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On the energy loss in Čerenkov radiation

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It is shown that the electromagnetic field energy associated with a charge-particle moving with uniform velocity since infinite remote past in an infinitely extended homogeneous medium is constant. The usual expression for the rate of loss of energy of a particle calculated from the total Poynting flux is shown to be negative of the volume integral of $c\boldsymbol{E}.\boldsymbol{j}$

Introduction

In the problem of Čerenkov radiation emitted by a charge-particle moving in a homogeneous medium with uniform velocity greater than the phase velocity of electromagnetic waves in the medium, it is usually assumed that the particle is moving since infinite remote past. In order to obtain the energy emitted by the particle per unit time one calculates the total Poynting flux across a surface enclosing the particle. The total flux is taken to be the rate of loss of energy per unit time. As the particle is moving from infinite remote past, if we take an overall picture of the electromagnetic field associated with the particle, we will find that the field quantities are moving in the forward direction with the same uniform velocity as that of the particle. Since the medium is homogeneous and isotropic the total field energy does not change with time, so long as hysterisis losses are neglected; of course, the total field energy, as in case of a point particle, may be infinite. As a matter of fact, the total flux across the bounded region is nothing but the volume integral of $-c\boldsymbol{E}\,\boldsymbol{j}$ The Poynting theorem states

$$\frac{\partial u}{\partial t}+\nabla . c\boldsymbol{E}\times\boldsymbol{H} = -c\boldsymbol{E}.\boldsymbol{j}. \qquad \ldots \quad (1)$$

where u is the energy density

$$u = \frac{\epsilon}{2}\boldsymbol{E}.\boldsymbol{E}+\frac{\mu}{2}\boldsymbol{H}.\boldsymbol{H}. \qquad \ldots \quad (2)$$

If the total energy,

$$U = \int u d\boldsymbol{r} \qquad \ldots \quad (3)$$

(the integral being over the entire space, is independent of time), one obtains on integrating equation (1)

$$\int c\boldsymbol{E}\times\boldsymbol{H}.d\boldsymbol{s} = -\int c.\boldsymbol{E}.\boldsymbol{j}\,d\boldsymbol{r}, \qquad \ldots \quad (4)$$

In the following section, it is explicitly shown that the usual expression for the rate of energy loss of the particle in the Čerenkov radiation, which is calculated from the total flux of the Poynting vector, is nothing but the right hand side of the above equation. In section 3, it is shown that with the usual expressions for $\boldsymbol{E}$ and $\boldsymbol{H}$ due to a particle emitting Čerenkov radiation the total field energy U (equations 2 and 3) is independent of time, *i.e.*, $\partial U/\partial t = 0$. It needs to be emphasized that this is true so long as the particle is assumed to move with uniform velocity from infinite remote past in an infinitely extended homogeneous medium. In the last section, we have tried to point out the difference between the energy flowing across a given section and the energy loss of the particle.

2. The Expression for the Volume Integral of $c\boldsymbol{E}.\boldsymbol{j}$.

Let the charge-current due to a particle moving with uniform velocity $\boldsymbol{v}$ in a homogeneous isotropic medium with dielectric constand ϵ and permeability μ, be given by

$$q = q_0\delta(\boldsymbol{r}-\boldsymbol{v}t),\ \boldsymbol{j} = \frac{\boldsymbol{v}}{c}\, q. \quad \dots (5)$$

The expressions for $\boldsymbol{E}$ and $\boldsymbol{H}$ due to the particle moving since infinite past are

$$\boldsymbol{E} = \gamma^2\nabla\Phi + \frac{1}{u^2}\, \boldsymbol{v}\times\boldsymbol{v}\times\Delta\Phi \quad \dots (6)$$

$$\boldsymbol{H} = \frac{\epsilon}{c}\, \boldsymbol{v}\times\boldsymbol{E}, \quad \dots (7)$$

where

$$\gamma^2 = \frac{v^2}{u^2} - 1, \quad u^2 = \frac{c^2}{\epsilon\mu}, \quad v = |\boldsymbol{v}| \quad \dots (8)$$

and

$$\Phi = \frac{q_0}{(2\pi)^3} \int \frac{1}{\epsilon}\, e^{i\boldsymbol{k}.(\boldsymbol{r}-\boldsymbol{v}t)} . \left\{ \frac{1}{\boldsymbol{k}\cdot\boldsymbol{k} - \left(\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{u}\right)^2} + i\pi \frac{\boldsymbol{k}\cdot\boldsymbol{v}}{|\boldsymbol{k}\cdot\boldsymbol{v}|}\, \delta\left(\boldsymbol{k}\cdot\boldsymbol{k} - \left(\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{u}\right)^2\right) \right\} d\boldsymbol{k} \quad \dots (9)$$

(Iwanenko & Sokolov 1953; Sen Gupta 1965, 1968). With these expressions let us now calculate the volume integral of $c\boldsymbol{E}\cdot\boldsymbol{j}$

$$c \int \boldsymbol{E}.\boldsymbol{j}d\boldsymbol{r} = -\frac{iq_0^2}{8\pi^3} \int \frac{\gamma^2}{\epsilon}\, \boldsymbol{k}\cdot\boldsymbol{v}\, e^{i\boldsymbol{k}.(\boldsymbol{r}-\boldsymbol{v}t)} .\delta(\boldsymbol{r}\text{-}\boldsymbol{v}t) \left\{ \frac{1}{\boldsymbol{k}\cdot\boldsymbol{k} - \left(\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{u}\right)^2} + i\pi \frac{\boldsymbol{k}\cdot\boldsymbol{v}}{|\boldsymbol{k}\cdot\boldsymbol{v}|}\, \delta\left(\boldsymbol{k}\cdot\boldsymbol{k} - \left[\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{u}\right)^2\right) \right\} d\boldsymbol{k}d\boldsymbol{r}. \quad \dots (10)$$

