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Heat and mass transfer and their thermodynamic coupling in boundary layer flow of a viscoelastic fluid past a flat plate

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Studies on the steady state heat and mass transfer of a viscoelastic fluid past a flat plate are presented in this paper. The plate is either porous such that blowing or suction of a diffusing fluid through the surface can be effected or it is made of such material, as sublimes into a hot boundary layer fluid, thus introducing a mass transfer process. According to Thermodynamics of Irreversible Processes, the fluxes of heat and mass are coupled. The effects of mass transfer on the temperature profiles have been studied, neglecting the effects of temperature gradient on concentration fields. It is observed that the skin friction, heat transfer and mass transfer coefficients increase with suction and decrease with injection The viscoelasticity of the fluid increases the skin friction and decreases the heat and mass transfer coefficients. The effect of thermodiffusion coefficient is to increase the temperature at a point.

1. INTRODUCTION

The importance of the effects of thermodynamic coupling between heat and mass transfer on heat transfer and adiabatic wall temperature in binary systems have been discussed in some recent papers (Baron 1962, Tewfik & Shirliffe 1962). Measurements, with helium injection into an incompressible turbulent boundary layer (Tewfik *et al* 1962) clearly demonstrated experimentally significant effects of the coupling in an important engineering application of injection process Subsequent measurements (Tewfik 1964) showed that the heat flux due to coupling was of the same order of magnitude and even exceeded the familiar Fourier's condition of heat flux in opposite directions

Following Ibbs & Grew (1952), we have neglected the Soret effect, *i.e.*, the effect of temperature on mass, which is very small. The general conservation equations, taking such coupling into consideration, were developed by Baron (1959) for two dimensional flow These equations have been extended in this paper for the viscoelastic fluid represented by Walters B″ model (Beard & Walters 1964). It is thought that the boundary layer flows are mostly developed in viscoelastic fluids, that are mobile and highly elastic. We shall regard the flow as a perturbation of a Newtonian viscous flow so that usual concept of boundary layer can be assumed to apply.

2. Governing Equations

Following Baron (1959), Beard & Waltors (1964) and Tewfik & Shirtiffe (1962) and neglecting the Soret effect, the governing equations for two dimensional boundary layer flow, heat and mass transfer may be obtained as :

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad \ldots (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}-k\left[u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}\right], \ldots (2)$$

$$u\frac{\partial w_1}{\partial x}+v\frac{\partial w_1}{\partial y}=D\frac{\partial^2 w_1}{\partial y^2}, \qquad \ldots (3)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+D_{12}D\frac{\partial}{\partial y}\left(T\frac{\partial w_1}{\partial y}\right), \qquad \ldots (4)$$

with the following boundary conditions

$$\left.\begin{aligned} &u=0,\ v=V_0,\ T=T_w=\text{constant},\ w_1=w_{1w} \text{ at } y=0\\ &u=U_1,\ T=T_\infty,\ w_1=w_{1\infty} \quad \text{at} \quad y\to\infty. \end{aligned}\right\} \qquad \ldots (5)$$

The value $v=V_0$ shows injection if $V_0>0$ and suction if $V_0<0$. In the above u, v are the velocities of the fluid along x and y axis, p isotropic pressure, k viscoelastic parameter, U_1 velocity outside the boundary, layer, D, D_{12} and α are coefficients of diffusion, thermodiffusion and the thermal diffusivity ($k^*/\rho c$).

3. Similarity Transformations

A search for the similarity solutions of the non-linear partial differential equations reveal that the only possible form for the pressure gradient in these circumstances is of the form $C.x$ where C is a constant, defined by $C=U_1^2$. Introducing the stream function $\phi(x, y)$ such that,

$$u=\frac{\partial\phi}{\partial y},\ v=-\frac{\partial\phi}{\partial x} \qquad \ldots (6)$$

where

$$\phi=(\nu U_1)^{\frac{1}{2}}xf(\eta)$$

and using the following nondimensional transformations :

$$f(\eta)=\frac{\phi(x,y)}{(\nu U_1)^{\frac{1}{2}}x},\ \theta=\frac{T(x,y)-T_w}{T_\infty-T_w} \qquad \ldots (7a)$$

$$W(\eta)=\frac{w_1(x,y)-w_{1w}}{w_{1\infty}-w_{1w}},\ \eta=\left(\frac{U_1}{\nu}\right)^{\frac{1}{2}}y, \qquad \ldots (7b)$$

