity of the liquid

$h(x)$ = film thickness

$\delta(x)$ = boundary layer thickness

M = Hartmann number

B = external magnetic field

ϕ = dimensionless variable $\left(\phi^2 = \dfrac{3U_0\nu}{gh_0^2}\right)$

ξ = dimensionless variable $\left(\xi^2 = 1+\dfrac{2gh_0\bar{x}}{U_0^2}\right)$

$K = \frac{5}{8}\phi^2 M^2$

INTRODUCTION

The solution of the laminar accelerated flow of a thin film falling along a vertical wall has been investigated by Haugen (1968). In his analysis the film is considered to have initial thickness h_0 and uniform velocity U_0. Due to gravitational acceleration, the velocity U_s in the core fluid outside the boundary layer increases along the length.

The problem considered here is the laminar accelerating flow of an incompressible conducting fluid along a vertical wall under the influence of gravity and in presence of transverse magnetic field when surface tension effects are neglected. The effects of magnetic field on the boundary layer thickness and on the film thickness respectively have been investigated. Also, for fully developed flow the uniform flow thickness is reached at infinite distance. When $M = 0$, *i.e*, when transverse magnetic field is not applied it reduces to the case of Haugen.

Equations of Motion

A coordinate system is chosen with the wall lying along the positive x-axis and the y-axis being perpendicular to it Assuming the rate of change of film thickness to be small, the velocity component normal to the surface of the wall will be small compared to the velocity component in the main direction of flow along the wall. Under these conditions, continuity and Navier-Stoke's equations may be written as

Continuity :

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \dots \quad (1)$$

Momentum :

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+g+\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B^2 u}{\rho}, \qquad \dots \quad (2)$$

Neglecting the pressure gradient, the integral form of the continuity equation and momentum equation becomes :

$$\int^{\delta(x)} u(x)\mathrm{d}y+U_s(x)[h(x)-\delta(x)] = \text{constant} \qquad \dots \quad (3)$$

and

$$-\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\delta(x)} u(x)^2\mathrm{d}y+U_s(x)\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\delta(x)} u(x)\mathrm{d}y+g\delta(x)+\frac{\sigma B^2}{\rho}\int_0^{\delta(x)}(U_s-u)\mathrm{d}y=\nu\left(\frac{\partial u}{\partial y}\right)_w, \qquad \dots \quad (4)$$

Method of Solution

We solve (4) by assuming a polynomial of second degree for the velocity function

$$u = a(x)+b(x)y^2 \qquad \dots \quad (5)$$

The coefficients $a(x)$ and $b(x)$ are evaluated by using the following conditions :

$$\left.\begin{array}{ll} u = 0 & \text{at } y = 0, \\ u = U_s \text{ and } \dfrac{\partial u}{\partial y} = 0 & \text{at } y = \delta \end{array}\right\} \quad \ldots (6)$$

The first condition of (6) is that of no-slip at the wall and the second is that at the free surface of the boundary layer.

Hence equation (5), becomes

$$\frac{u}{U_s} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 . \quad \ldots (7)$$

By using equation (7), equation (4) becomes

$$-\frac{d}{dx}\left(\frac{8\delta U_0^2}{15}\right) + U_s \frac{d}{dx}\left(\frac{2\delta U_s}{3}\right) + g\delta(x) + \frac{\sigma B^2}{\rho}\frac{U_s\delta}{3} = \frac{2U_s\nu}{\delta}, \quad \ldots (8)$$

or

$$\frac{2}{15}U_s^2\frac{d\delta}{dx} - \frac{2\delta U_s}{5}\frac{dU_s}{dx} + g\delta = \frac{U_s\nu}{3\delta}\left(6 - \frac{\sigma B^2\delta^2}{\mu}\right), \quad \ldots (9)$$

For the case of zero pressure gradients, Euler's equation may be written as

$$\frac{d(\frac{1}{2}U_s^2)}{dx} = g, \quad \ldots (10)$$

i e.,

$$U_s = (U_0^2 + 2gx)^{\frac{1}{2}}. \quad \ldots (11)$$

Substituting the values of U_s in equation (9), we have

$$\frac{2}{15}(U_0^2+2gx)\frac{d\delta}{dx} + \frac{3g\delta}{5} = \frac{\nu(U_0^2+2gx)^{\frac{1}{2}}}{3\delta}\left(6 - \frac{\sigma B^2\delta^2}{\mu}\right). \quad \ldots (12)$$

Changing equation (12) into non-dimensional form, we have

$$\frac{d\bar{\delta}}{d\xi} + \frac{9\bar{\delta}}{2\xi} - \frac{5}{6}\frac{\phi^2}{\bar{\delta}}(6 - M^2\bar{\delta}^2) = 0, \quad \ldots (13)$$

where

$$\left.\begin{array}{ll} \bar{\delta} = \dfrac{\delta}{h_0}, & \bar{x} = \dfrac{x}{h_0}, \\ \phi^2 = \dfrac{3U_0\nu}{gh_0^2}, & \xi^2 = 1 + \dfrac{2gh_0\bar{x}}{U_0^2}, \\ M^2 = \dfrac{\sigma B^2 h_0^2}{\mu}. & \end{array}\right\} \quad \ldots (14)$$

Determination of boundary layer thickness :

Case I : when K is small.