The first term in the bracket does not contribute (considered as an improper integral), as the presence of the factor $\boldsymbol{k}.\boldsymbol{v}$ makes it an odd function of $(\boldsymbol{k}.\boldsymbol{v}.)/v$. The second term may be written as

$$\frac{q_0^2}{8\pi^2}\int\frac{\gamma^2}{\epsilon}\,|\boldsymbol{k}.\boldsymbol{v}|\,\delta\left(k_1^2+k_2^2-\gamma^2\left(\frac{\boldsymbol{k}.\boldsymbol{v}}{v}\right)^2\right)dk_1dk_2d\left(\frac{\boldsymbol{k}.\boldsymbol{v}}{v}\right). \quad \ldots \quad (11)$$

i) $\gamma^2>0$, from equation (8), $v>u$ *i.e.*, the case of Čerenkov radiation, the expression (11) becomes

$$\frac{q_0}{8\pi}\int\frac{\gamma^2}{\epsilon}\,|\boldsymbol{k}.\boldsymbol{v}|.\;d\left(\frac{\boldsymbol{k}.\boldsymbol{v}}{v}\right). \quad \ldots \quad (12)$$

On writing ω for $\boldsymbol{k}.\boldsymbol{v}$ one obtains finally

$$-c\int \boldsymbol{E}.\boldsymbol{j}d\boldsymbol{r}=\frac{q_0^2v}{4\pi c^2}\int\mu\left(1-\frac{u^2}{v^2}\right)\omega d\omega \quad \ldots \quad (13)$$

On taking account of the dispersion of the medium the upper limit of the above integral is ω_0 such that $u(\omega)>v$ as $\omega>\omega_0$. It is exactly the same as the expression which is usually quoted for the rate of loss of energy by the particle calculated from the total Poynting flux (Iwanenko & Sokolov 1953). Hence, we obtain equation (4), which in turn implies that $\partial U/\partial t=0$ from equations (1) and (3). In case of non-dispersive medium the integral on the right hand side of (13) is unbounded which is also expected as $\boldsymbol{E}$ on the left hand is the self-field and diverges strongly at the position of the charge.

ii) $\gamma^2<0$ *i.e.*, $v<u$, the integral (11) has no contribution. Hence, the volume integral of $\boldsymbol{E}.\boldsymbol{j}$ is zero. It is also well known that the total Poynting flux across a surface enclosing the charge is zero. Thus, we obtain again $\partial U/\partial t=0$.

3. The Total Field Energy

The field quantities may be better expressed in spherical polar coordinates, with the instantaneous position of the particle as pole and the line of flight of the particle as axis. Thus,

$$\boldsymbol{E}=\frac{q_0u^3\gamma^2}{2\pi\epsilon v^3}\;\frac{\boldsymbol{R}}{R^3\left(\cos^2\theta-\frac{u^2}{v^2}\gamma^2\right)^{3/2}} \quad \ldots \quad (14)$$

i) $\gamma^2>0$. The expression for the field energy (equations (2) and (3)) integrated over the region of Čerenkov cone is

$$U=\lim_{\substack{\theta'\to\theta_0\\ \eta\to 0}}\frac{q_0^2u^4\gamma^4}{4\pi\epsilon v^6}\int_0^{\theta'}d\theta\int_\eta^\infty\frac{dR}{R^3}\;\frac{u^2+v^2\sin^2\theta}{\left(\cos^2\theta-\frac{u^2}{v^2}\gamma^2\right)^3} \quad \ldots \quad (15)$$

where $\cos\theta_0 = u\gamma/v$. The upper limit of R integration does not contribute, the lower limit is divergent but independent of time Again, the upper limit of θ integration is divergent but independent of time.

ii) $\gamma^2 < 0$: The expression (15) is similar, with only change of upper limit of θ integration which is now π. Thus, in both the cases

$$\frac{\partial U}{\partial t} = 0. \tag{16}$$

Hence, the total field energy is constant, though, it is unbounded. This divergence is due to the self-energy which depends also on the particle velocity and the electromagnetic properties of the material medium. The fact that the total energy is independent of time is also expected from simple ground as has been noted in the introduction. As the particle is moving with uniform velocity from infinite past, the field quantities are only translated with the same uniform velocity with the evolution of time. Since the total energy, which is obtained by integrating over the whole space, is invariant with respect to this translation, it remains the same.

In the conclusion, we wish to point out that the total Poynting flux gives only the flow across the surface. It is not proper to take this as the rate of loss of energy by the particle due to the radiation. Because the particle can only lose energy to the field and we have shown the total field energy due to a particle moving with uniform velocity ($v < u$ or $v < u$) since infinite past, is constant. On the other hand, in the usual formulation of the Čerenkov radiation, the particle is assumed to be moving with uniform velocity from infinite past and it is said to be losing energy. Those two statements are in general contradictory They are agreeable only when the rate of loss of the energy of the particle is zero as we have shown above. It seems a reformulation of the problem of energy loss of the particle is necessary.

Reference

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