the governing equations (1–4) and the boundary conditions (5) become,

$$f'''+ff''+1-f'^2+k[ff^{iv}-2f'f'''+f''^2] = 0 \qquad \dots (8)$$

$$\theta''+(Pr)f\theta'+\beta(Sc)(\theta'W'+\theta W'') = 0 \qquad \dots (9)$$

$$W''+(Sc)fW' = 0 \qquad \dots (10)$$

and

$$\left.\begin{aligned} &f' = 0,\ f = -\frac{V_0}{(U_1\nu)^{\frac{1}{2}}} = f^0,\ W = 0,\ \theta = 0 \text{ at } \eta = 0 \\ &f' = 1,\ \theta = 1,\ W = 1 \quad \text{at} \quad \eta = \infty, \end{aligned}\right\} \qquad \dots (11)$$

where $(Pr) = (\mu c/K)$ Prandtl number, $(Sc) = (\nu/D)$ Schmict number, β thermodiffusion coefficient, k viscoelastic parameter.

It may be noted that $f^0 < 0$ indicates injection or blowing of a foreign or coolant fluid across the porous wall into the mainstream fluid, whereas $f^0 > 0$ depicts suction from the boundary layer fluid or mass transfer towards the wall.

4. Solution of the Problem

A. *Momentum Transfer* : It has been observed that the presence of elasticity in the fluid yields a fourth order differential equation (8), whereas in viscous case ($k = 0$), the order of the equation is three. It would thus appear that an additional boundary condition must be imposed to obtain a solution. However, in the derivation of equation (2), we have implicitly neglected the terms of the order of k^2. It is therefore reasonable to seek a solution for f in the form

$$f = f_0+kf_1+k^2(0). \qquad \dots (12)$$

Substituting in equation (8) and equating the coefficients of k and independent terms on both sides we get :

$$f_0'''+f_0f_0''+1-f_0'^2 = 0 \qquad \dots (13)$$

and

$$f_1'''+f_0f_1''+f_1f_0''-2f_0'f_1'+f_0f_0^{iv}-2f_0'f_0'''+f_0''^2 = 0, \qquad \dots (14)$$

with the boundary conditions,

$$f_0(0) = f^0,\ f_0'(0) = 0,\ f_0'(\infty) = 1 \qquad \dots (15)$$

and

$$f_1(0) = f_1'(0) = 0, \quad f_1'(\infty) = 0. \qquad \dots (16)$$

The nonlinear part of the equation (13) is separated from the linear part and sequences of approximations are thus constructed using the Laplace transformation as follows :

$$f_{0,1} = f^0+\frac{\alpha_1}{2!}\eta^2-\frac{1}{3!}\eta^3 \qquad \dots (17)$$

and

$$f_{0,2} = f^0+\frac{\alpha_1}{2!}\eta^2-\frac{(1-\alpha_1 f^0)}{3!}\eta^3+\frac{f^0}{4!}\eta^4+\frac{\alpha_1^2}{5!}\eta^5-\frac{2\alpha_1}{6!}\eta^6+\frac{2}{7!}\eta^7, \qquad \dots (18)$$

using two boundary conditions at $\eta = 0$ ($f_0 = f_0' = 0$) and an additional condition $f_0''(0) = \alpha_1$ at $\eta = 0$. The value of α_1 is determined by using the condition $f_0' = 1$, at $n \to \infty$ by Meksyn's (1961) method of steepest descent.

Proceeding exactly in a similar manner, we construct the sequences $\{f_{1,n}\}$ for f_1 from the equation (15) as follows :

$$f_{1,1} = \frac{\alpha_2}{2\,!}\,\eta^2 \qquad \dots (19)$$

and

$$f_{1,2} = \frac{\alpha_2}{2\,!}\,\eta^2 - \frac{(\alpha_1{}^2 + f^0\alpha_1)}{3\,!}\,\eta^3 + \frac{2\alpha_2\alpha_1}{5\,!}\,\eta^5 - \frac{2\alpha^2}{6\,!}\,\eta^6 \qquad \dots (20)$$

The unknown parameter $f_1''(0) = \alpha_2$ has been evaluated by using the method of steepest descent (Meksyn 1961).

The values of α_1 and α_2 are given in table 1. The corresponding values given by Hartree (1937) for viscous fluid and Beard & Walters (1964) for visco-elastic fluids are also given for comparison.

