Indian J. Phys. **47** 423-431 (1973)

Heat transfer in a second-order fluid with suction and constant heat sources

R. S. AGARWAL

Department of Applied Mathematics, University of Roorkee, Roorkee

AND

S. P. S. BHATIA*

School of Basic Sciences, U. P. Agriculture University, Pantnagar, Nainital

(*Received* 30 *June* 1972, *revised* 9 *October* 1972)

The heat transfer in the laminar boundary layer due to the flow of a second order fluid over a flat plate subjected to suction in the presence of constant heat sources has been discussed. The solution of the problem is sought by expanding the flow and temperature functions in powers of $1/\lambda$, λ being the suction parameter. Besides other results it is interesting to observe that an increase in elastico-viscous parameter, γ, decreases the temperature near the plate irrespective of suction.

1. INTRODUCTION

Due to its great applicability to the space vehicle re-entry problems, the study of heat transfer in presence of heat sources has acquired newer dimensions. Quite a few analytical studies have been carried out for various forms of heat generation (Low 1955, Fay & Riddell 1958, Sparrow & Cess 1960). Sastri (1965*a*,*b*) analysed suction effects on heat transfer in the presence of constant as well as temperature dependent heat sources. In the present paper we have studied the heat phenomenon in a second order fluid over a flat plate, when the latter is subjected to suction and the fluid flows with constant heat sources in it. The cross-viscosity of the fluid does not produce any modification due to the problem being two dimensional (Srivastava 1959). The effects of elastico-viscosity on the flow and temperature fields have been investigated and the Nusselt number on the plate has been found.

2. FORMULATION OF THE PROBLEM

Consider on a flat plate the steady flow of an incompressible second order fluid characterized by the rheological equation,

$$\tau_{ij} = 2\mu_1 d_{ij} - 2\mu_2 e_{ij} + 4\mu_3 C_{ij}, \tag{1}$$

---

* Present address : Electrical Engineering Department, Oregon State University, U. S. A.

where μ_1, μ_2*, μ_3 are the material constants and,

$$d_{ij} = \tfrac{1}{2}(v_{i,j}+v_{j,i}),$$
$$e_{ij} = \tfrac{1}{2}(a_{i,j}+a_{j,i}+2v^m{}_{,i}v_{m,j}),$$
$$C_{ij} = d_{i\alpha}d^{\alpha}{}_{j},$$

v_i, a_i being the velocity and acceleration vectors, respectively.

In a cartesian system of reference, x-axis is taken along the plate and y-axis perpendicular to it. In the absence of body forces, the boundary layer equations of motion and continuity are :

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu_1 \frac{\partial^2 u}{\partial y^2} - \nu_2\left[\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial u}{\partial y}\cdot\frac{\partial^2 v}{\partial y^2}+v\frac{\partial^3 u}{\partial y^3}\right], \qquad \ldots (2)$$

$$\frac{\partial u}{\partial x}+\frac{\partial y}{\partial y} = 0, \qquad \ldots (3)$$

where $\nu_1 (= \mu_1/\rho)$ is the kinematic coefficient of viscosity and $\nu_2 (= \mu_2/\rho)$, the coefficient of elastico-viscosity while ρ is the density of the fluid.

Also the energy equation describing the transport of thermal energy in the presence of constant heat sources reduces to the form,

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = \frac{\nu_1}{\sigma}\frac{\partial^2 T}{\partial y^2}+\frac{h}{\rho C_v}, \qquad \ldots (4)$$

where h is the amount of heat generated per unit volume per unit time and is constant, C_v is the specific heat at constant volume and σ is the Prandtl number. The term due to viscous dissipation has been assumed negligible in comparison to the heat transfer across the plate.

The boundary conditions on velocity are :

$$\left.\begin{aligned} &u = 0, \quad v = -v_0(x), && \text{at } y = 0;\\ &u \to u_1 \text{ (constant)} && \text{as } y \to \infty;\end{aligned}\right\} \qquad \ldots (5)$$

and those on temperature are,

$$\left.\begin{aligned} &T = T_w \text{ (constant)}, && \text{at } y = 0;\\ &T \to T_1(x) && \text{at } y \to \infty;\end{aligned}\right\} \qquad \ldots (6)$$

where $v_0(x)$ and u_1 represent, respectively, the velocity of suction and that of the main stream, T_w is the constant temperature on the plate and $T_1(x)$ is the temperature at infinity.