The solution of equation (13) is

$$\bar{\delta}^2 = \frac{\phi^2(\xi+\frac{10}{11}K\xi^2)}{1+K\xi} + c\xi^{-9}(1+K\xi)^{-1}.$$

at $\xi=1$, $\bar{\delta}=0$; therefore $c = -\phi^2(1+\frac{10}{11}K)$

Hence,

$$\bar{\delta} = \frac{\phi}{\sqrt{1+K\xi}}\left[\xi+\frac{10}{11}K\xi^2-\xi^{-9}\left(1+\frac{10}{11}K\right)\right]^{\frac{1}{2}} \qquad \ldots (15)$$

Case II. When K is large:

The solution of (13) is

$$\bar{\delta}^2 = 10\phi^2\left[\frac{1}{K}-\frac{9}{K^2}\xi^{-1}+\frac{9.8}{K^3}\xi^{-2}-\frac{9.8.7}{K^4}\xi^{-3}+\frac{9.8.7.6}{K^5}\xi^{-4}-\frac{9.8.7.6.5}{K^6}\xi^{-5}+\right.$$
$$\left.+\frac{9.8.7.6.5.4}{K^7}\xi^{-6}-\frac{9.8.7.6.5.4.3}{K^8}\xi^{-7}+\frac{\lfloor 9}{K^9}\xi^{-8}-\frac{\lfloor 9}{K^{10}}\xi^{-9}\right]+c\xi^{-9}e^{-k\xi}$$

at $\xi=1$, $\bar{\delta}=0$; therefore,

$$c = -10\phi^2\left[\frac{1}{K}-\frac{9}{K^2}+\frac{9.8}{K^3}-\frac{9.8.7}{K^4}+\frac{9.8.7.6}{K^5}-\frac{9.8.7.6.5}{K^6}+\right.$$
$$\left.+\frac{9.8.7.6.5.4}{K^7}-\frac{9.8.7.6.5.4.3}{K^8}+\frac{\lfloor 9}{K^9}-\frac{\lfloor 9}{K^{10}}\right]e^k$$

or

$$\bar{\delta} = \sqrt{10}\phi\left[\frac{1}{K}(1-\xi^{-9}e^{-\xi})-\frac{9}{K^2}(\xi^{-1}-\xi^{-9}e^{-\xi})+\frac{9.8}{K^3}(\xi^{-2}-\xi^{-9}e^{-\xi})-\right.$$
$$-\frac{9.8.7}{K^4}(\xi^{-3}-\xi^{-9}e^{-\xi})+\frac{9.8.7.6}{K^5}(\xi^{-4}-\xi^{-9}e^{-\xi})-\frac{9.8.7.6.5}{K^6}(\xi^{-5}-\xi^{-9}e^{-\xi})+$$
$$+\frac{9.8.7.6.5.4}{K^7}(\xi^{-6}-\xi^{-9}e^{-\xi})-\frac{9.8.7.6.5.4.3}{K^8}(\xi^{-7}-\xi^{-9}e^{-\xi})+$$
$$\left.+\frac{\lfloor 9}{K^9}(\xi^{-8}-\xi^{-9}e^{-\xi})-\frac{\lfloor 9}{K^{10}}(\xi^{-9}-\xi^{-9}e^{-\xi})\right]^{\frac{1}{2}} \qquad \ldots (16)$$

Determination of film thickness :

Using equation (3), the film thickness $h(x)$ is given by

$$h(x) = \delta(x)+\frac{h_0 U_0}{U_s}-\delta\int_0^1 \frac{u}{U_s}\,dy, \qquad \dots (17)$$

$$\frac{U_s}{U_0} = \xi, \qquad \dots (18)$$

Therefore,

$$\bar{h} = \frac{h}{h_0} = \xi^{-1}+\frac{\delta}{3} \qquad \dots (19)$$

Thus equations (15), (16) and (19) determines the boundary layer thickness and film thickness, respectively and is plotted in figures 1 and 2.

