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On mechanical response in a piezoelectric plate characterized by a diffusion and subjected to a prescribed polarization gradient

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The paper is concerned with the mechanical response in a piezoelectric plate transducer under constant voltage input across it; the plate being characterized by a prescribed diffusion and subjected to an exponentially decaying polarization gradient.

INTRODUCTION

The studies of piezoelectric transducer from the point of view of mechanics of continuous media are largely due to Redwood (1961), Mason (1950), Filipczynski (1956), Sinha (1963, 1965 and 1967). Giri (1966), Das (1967), Chakrabarty (1968, 1969) and others. This article is a similar attempt seeking to investigate the mechanical response due to a constant voltage input under a prescribed diffusion and characterised by an exponentially decaying polarization gradient. The analysis presented here conforms to the line of investigation undertaken by the above-mentioned authors. It has been found that the method of Laplace transform facilitates the solution of the problem.

PROBLEM, FUNDAMENTAL EQUATION AND BOUNDARY CONDITION

Let $x = 0$ and $x = X$ be the extremities of a piezoelectric plate in the direction of its thickness (taken as the X-axis), excited by a constant voltage ϕ_0 at its ends. The transducer is characterised by an exponentially decaying polarization gradient. Let there be a cavity excited by a field defined by e^{-kx}. $\delta(t)$, k being a constant and $\delta(t)$ is the well known Dirac's Delta function. The plate is taken as rigidly backed at $x = X$.

The equations of state for the plate, (Redwood 1961) are

$$T_1 = c_{11}\frac{\partial \xi}{\partial x} + e_{11}E_1 \tag{1}$$

$$P_1 = e_{11}\frac{\partial \xi}{\partial x} + k_{11}E_1 \tag{2}$$

where T_1 = stress, c_{11} = average stiffness co-efficient, ξ = mechanical deformation at any point x, e_{11} = piezoelectric constant, P_1 = polarization vector and k_{11} = dielectric susceptibility.

In accordance with our assumption, we have

$$\frac{\partial P_1}{\partial x} = -P_0 e^{-rt} \tag{3}$$

so that by suitable choice

$$P_1 = -P_0 x e^{-rt} \tag{4}$$

where P_0 is a constant and r is a decaying factor. This assumption is justified by experimental facts (Swann, 1950).

Eliminating E_1 from (1) and (2) we have

$$T_1 = a\frac{\partial \xi}{\partial x} + \frac{e_1}{k_{11}} P_1 \tag{5}$$

where $a = c_{11} - e_{11}{}^2/k_{11}$

Differentiating (5) and using (4) we get

$$\frac{\partial T_1}{\partial x} = a\frac{\partial^2 \xi}{\partial x^2} - \frac{e_{11}P_0}{k_{11}} e^{-rt} \tag{6}$$

Using (6) with Newton's equation

$$\rho\frac{\partial^2 \xi}{\partial t^2} - \frac{\partial T_1}{\partial x} = e^{-kx}.\,\delta(t)$$

we have

$$\rho\frac{\partial^2 \xi}{\partial t^2} - a\frac{\partial^2 \xi}{\partial x^2} + \frac{e_{11}P_1}{k_{11}} e^{-rt} = e^{-kx}.\,\delta(t) \tag{7}$$

Equation (7) constitutes the fundamental equation of the problem.

The boundary conditions are that both the displacement and force are continuous at the end $x = 0$ of the transducer. Also the transducer being rigidly backed at $x = X$, the mechanical displacement at $x = X$ must be zero.

Solution of the Problem

Taking Laplace transform (7) of parameter $s(s > 0)$

who have $$\frac{\partial^2 \xi}{\partial x^2} - \frac{\rho}{a} s^2 \xi = \frac{1}{a}\left[\frac{e_{11}P_0}{k_{11}}.\,\frac{1}{s+r} - e^{-kx}\right] \quad \ldots \tag{8}$$

Solving (8) we get

$$\xi = Ae^{-\frac{s}{v}x} + Be^{-\frac{s}{v}x} - \frac{e_{11}P_0}{k_{11}\rho.\; s^2(s+r)} + \frac{e^{-kx}}{\rho\{s^2-(kv)^2\}} \quad \ldots \tag{9}$$

where A and B are constants of integration and $v^2 = a/\rho$, and it is assumed that $\xi(0) = 0$, $\xi'(0) = 0$.