---

* A negative sign with μ_2 has been taken following Markovitz (1964).

Defining a stream function,

$$\psi(x, y) = (\nu_1 u_1 x)^{\frac{1}{2}} f(\eta), \qquad \ldots (7)$$

where $\eta[=(u_1/\nu_1 x)^{\frac{1}{2}} y]$ is the dimensionless distance perpendicular to the plate, we have,

$$u = \frac{\partial \psi}{\partial y} = u_1 f',$$

$$v = -\frac{\partial \psi}{\partial x} = -\tfrac{1}{2}(u_1 \nu_1/x)^{\frac{1}{2}}[f-\eta f'], \qquad \ldots (8)$$

where a prime denotes differentiation with respect to η.

Substituting u and v from (8) in (2) and (3), the latter is identically satisfied while the former reduces to the form,

$$ff''+2f'''-\alpha[f''^2-ff^{iv}-2f'f'''] = 0, \qquad \ldots (9)$$

where $\alpha = (= \nu_2 u_1/\nu_1 x)$ is a dimensionless parameter representing the elastico-viscous effects in the flow under consideration.

Introducing a new variable ξ such that

$$\xi = \lambda \eta, \qquad (10)$$

equation (9) transforms to,

$$2\lambda \dddot{f} + f\ddot{f} + \alpha\lambda^2[f\ddddot{f} + 2\dot{f}\dddot{f} - \ddot{f}^2] = 0, \qquad (11)$$

where a dot denotes differentiation with respect to ξ.

The boundary conditions on f now are :

$$\left.\begin{aligned} &\dot{f} = 0, \quad f = \lambda, && \text{at } \xi = 0; \\ &\dot{f} = 1/\lambda && \text{as } \xi \to \infty; \end{aligned}\right\} \qquad (12)$$

$\lambda\,[= 2v_0(x/u_1\nu_1)^{\frac{1}{2}}]$ being the suction parameter and is greater than 3 ($\lambda = 3$ would lead to inconsistent results, Sastri 1965*a*).

3. Solution of the Problem

We assume the Blasius function in the form

$$f(\xi) = \lambda + \frac{1}{\lambda} f_1(\xi) + \frac{1}{\lambda^2} f_2(\xi) + \frac{1}{\lambda^3} f_3(\xi) + \frac{1}{\lambda^4} f_4(\xi) + \ldots \qquad \ldots (13)$$

With the help of (13) and (12) the equation (11) leads to,

$$f_1 = -2+\xi+2\exp(-\xi/2),$$

$$f_2 = \frac{\gamma}{2}\left[1-\left(1+\frac{\xi}{2}\right)\exp(-\xi/2)\right],$$

$$f_3 = \left(5+\frac{\gamma^2}{8}\right)+\exp(-\xi/2)\left[\frac{\gamma^2}{64}(\xi^2-4\xi-8)-\tfrac{1}{2}(\xi^2+4\xi+12)\right]+\exp(-\xi),$$

$$f_4 = -\frac{5}{2}\gamma+\frac{1}{16}\gamma^3+\frac{1}{4}\gamma\exp(-\xi)(2-\xi)-\gamma\exp(-\xi/2)$$

$$-\left[\frac{\gamma^2}{1536}(\xi^3-18\xi^2+48\xi+96)-\frac{1}{16}(\xi^3+28\xi+32)\right], \qquad \ldots (14)$$

where γ $(=\alpha\lambda^3)$ is the new elastico-viscous parameter

Now the motion being one dimensional outside the boundary layer, the temperature will satisfy the equation,

$$u_1\frac{\partial T_1}{\partial x} = \frac{h}{\rho C_v}, \qquad \ldots (15)$$

which yields,

$$T_1-T_{10} = \frac{h}{\rho C_v u_1} \qquad \ldots (16)$$

where $T_{10} = T_1(0)$.