Table 1

	For the present analysis							Beard & Walters' solution for visco-elastic fluid	Hartree's solution for viscous fluid
f^0	0	.1	.2	.5	−.1	−.2	−.5		
α_1	1.2348	1.4155	1.6163	2.2617	1.0557	0.8946	0.04676	1.2326	1.232
α_2	1.2976	1.1769	1.4490	0.9710	1.1549	1.0547	0.2724	1.1390	—

Skin friction : The dependence of skin friction on blowing or suction values are given by

$$f''(0) = f_0''(0) + Kf_1''(0) = \alpha_1 + k\alpha_2 \qquad \dots (21)$$

The values of skin friction [*i.e.*, $f''(0)$] for various values of suction and injection are given in table 2.

Table 2

f^0	0	.1	.2	.5	−.1	−.2	−.5
$H = 0$	1.2348	1.4155	1.6165	2.2617	1.0557	0.8946	0.4676
$H = 2$	1.4943	1.7694	1.9161	2.4559	1.2867	1.1047	0.5221

B. *Mass Transfer* : Separating nonlinear terms from linear ones of the equation (10), one gets the sequences of approximations using the Laplace transformation as follows :

$$W_1 = \delta\eta, \quad \dots (22)$$

and

$$W_2 = \delta\eta - \delta(Sc)\Big[\frac{f^0}{2!}\eta^2 + \frac{(\alpha_1 + K\alpha_2)}{4!}\eta^4 - \frac{\{(1-\alpha_1 f^0) + K(\alpha_1^2 + f^0\alpha_2)\}}{5!}\eta^5$$

$$+\frac{f^0}{6!}\eta^6 + \frac{(\alpha_1 + 2K\alpha_1\alpha_2)}{7!}\eta^7 - \frac{2(\alpha_1 + K\alpha_2)}{8!}\eta^8 + \frac{2}{9!}\eta^9\Big], \quad \dots (23)$$

where δ is an unknown constant and the value of δ has been evaluated by Meksyn's steepest descent method (1961). The values of δ are given in table 3.

Table 3

f^0	0	.1	.2	5	−.1	− 2	−.5
$k = 0$	0.5666	0.6822	0.6931	0 8339	0.4859	0 3878	0.1022
$k = .2$	0.6016	0.6625	0 6902	0 7905	0.5191	0 4761	0 1234

Mass transfer coefficient : Ratio of mass transfer coefficient $[h_0]$ with that of zero blowing or suction $[h_0]_{f^0=0}$ is given by

$$\frac{h_0}{[h_0]_{f^0=0}} = \frac{W'(0)}{[W'(0)]_{f^0=0}} \quad \dots (24)$$

Values of $h_0/[h_0]_{f^0=0}$ for various values of suction and injection are given in table 4

Table 4

f^0	0	1	.2	.5	−.1	−.2	−.5
$k = 0$	1.0000	1 115	1.223	1 471	0.8572	0.6810	0.1803
$k = .2$	1.0000	1.101	1.148	1.314	0.8630	0.7914	0.2051

C. *Heat Transfer* : In the present case, we disregard the terms due to dissipation and include the thermodynamic coupling effect on temperature due to concentration fields. Using the transformation

$$\theta = \theta_0 + \beta(Sc)\theta_1 \quad \dots (25)$$

in the equation (9) and equating the terms independent of (Sc) and its coefficients on both sides we get

$$\theta_0'' + (Pr)f\theta_0' = 0 \quad \dots (26)$$

and

$$\theta_1'' + (Pr)f\theta_1' + W'\theta_0' + \theta_0 W'' = 0, \quad \dots (27)$$

with boundary conditions

$$\theta_0(0) = 0,\ \theta_0(\infty) = 1 \quad \text{and} \quad \theta_1(0) = 0,\ \theta_1(\infty) = 0 \qquad \dots (28)$$