To evaluate A and B from the boundary conditions, we attach two mechanical systems 1 and 2 as assumed by Rodwood (1961) to the extremities $x = 0$ and $x = X$. We denote the corresponding entities, A_1, B_1 and A_2, B_2 by the symbols 1 and 2, respectively. In that case, since the transducer is rigidly backed,

$$A_2 = B_2 = 0 \quad \text{and} \quad A_1 = 0 \tag{10}$$

Also continuity of displacement at $x = 0$, gives

$$\bar{\xi}(0) = \bar{\xi}_1(0)$$

i.e.
$$A+B-\frac{e_{11}P_0}{k_{11}\rho}\cdot\frac{1}{s^2(s+r)}+\frac{1}{\rho\{s^2-(kv)^2\}}= B_1 \quad \dots \tag{11}$$

where
$$\bar{\xi}_1(x) = B_1 e^{\frac{s}{v}x} \quad \dots \tag{12}$$

Again for a rigidly backed transducer $\bar{\xi}(X) = 0$.

Hence
$$Ae^{-\frac{s}{v}X}+Be^{-\frac{s}{v}X}-\frac{e_{11}P_0}{k_{11}\rho}\cdot\frac{1}{s^2(s+r)}+\frac{e^{-kX}}{\rho\{s^2-(kv)^2\}}=0 \quad \dots \tag{13}$$

Using (1), (3) and the relation $E_1 = D_1 - 4\pi P_1$, we have

$$T_1 = c_{11}\frac{\partial\xi}{\partial x}+e_{11}\frac{Q_1}{YZ}+4\pi e_{11}P_0 xe^{-rt} \quad \dots \tag{14}$$

where $D_1 = \dfrac{Q_1}{YZ}$, Y is length of the transducer along Y-axis, Z its length along Z-axis, and Q_1 is the charge.

Taking Laplace transform of (14) and using (9) we have

$$\bar{T}_1(0) = c_{11}\left[-\frac{s}{v}A+\frac{s}{v}B-\frac{k}{\rho}\cdot\frac{1}{s^2-(kv)^2}\right]+e_{11}\frac{\bar{Q}_1}{YZ}\cdot \quad \dots \tag{15}$$

If T stands for the stress for the medium towards the left of the transducer (*i.e.*, at $x = 0$) then from (12)

$$\bar{T} = \frac{sc}{v}B_1 e^{\bar{v}} \quad \dots \tag{16}$$

c being a similar constant as c_{11}.

Hence
$$\bar{T}(0) = \frac{sc}{v}B_1 \quad \dots \tag{17}$$

For continuity of stress at $x = 0$, we have from (15) and (17)

$$c_{11}\left[-\frac{s}{v}A+\frac{s}{v}B-\frac{k}{\rho}\cdot\frac{1}{s^2-(kv)^2}\right]+e_{11}\frac{\overline{Q}_1}{YZ}=s\frac{c}{v}B_1 \quad \ldots (18)$$

Again

$$\overline{V}=-(\overline{V}_X-\overline{V}_0)=\int_0^X \overline{E}dx$$

$$=\int_0^X (\overline{D}_1-4\pi\overline{P}_1)dx$$

Hence,

$$\overline{\phi}_0=\frac{\overline{Q}_1}{YZ}X+\frac{4\pi P_0}{s+r}\cdot\frac{X^2}{2} \quad \ldots (19)$$

since the voltage across the transducer is given by ϕ_0

Substituting the value of B_1 from (11) in (18) and then eliminating $\overline{Q}_1$ from (18) and (19) we get a relation connecting A and B. Using this relation along with (13) we get the values of A and B. The values of A and B are given by the following expressions

$$A=\frac{l_2m_3-l_3e^{-\frac{s}{v}X}}{l_1e^{\frac{s}{v}X}-l_2e^{-\frac{s}{v}X}} \quad \text{and} \quad B=\frac{l_3e^{\frac{s}{v}X}-l_1m_3}{l_1e^{\frac{s}{v}X}-l_2e^{-\frac{s}{v}X}} \quad \ldots (20)$$

where

$$\left.\begin{aligned}
l_1&=-\frac{1}{v}(c_{11}+c), \quad l_2=-\frac{1}{v}(c-c_{11})\\
l_3&=\left[\frac{c}{\rho v}-\frac{c_{11}k}{\rho}\right]\cdot\frac{1}{s\{s^2-k^2v^2\}}+\frac{e_{11}P_0c}{v}\cdot\frac{1}{s^2(s+r)}-\frac{c}{\rho v}\cdot\frac{1}{s^2-k^2v^2}\\
&\quad+\frac{e_{11}\phi_0}{s^2}-\frac{2\pi P_0e_{11}X^2}{s(s+r)}\\
m_3&=-\frac{e_{11}P_0}{k_{11}\rho}\,\frac{1}{s^2(s+r)}+\frac{e^{-kX}}{\rho\{s^2-(kv)^2\}}
\end{aligned}\right\} \quad (20A)$$