Substituting $t = (T-T_w)/(T_{10}-T_w)$ and using (8), the equation (4) takes the form,

$$\frac{1}{\sigma}\frac{\partial^2 t}{\partial\eta^2}+\frac{1}{2}f\frac{\partial t}{\partial\eta}-xf'\frac{\partial t}{\partial x}+\frac{hx}{\rho C_v u_1(T_{10}-T_w)} = 0 \qquad \ldots (17)$$

The boundary conditions (6) can be rewritten as,

$$t = 0, \qquad \text{at } \eta = 0;$$

$$t = 1+\frac{hx}{\rho C_v u_1(T_{10}-T_w)}, \quad \text{as } \eta\to\infty \qquad \ldots (18)$$

Introducing a non-dimensional variable,

$$r = \frac{hx}{\rho C_v u_1(T_{10}-T_w)},$$

which can be interpreted as a non-dimensional longitudinal coordinates, we can assume the non-dimensional temperature in the following form

$$t(x,\eta) = t_0(\eta)+rt_1(\eta). \qquad \ldots (19)$$

Substituting t from (19) into (17) and equating to zero the terms independent of r and coefficient of r, the equations to determine t_0 and t_1 in terms of ξ are obtained as

$$\frac{1}{\sigma}\ddot{t}_0+\frac{f}{2\lambda}\dot{t}_0=0, \qquad \dots (20)$$

$$\frac{1}{\sigma}\ddot{t}_1+\frac{f}{2\lambda}\dot{t}_1-\frac{f^{\cdot}}{\lambda}t_1+\frac{1}{\lambda^2}=0. \qquad \dots (21)$$

The corresponding boundary conditions are,

$$\left.\begin{aligned} t_0&=0, & t_1&=0, & &\text{at } \xi=0,\\ t_0&=1, & t_1&=1, & &\text{as } \xi\to\infty. \end{aligned}\right\} \qquad \dots (22)$$

We assume,

$$t_0=\sum_{n=0}^{\infty}\frac{1}{\lambda^n}\,t_{0n}(\xi),$$

$$t_1=\sum_{n=0}^{\infty}\frac{1}{\lambda^n}\,t_{1n}(\xi). \qquad \dots (23)$$

With the help of (23) and (22), the equations (20) and (21) yield,

$$t_{00}=1-\exp(-\sigma\xi/2),$$

$$t_{01}=0,$$

$$t_{02}=\frac{1}{4}\,[\sigma\xi^2-4(\sigma-1)\xi]\exp(-\sigma\xi/2)+\frac{2\sigma^2}{\sigma+1}\{\exp(-\sigma\xi/2)-\exp(-(\sigma+1)\xi/2)\},$$

$$\begin{aligned} t_{03}=\frac{\gamma\sigma}{4(\sigma+1)^2}&[\{(\sigma+1)^2\xi-2\sigma(2\sigma+3)\{\exp(-\sigma\xi/2)\\ &+\{(\sigma+1)\xi+2(2\sigma+3)\{\exp(-(\sigma+1)\xi/2)], \end{aligned}$$

$$\begin{aligned} t_{04}=\frac{\sigma^2}{2}\Bigg[\Bigg\{&\frac{1}{16}\,\xi^4-\frac{(\sigma-1)}{2\sigma}\,\xi^3+\frac{(2\sigma^3-\sigma^2+2)}{\sigma^2(\sigma+1)}\,\xi^2-\\ &-\left(\frac{\gamma^2}{8\sigma}+\frac{(4\sigma^4+\sigma^3+9\sigma^2-4\sigma-8)}{\sigma^3(\sigma+1)}\right)\xi+\frac{\sigma(\sigma+2)}{4(\sigma+1)^3}\gamma^2\\ &+\frac{(4\sigma^4+35\sigma^3+102\sigma^2+95\sigma-16)}{\sigma(\sigma+2)(\sigma+1)^2}\Bigg\}\exp\,(-\sigma\xi/2)\\ &+\Bigg\{\left(\frac{\gamma^2}{32(\sigma+1)}-1\right)\xi^2+\left(\frac{\gamma^2}{8(\sigma+1)^2}+\frac{4(\sigma-3)}{(\sigma+1)}\right)\xi-\\ &-\frac{(\sigma+2)}{4(\sigma+1)^3}\gamma^2-\frac{(8\sigma^3+28\sigma^2+52\sigma-8)}{\sigma(\sigma+1)^2}\Bigg\}\exp\,(-(\sigma+1)\xi/2)\\ &+\frac{(4\sigma+1)}{(\sigma+2)}\exp\,-(\sigma+2)\xi/2)\Bigg], \end{aligned} \qquad \dots (24)$$