Assuming $\theta_0'(0) = \gamma_1$ and $\theta_1'(0) = \gamma_2$ and using the same technique, we construct sequences, $\{\theta_{0,n}\}$ and $\{\theta_{1,n}\}$ as follows :

$$\theta_{0,1} = \gamma_1 \eta, \qquad \dots (29)$$

$$\theta_{0,2} = \gamma_1 \Big[\eta - \frac{(Pr)f^0}{2\,!}\eta^2 - \frac{(Pr)(\alpha_1 + K\alpha_2)}{4\,!}\eta^4 + \frac{(Pr)\{(1-\alpha_1 f^0) + k(\alpha_1^2 + f^0\alpha_2)\}}{5\,!}\eta^5$$

$$- \frac{(Pr)f^0}{6\,!}\eta^6 - \frac{(Pr)(\alpha_1^2 + 2K\alpha_1\alpha_2)}{7\,!}\eta^7 + \frac{2(\alpha_1 + K\alpha_2)(Pr)}{8\,!}\eta^8 - \frac{2(Pr)}{9\,!}\eta^9\Big], \qquad \dots (30)$$

and

$$\theta_{1,1} = \gamma_2 \eta, \qquad \dots (31)$$

$$\theta_{1,2} = \gamma_2 \eta - \frac{(Pr)\gamma_2 + \delta\gamma_1}{2\,!} f^0 \eta^2 + \frac{2\delta(Sc)\gamma_1 f^0}{3\,!}\eta^3 - \frac{(Pr)\gamma_2(\alpha_1 + K\alpha_2)}{4\,!}\eta^4$$

$$+ \frac{(Pr)\gamma_2(1-\alpha_1 f^0) + k(\alpha_1^2 + f^0\alpha_2) + 4\delta(Sc)\gamma_1(\alpha_1 + K\alpha_2)}{5\,!}\eta^5$$

$$- \frac{(Pr)\gamma_2 f^0 + 5\delta(Sc)\gamma_1\{(1-\alpha_1 f^0)K(\alpha_1^2 + f^0\alpha_2)\}}{6\,!}\eta^6$$

$$- \frac{(Pr)\gamma_2(\alpha_1^2 + 2K\alpha_1\alpha_2) - 6 f^0\delta(Sc)\gamma_1}{7\,!}\eta^7$$

$$+ \frac{2(Pr)\gamma_2(\alpha_1 + K\alpha_2) + 7\delta(Sc)\gamma_1(\alpha_1^2 + 2K\alpha_1\alpha_2)}{8\,!}\eta^8$$

$$- \frac{2(Pr)\gamma_2 + 16\delta(Sc)\gamma_1(\alpha_1 + K\alpha_2)}{9\,!}\eta^9 + \frac{9\delta(Sc)\gamma_1}{10\,!}\eta^{10}, \qquad \dots (32)$$

The values of γ_1 and γ_2 are evaluated by the same mothod Maksyn (1961) *et al* 1962), using the conditions $\theta_0(\infty) = 1$ and $\theta_1(\infty) = 0$. The values of γ_1 and γ_2 are given in table 5 for $(Pr) = (Sc) = 1$.

Table 5

γ	f^0	0	.1	.2	.5	−.1	−.2	−.5
γ_1	$k = 0$	0.5666	0.6322	0.6931	0.8339	0.4859	0.3878	0.1022
γ_1	$k = .2$	0.6016	0.6625	0.6902	0 7905	0.5191	0.4761	0.1234
γ_2	$k = 0$	0.0000	0 0287	0.0966	0.2433	1.1728	0.0439	−0.0059
γ_2	$k = .2$	0.0000	0.0178	0.0764	0.2494	0.2408	0.0282	−0.0086

Heat transfer coefficient: Ratio of heat transfer coefficient $[h]$ to that at zero blowing or suction $[h]_{f0=0}$, *i.e.*,

$$\frac{h}{[h]_{f0=0}} = \frac{\theta'(0)}{[\theta'(0)]_{f0=0}} \quad \dots (33)$$

are given for various values of suction and injection in table 6.

Table 6

	f^0	0	.1	.2	.5	−.1	−.2	−.5
Without coupling effect	$k = 0$	1.0000	1.115	1.223	1.471	0.8572	0.6810	0.1803
	$k = .2$	1.0000	1.101	1.148	1.314	0.8630	0.7914	0.2051
With coupling effect	$k = 0$	1.0000	1.180	1.240	1.515	0.8350	0.6766	0.1775
	$k = .2$	1.0000	1.165	1 160	1 355	0.8460	0.7368	0.2036

5. Discussion and results

The results of above analysis are shown in tables (1–6). It may be observed that the skin friction, heat transfer and mass transfer coefficients increase with suction and decrease with injection. The effect of viscoelastic parameter (k) is to increase skin friction and to decrease heat and mass transfer coefficients. It has also been observed that the thermodynamic coupling increases the temperature at a point.

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