Substituting the values of A and B in (9) we have

$$\xi=\frac{\left(l_2m_3-l_3e^{-\frac{s}{v}X}\right)e^{-\frac{s}{v}x}+\left(l_3e^{-\frac{s}{v}X}-l_1m_3\right)e^{-\frac{s}{v}X}}{l_1e^{\frac{s}{v}X}-l_2e^{-\frac{s}{v}X}}+[\overline{m}_3]_{X=x} \quad (20B)$$

It is simple to detect that $\xi=0$ when $x=X$. This must be the case, since the transducer is rigidly backed at $x=X$.

The inversion being too complicated, we find it for a particular value of x, say $X/2$. This is to be done by the methods of approximation. We restrict to approximation followed by Redwood (1961) and find

$$\bar{\xi}_{\frac{X}{2}} = \frac{l_2}{l_1} m_3 e^{-\frac{3sX}{2v}} - \frac{l_3}{l_1} e^{-\frac{3sX}{2v}} - m_3 e^{-\frac{sX}{2v}} - \frac{l_3}{l_1} e^{-\frac{sX}{2v}} + [m_3]_{X=\frac{X}{2}} \tag{21}$$

Substituting for l_3 and m_3 and taking inverse transform we have

$$\xi_{\frac{X}{2}} = \frac{l_2}{l_1} \cdot \frac{e_{11}P_0}{k_{11}\rho} \left[\frac{1}{r}\left(t - \frac{3X}{2v}\right) - \frac{1}{r^2} + \frac{1}{r^2} e^{-r\left(t-\frac{3X}{2v}\right)} \right] + \frac{l_2 e^{-\frac{kX}{2}}}{l_1\rho} \cdot \frac{\sinh\left[kv\left(t - \frac{3X}{2v}\right)\right]}{kv}$$

$$- \frac{1}{l_1} \left[\left(\frac{c_{11}k}{\rho} - \frac{c}{\rho v} \right) \cdot \frac{1}{k^2v^2} \left\{ 1 - \cosh kv\left(t - \frac{3X}{2v}\right) \right\} + \frac{e_{11}P_0 c}{v} \left\{ \frac{1}{r}\left(t - \frac{3X}{2v}\right) \right. \right.$$

$$\left. - \frac{1}{r^2} + \frac{1}{r^2} e^{-r\left(t-\frac{3X}{2v}\right)} \right\} - \frac{c}{\rho v} \cdot \frac{\sinh kv\left(t - \frac{3X}{2v}\right)}{kv} + e_{11}\phi_0\left(t - \frac{3X}{2v}\right)$$

$$\left. - 2\pi P_0 e_{11} X^2 \left\{ \frac{1}{r} - \frac{1}{r} e^{-r\left(t-\frac{3X}{2v}\right)} \right\} \right] - \frac{1}{l_1} \left[\left(\frac{c_{11}k}{\rho} - \frac{c}{\rho v} \right) \frac{1}{k^2v^2} \right.$$

$$\left\{ 1 - \cosh kv\left(t - \frac{X}{2v}\right) \right\} + \frac{e_{11}P_0 c}{v} \left\{ \frac{1}{r}\left(t - \frac{X}{2v}\right) - \frac{1}{r^2} + \frac{1}{r^2} e^{-r\left(t-\frac{X}{2v}\right)} \right\}$$

$$\left. - \frac{c}{\rho v} \cdot \frac{\sinh kv\left(t - \frac{X}{2v}\right)}{kv} + e_{11}\phi_0\left(t - \frac{X}{2v}\right) - 2\pi P_0 e_{11} X^2 \left\{ \frac{1}{r} - \frac{1}{r} e^{-r\left(t-\frac{X}{2v}\right)} \right\} \right]$$

$$+ \frac{e_{11}P_0}{k_{11}\rho} \left[\frac{1}{r}\left(t - \frac{X}{2v}\right) - \frac{1}{r^2} + \frac{1}{r^2} e^{-r\left(t-\frac{X}{2v}\right)} \right] - \frac{e^{-kX}}{\rho kv} \sinh\left\{kv\left(t - \frac{X}{2v}\right)\right\}$$

$$- \frac{e_{11}P_0}{k_{11}\rho} \left\{ \frac{1}{r} t - \frac{1}{r^2} + \frac{1}{r^2} e^{-rt} \right\} + \frac{e^{-k\frac{X}{2}}}{\rho kv} \sinh kvt \tag{22A}$$

for $t > \dfrac{3X}{2v}$.