and

$$t_{10} = 1-\exp(-\sigma\xi/2), \qquad t_{11} = 0,$$

$$t_{12} = \frac{4\sigma}{(\sigma-1)} \exp(-\xi/2)+\Big[\frac{\sigma}{4} \xi^2-(\sigma-3)\xi+\frac{2\sigma^2(\sigma-5)}{(\sigma^2-1)} \exp(-\sigma\xi/2)$$

$$-\frac{2\sigma(\sigma-2)}{(\sigma+1)} \exp(-(\sigma+1)\xi/2),$$

$$t_{13} = \gamma\sigma\Big[-\frac{1}{(\sigma-1)}\Big\{\frac{1}{2}\xi+\left(\frac{\sigma-2}{\sigma-1}\right)\Big\} \exp(-\xi/2)$$

$$+\Big\{\frac{1}{4}\xi-\frac{1}{2(\sigma^2-1)^2}(2\sigma^4-5\sigma^3-4\sigma^2+15\sigma)\Big\} \exp-\sigma/2$$

$$+\frac{1}{2(\sigma+1)}\Big\{\frac{1}{2}(\sigma-2)\xi+\frac{(2\sigma^2+\sigma-4)}{(\sigma+1)}\Big\} \exp-(\sigma+1)\xi/2\Big],$$

$$t_{14} = \frac{2\sigma}{(\sigma-1)}\Big[\left(\frac{\gamma^2}{64}-\frac{1}{2}\right)\xi^2-\left(\frac{\sigma\gamma^2}{16(\sigma-1)}+4\right)\xi-$$

$$-\frac{(\sigma^2-3\sigma+1)}{8(\sigma-1)^2}\gamma^2-14\Big]\exp(-\xi/2)+\frac{2\sigma(3\sigma-1)}{(\sigma-1)(\sigma-2)}\exp(-\xi)$$

$$+\Big[-\frac{\sigma^2}{32}\xi^4+\frac{\sigma(\sigma-3)}{4}\xi^3-\frac{(2\sigma^4-11\sigma^3+11\sigma^2+6\sigma-12)}{2(\sigma\sigma^2-1)}\xi^2$$

$$+\Big\{\frac{\sigma\gamma^2}{16}+\frac{(4\sigma^5-27\sigma^4+72\sigma^3-53\sigma^2-12\sigma+48)}{2\sigma(\sigma^2-1)}\xi$$

$$+\frac{\gamma^2}{8(\sigma^2-1)^3}(\sigma^7-\sigma^6-3\sigma^5+5\sigma^4-22\sigma^3+4\sigma)-\frac{1}{2(\sigma+1)(\sigma^2-1)(\sigma^2-4)}$$

$$\times(4\sigma^7-9\sigma^6-3\sigma^5-181\sigma^4+23\sigma^3+1318\sigma^2-480\sigma)\Big]\exp(-\sigma\xi/2)$$

$$-\frac{2\sigma}{(\sigma+1)}\Big[\Big\{\frac{(\sigma-2)}{128}\gamma^2-\frac{1}{4}(\sigma^2-\sigma-2)\Big\}\xi^2+\Big\{\frac{3\sigma\gamma^2}{32(\sigma+1)}$$

$$+(\sigma^2-7\sigma+10)\Big\}\xi-\frac{(\sigma^3-6\sigma-2)}{16(\sigma+1)^2}\gamma^2$$

$$-\frac{1}{(\sigma^2-1)}(2\sigma^4-7\sigma^3+24\sigma^2-55\sigma+44)\Big]\exp(-(\sigma+1)\xi/2)$$

$$-\frac{(4\sigma^4-11\sigma^3+5\sigma^2-4\sigma)}{2(\sigma+1)(\sigma+2)}\exp-(\sigma+2)\xi/2). \qquad \ldots (25)$$

The heat flux q from the plate is given by,

$$q = k\left(\frac{\partial T}{\partial y}\right)_{y=0},$$

$$= k(T_{10}-T_w)(u_1/x\nu_1)^{\frac{1}{2}}\dot{t}'(0), \qquad (26)$$

k being the thermal conductivity of the fluid.