Solution in Fully Developed Region

Since, in fully developed region the film thickness $h(x)$ and boundary layer thickness $\delta(x)$ coincide, so in this region the continuity equation (3) and momentum equation (4), becomes

$$\int^{\delta(x)} u\,dy = h_0 U_0 = \frac{2\delta U_s}{3} \qquad \dots (20)$$

and

$$-\frac{d}{dx}\int_0^{\delta(x)} u(x)^2 dy + U_s(x)\frac{d}{dx}\int^{\delta(x)} u(x)dy + g\delta(x) + \frac{\sigma B^2}{\rho}\int_0^{\delta(x)} (U_s-u)dy$$

$$= \nu\left(\frac{\partial u}{\partial y}\right)_\omega . \qquad \dots (21)$$

Substituting (20) in (21), we have

$$\frac{d\bar{\delta}}{d\bar{\xi}} = \tfrac{5}{6}\,\xi(\phi^2-\delta^3-\tfrac{1}{6}M^2\phi^2\bar{\delta}^2) \qquad \dots (22)$$

Solving equation (22) numerically when $M^2 = 1{\cdot}0$, we have

$$\xi^2 = \tfrac{6}{17}\log\frac{5\bar{\delta}^2+6\delta+6}{(\delta-1)^2}+\frac{48}{17\sqrt{21}}\tan^{-1}\frac{5\bar{\delta}^2+3}{\sqrt{21}}-c_1 \qquad \dots (23)$$

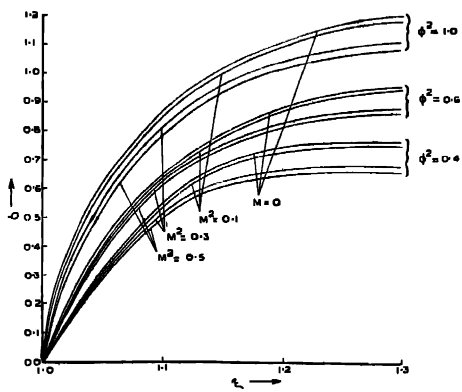
where

$$c_1 = -\xi^{*2}+\tfrac{6}{17}\log\frac{5\delta^{*2}+6\delta^{*2}+6}{(\bar{\delta}^*-1)^2}+\frac{48}{17\sqrt{21}}\tan^{-1}\frac{5\bar{\delta}^{*2}+3}{\sqrt{21}}$$

where $\delta = \xi$, *i.e.*, at on set of fully developed flow.

CONCLUSIONS

In the present work, MHD laminar flow of an electrically conducting fluid along a vertical wall in presence of transverse magnetic field is considered, and a number of analytical solutions have been presented, which are given in equations (7), (15), (16), (19) and (23). The analysis provides a better understanding of the film flow of the type that generally occurs in cooling films and which is often found in chemical process packing towers. In the present problem, only the magnetic force, the viscous force and the gravitational force are the important forces; hence Hartmann number M and ϕ are the important parameters. In equation (15) as K tends to zero, we have the boundary layer thickness in ordinary hydrodynamics From figure 1, it is clear that (i) for fixed value of ϕ^2 and for a



..1 BOUNDARY LAYER THICKNESS VERSUS DIMENSIONLESS VARIABLE ξ

particular value of Hartmann number the boundary layer thickness $\bar{\delta}$ increases as ξ increases; (ii) for fixed value of ξ the boundary layer thickness decreases as Hartman number increases; (iii) for fixed value of ξ and Hartman number the boundary layer thickness increases as ϕ^2 increases. From figure 2, it is clear that (i) for fixed value of ξ the film thickness increases as ϕ^2 increases and (ii) for fixed value of ξ the film thickness decreases as Hartman number increases. When magnetic field is very large, it is clear from equation (16) that the boundary layer

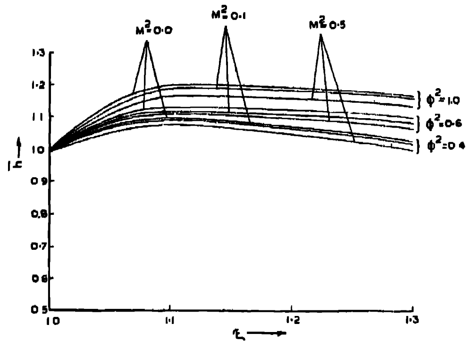


FIG 2 FILM THICKNESS VERSUS DIMENSIONLESS VARIABLE ξ

thickness becomes zero. From equation (23) it is clear that for $\bar{\delta} = 1$, *i.e.*, for fully developed flow the uniform flow thickness is reached at infinite distance.

Reference

Haugen R. 1968 *Journal of Applied Mechanics*, **35**, *series E*, *no* 4, 630