and
$$\xi_{\frac{X}{2}} = -\frac{1}{t_1} \left[\left(\frac{c_{11}k}{\rho} - \frac{c}{\rho v} \right) \cdot \frac{1}{k^2v^2} \left\{ 1 - \cosh kv\left(t - \frac{X}{2v}\right) \right\} \right.$$

$$+ \frac{e_{11}P_0 c}{v} \left\{ \frac{1}{r}\left(t - \frac{X}{2v}\right) - \frac{1}{r^2} + \frac{1}{r^2} e^{-r\left(t-\frac{X}{2v}\right)} \right\} - \frac{c}{\rho v} \cdot \frac{\sinh kv\left(t - \frac{3X}{2v}\right)}{kv}$$

$$+e_{11}\phi_0\left(t-\frac{3X}{2v}\right)-2\pi P_0 e_{11}X^2\left\{\frac{1}{r}-\frac{1}{r}e^{-r\left(t-\frac{X}{2v}\right)}\right\}\Bigg]$$

$$+\frac{e_{11}P_0}{k_{11}\rho}\left[\frac{1}{r}\left(t-\frac{X}{2v}\right)-\frac{1}{r^2}+\frac{1}{r^2}e^{-r\left(t-\frac{X}{2v}\right)}\right]-\frac{e^{-kX}}{\rho kv}\sinh\left\{kv\left(t-\frac{X}{2v}\right)\right\}$$

$$-\frac{e_{11}P_0}{k_{11}\rho}\left[\frac{1}{r}t-\frac{1}{r^2}+\frac{1}{r^2}e^{-\gamma t}\right]+\frac{e^{-k\frac{X}{2}}}{\rho kv}\sinh kvt \tag{22B}$$

for $\frac{X}{2v}<t<\frac{3X}{2v}$.

and

$$\xi_{\frac{X}{2}}=\frac{1}{\rho kv}e^{-k\frac{X}{2}}\,.\,\sinh kvt-\frac{e_{11}P_0}{k_{11}\rho}\left[\frac{1}{r}t-\frac{1}{r^2}+\frac{1}{r^2}e^{-rt}\right] \tag{22C}$$

for $t<\frac{X}{2v}$.

Relations (22A) (22B) and (22C) gives the expression for the mechanical response at $x=\frac{X}{2}$. This shows that the mechanical response is partly constant, partly linear and it partly decays exponentially with time. It also varies as hyperbolic sine function of time.

DISPLACEMENT OF THE SURFACE at $x=0$

When $x=0$, equation (20B) takes the form

$$\bar{\xi}_{x=0}=\frac{\left((l_2m_3-l_3e^{-\frac{s}{v}X}\right)+\left(l_3e^{-\frac{s}{v}X}-l_1m_3\right)}{l_1e^{-\frac{s}{v}X}-l_2e^{-\frac{s}{v}X}}+[\bar{m}_3]_{X=0}$$

Following the same approximations, we have

$$\bar{\xi}_{x=0}=\frac{l_2-l_1}{l_1}m_3e^{-\frac{s}{v}X}-\frac{1}{l_1}l_3+\frac{1}{l_1}\,.\,l_3e^{-2\frac{s}{v}X}+[m_3]_{X=0}$$

After taking inverse transform we get

$$\xi_{x=0}=\frac{1}{l_1}\Big[\left(\frac{c_{11}k}{\rho}-\frac{c}{\rho v}\right)\cdot\frac{1}{k^2v^2}\left\{1-\cosh kv\left(t-\frac{2X}{v}\right)\right\}+\frac{e_{11}P_0c}{v}\left\{\frac{1}{r}\left(t-\frac{2X}{v}\right)\right.$$

$$-\frac{1}{r^2}+\frac{1}{r^2}e^{-r\left(t-\frac{2x}{v}\right)}\Bigg\}-\frac{c}{\rho k v^2}\sinh kv\left(t-\frac{2X}{v}\right)+e_{11}\phi_0\left(t-\frac{2X}{v}\right)$$

$$-2\pi P_0 e_{11}X^2\left\{\frac{1}{r}-\frac{1}{r}e^{-r\left(t-\frac{2X}{v}\right)}\right\}\Bigg]-\frac{1}{l_1}\left[\left(\frac{c_{11}k}{\rho}-\frac{c}{\rho v}\right)\frac{1}{k^2v^2}\left(1-\cosh kvt\right)\right.$$