The Nusselt number at any x is then,

$$Nu \qquad \frac{\quad}{T_1-T_w}\cdot\frac{\quad}{h}$$

$$= \lambda(Re)^{\frac{1}{2}}\left[\frac{\dot{t}_0(0)+r\dot{t}_1(0)}{1+r}\right]. \qquad \dots (27)$$

4. DISCUSSION OF RESULTS

The values of dimensionless temperature t at different ξ, the dimensionless distance perpendicular to the plate have been calculated for $\sigma = 0{\cdot}5$. The temperature profiles for various γ for $n = 4$ and 10 are, respectively, shown in figures 1 and 2 It is clear that the temperature increases continuously from the plate upwards to an asymptotic value Also an increase in the elastico-viscous parameter γ is followed by a decrease in temperature near the plate. Away from the plate the situation gets reversed With increase in suction the fall or rise in temperature with γ is less pronounced.

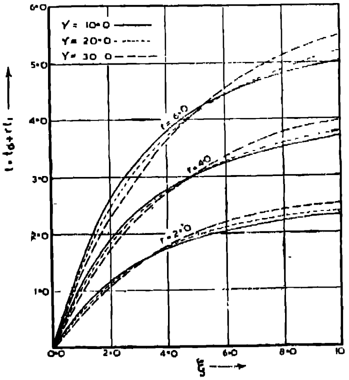

FIG 1. RESPONSE OF TEMPERATURE TO AN INCREASE IN SECOND ORDER EFFECTS (λ = 4 0)

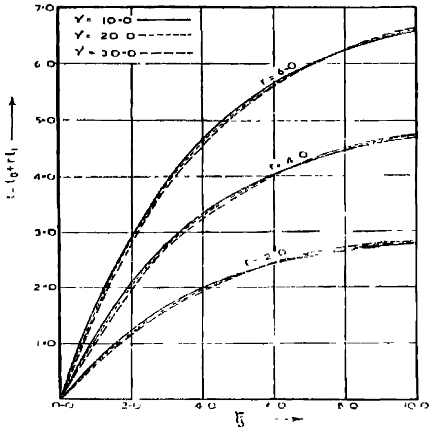

FIG 2 _RESPONSE OF TEMPERATURE TO AN INCREAS IN SECOND ORDER EFFECTS ($\lambda = 10{\cdot}0$)

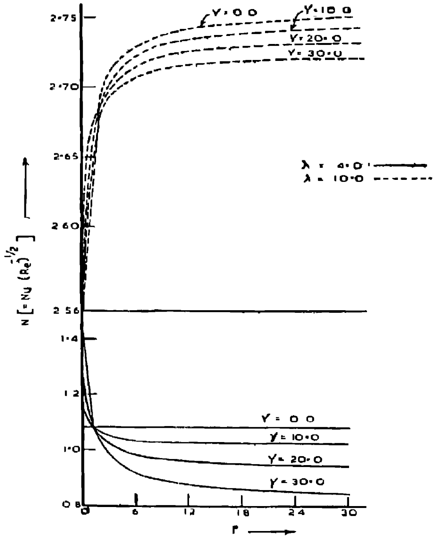

FIG. 3_HEAT TRANSFER PARAMETER $N\left[= N_u\,(R_e)^{-1/2}\right]$ vs THE DIMENSIONLESS LONGITUDNAL COORDINATE r

From figure 3 it can be seen that the heat transfer parameter steadily assumes an asymptotic value. It is also concluded that the cross viscous forces tend to decrease the rate of heating from the plate.

ACKNOWLEDEMENTS

The authors are thankful to Prof. C. Prasad, Head, Department of Mathematics, University of Roorkee for constant encouragement. Thanks are also due to Dr. S K. Sharma, Professor, U.P. Agricultural University, Pantnagar, Nainital for useful suggestions.

REFERENCES

Fay J A & Riddell F. R 1958 *J Aero, Sci.* **25**, 73.
Low G M. 1955 *J. Aero. Sci* **22**, 329.
Markovitz H 1964 *4th Intern. Cong Rheology* John Wiley & Sons, 189
Sastri K S. 1954a *Appl Sci. Res.* A14, No. **1–2**, 126.
Sastri K S. 1965b *Jour Phy. Soc. Japan* 20, **9**, 1711.
Sparrow E. W. & Cess R. D. 1960 *Appl. Sci. Res.* **A10.**
Srivastava A C. 1959 *Ph.D. Thesis, Indian Institute of Technology, Kharagpur*