$$+\frac{e_{11}P_0c}{v}\left\{\frac{1}{r}t-\frac{1}{r^2}+\frac{1}{r^2}e^{-rt}\right\}-\frac{c}{\rho k v^2}\sinh kvt+e_{11}\phi_0 t-2\pi P_0 c_{11}X^2\left(\frac{1}{r}-\frac{1}{r}e^{-rt}\right)\Bigg]$$

$$+\frac{l_2-l_1}{l_1}\left[-\frac{e_{11}P_0}{k_{11}\rho}\left\{\frac{1}{r}\left(t-\frac{x}{v}\right)-\frac{1}{r^2}+\frac{1}{r^2}e^{-r\left(t-\frac{x}{v}\right)}\right\}+\frac{1}{\rho kv}e^{-kX}\cdot\sinh kv\left(t-\frac{X}{v}\right)\right]$$

$$-\frac{e_{11}P_0}{k_{11}\rho}\left[\frac{1}{r}t-\frac{1}{r^2}+\frac{1}{r^2}e^{-rt}\right]+\frac{1}{\rho kv}\sinh kvt \qquad \left[\text{for } t>\frac{2X}{v}\right] \qquad \ldots \quad (23A)$$

and

$$\xi_{x=0}=-\frac{1}{l_1}\left[\left(\frac{c_{11}k}{\rho}-\frac{c}{\rho v}\right)\cdot\frac{1}{k^2v^2}\left(1-\cosh kvt\right)+\frac{e_{11}P_0c}{v}\left(\frac{1}{r}t-\frac{1}{r^2}+\frac{1}{r^2}e^{-rt}\right)\right.$$

$$-\frac{c}{\rho k v^2}\sinh kvt+e_{11}\phi_0 t-2\pi P_0 e_{11}X^2\left(\frac{1}{r}-\frac{1}{r}e^{-rt}\right)\Bigg]+\frac{l_2-l_1}{l_1}$$

$$\left[-\frac{e_{11}P_0}{k_{11}\rho}\left\{\frac{1}{r}\left(t-\frac{X}{v}\right)-\frac{1}{r^2}+\frac{1}{r^2}e^{-r\left(t-\frac{X}{v}\right)}\right\}+\frac{e^{-kX}}{\rho kv}\sinh kv\left(t-\frac{X}{v}\right)\right]$$

$$-\frac{e_{11}P_0}{k_{11}\rho}\left[\frac{1}{r}t-\frac{1}{r^2}+\frac{1}{r^2}e^{-rt}\right]+\frac{1}{\rho kv}\sinh kvt \qquad \ldots \quad (23B)$$

for $$\frac{X}{v}<t<\frac{2X}{v}$$

and

$$\xi_{x=0}=-\frac{1}{l_1}\left[\left(\frac{c_{11}k}{\rho}-\frac{c}{\rho v}\right)\cdot\frac{1}{k^2v^2}\left(1-\cosh kvt\right)+\frac{e_{11}P_0v}{v}\left(\frac{1}{r}t-\frac{1}{r^2}+\frac{1}{r^2}e^{-rt}\right)\right.$$

$$-\frac{c}{\rho k v^2}\sinh kvt+e_{11}\phi_0 t-2\pi P_0 e_{11}X^2\left(\frac{1}{r}-\frac{1}{r}e^{-rt}\right)\Bigg]$$

$$-\frac{e_{11}P_0}{k_{11}\rho}\left[\frac{1}{r}t-\frac{1}{r^2}+\frac{1}{r^2}e^{-rt}\right]+\frac{1}{\rho kv}\sinh kvt \qquad \text{for } t<\frac{X}{v}. \qquad \ldots \quad (23C)$$

Evidently the response omitted by the transducer is by and large similar to the one discussed earlier at the point $x=\frac{X}{2}$.

The response is evidently zero at $t=0$.

Discussion

It is clear from each of the above expressions that the mechanical response of the transducer owing to prescribed electric excitations is dominated by the decay factor in the polarization gradient, whatever be the range of time. Moreover, for a large decay factor the responses do not exhibit the transient characteristics and remain uninfluenced by the polarization constant. In the general case, it is obvious from above, the effects due to the prescribed polarization are not coupled with those due to prescribed diffusion Further, while the prescribed polarization gradient accentuates the damping part of the response, the diffusion does not necessairly do so, for large values of r and k.

All these facts bring out some of the physical aspects of the vibration of the transducer which will hold with or without numerical calculations.

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