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Linear flow of heat in a semi-infinite-finite solid

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A problem on conduction of heat in semi-infinite–finite solid has been solved following Heaviside's Operational method Unlike Laplace transformation methods which involve complicated transformations and solutions, the present method finds the correct solution in a simple way. Expressions for temperature distribution in a finite and infinite solid are easily obtained Special cases of a thin film attached to a long solid having wide applications in Engineering to the theory of thin films have been worked out from the general theory.

Introduction

In solving the general problem of heat conduction through a semi-infinite-finite solid we take the following simplifying assumptions.

1 Heat flow through the solid is linear *i.e.*, one dimensional.

2. The media in the two regions are isotropic as regards conductivity, density, specific heat.

3. We neglect the loss of heat in our calculations.

4. There is no thermal resistance at the point of contact. Symbols used are as follows :

v_1, k_1, ρ_1, c_1 and h_1 are the temperature, conductivity, density, specific heat and diffusivity respectively, in the finite region, *i.e.* $-l < x < 0$, and v_2, k_2, ρ_2, c_2 and h_2 are the corresponding quantities in the infinite region, *i.e.* $x > 0$.

l = Length of the finite region.

x = Variable measured along the direction of propagation of heat-flux.

V = Temperature of the source, *i.e.* at $x = -l$.

t = Variable time, and

$$D = \frac{d}{dt}.$$

Method of Solution

The differential equations to be solved are

$$\frac{\partial^2 v_1}{\partial x^2} - \frac{1}{h_1}\frac{\partial v_1}{\partial t} = 0, \qquad -l < x < 0, \quad t > 0 \tag{1}$$

$$\frac{\partial^2 v_2}{\partial x^2} - \frac{1}{h_2}\frac{\partial v_2}{\partial t} = 0, \qquad x > 0, \quad t > 0 \tag{2}$$

Assuming that there is no contact resistance at the surface of separation $x = 0$, the boundary conditions are

$$k_1 \frac{\partial v_1}{\partial x} = k_2 \frac{\partial v_2}{\partial x}, \qquad x = 0, \qquad t > 0 \qquad \dots (3)$$

$$v_1 = v_2, \qquad x = 0, \qquad t > 0 \qquad \dots (4)$$

With initial temperature zero and $x = -l$ kept at V for $t > 0$ equations (1) and (2) in operational form becomes,

$$\frac{\partial^2 v_1}{\partial x^2} - \frac{D}{h_1} v_1 = 0, \qquad -l < x < 0 \qquad \dots (5)$$

$$\frac{\partial^2 v_2}{\partial x_2} - \frac{D}{h_2} v_2 = 0, \qquad x > 0 \qquad \dots (6)$$

Let us put $\frac{D}{h_1} = q_1^2$ and $\frac{D}{h_2} = q_2^2$ then the equations (5) and (6) become

$$\frac{\partial^2 v_1}{\partial x^2} - q_1^2 v_1 = 0 \qquad \dots (7)$$

$$\frac{\partial^2 v_2}{\partial x^2} - q_2^2 v_2 = 0 \qquad \dots (8)$$

The solutions of equations (7) and (8) are

$$v_1 = A \cosh q_1 x + B \sinh q_1 x \qquad \dots (9)$$

$$v_2 = C \cosh q_2 x + D \sinh q_2 x \qquad \dots (10)$$

Where A, B, C and D constants to be determined from boundary conditions in (3) and (4) and are as follows :

$$A = \frac{V}{\cosh q_1 l + \frac{k_2 q_2}{k_1 q_1} \sinh q_1 l}$$

$$B = -\frac{k_2 q_2 V}{k_1 q_1 \cosh q_1 l + k_2 q_2 \sinh q_1 l}$$

$$C = \frac{q_1 k_1 V}{k_1 q_1 \cosh q_1 l + q_2 k_2 \sinh q_1 l}$$

$$D = -\frac{k_1 q_1 V}{k_1 q_1 \cosh q_1 l + k_2 q_2 \sinh q_1 l}$$

Now substitution in equation (9) and (10) yields

$$v_1 = V\,\frac{\cosh q_1x-\sigma\sinh q_1x}{\cosh q_1l+\sigma\sinh q_1l} \qquad \dots \quad (9\cdot1)$$

$$v_2 = V\,\frac{e^{-q_2x}}{\cosh q_1l+\sigma\sinh q_1l} \qquad \dots \quad (10\cdot1)$$

where σ is a constant and given by

$$\sigma = \frac{k_2q_2}{k_1a_1} = \left(\frac{k_2\rho_2c_2}{k_1\rho_1c_1}\right)^{\frac{1}{2}}$$

Expanding the hyperbolic sines and cosines and simplifying,

$$v_1 = \frac{V.\,e^{-q_1(x+l)}\,(1-m\,e^{2q_1x})}{(1-m\,e^{-2q_1l})} \qquad \dots \quad (11)$$

$$v_2 = \frac{2V}{1+\sigma}\,\frac{e^{-q_2x}\,e^{-q_1l}}{(1-m\,e^{-2q_1l})} \qquad \dots \quad (12)$$

where $m = \dfrac{\sigma-1}{\sigma+1}$, a constant.

To know the variation of v_1 and v_2 with time we express q_1 and q_2 in terms of D, operating on Heaviside unit function $H(t)$ and remembering that

$$H(t) = 0, \qquad t<0;$$
$$H(t) = 1, \qquad t>0.$$

the equations (11) and (12) turn out to be

$$v_1 = V[\{e^{-D^{\frac{1}{2}}(x+l)/(h_1)^{\frac{1}{2}}}+me^{-D^{\frac{1}{2}}(x+3l)/(h_1)^{\frac{1}{2}}}+m^2e^{-D^{\frac{1}{2}}(x+5l)/(h_1)^{\frac{1}{2}}}+\dots\}$$
$$-\{me^{-D^{\frac{1}{2}}(l-x)/(h_1)^{\frac{1}{2}}}+m^2e^{-D^{\frac{1}{2}}(3l-x)/(h_1)^{\frac{1}{2}}}+m^3e^{-D^{\frac{1}{2}}(5l-x)/(h_1)^{\frac{1}{2}}}+\dots\}]\,H(t) \qquad \dots \quad (13)$$

$$v_2 = \frac{2V}{1+\sigma}\left[e^{-D^{\frac{1}{2}}\left\{\frac{x}{(h_2)^{\frac{1}{2}}}+\frac{l}{(h_1)^{\frac{1}{2}}}\right\}}+me^{-D^{\frac{1}{2}}\left\{\frac{x}{(h_2)^{\frac{1}{2}}}+\frac{3l}{(h_1)^{\frac{1}{2}}}\right\}}\right.$$
$$\left.+m^2\,.\,e^{-D^{\frac{1}{2}}\left\{\frac{x}{(h_2)^{\frac{1}{2}}}+\frac{5l}{(h_1)^{\frac{1}{2}}}\right\}}+\dots\right]H(t) \qquad \dots \quad (14)$$

$$\therefore\quad v_1 = V\sum_{n=0}^{\infty}(m)^n\left\{\operatorname{erfc}\frac{(2n+1)l+x}{2(h_1t)^{\frac{1}{2}}}-m.\operatorname{erfc}\frac{(2n+1)l-x}{2(h_1t)^{\frac{1}{2}}}\right\} \qquad \dots \quad (15)$$

and $$v_2 = \frac{2V}{1+\sigma} \sum_{n=1}^{\infty} (m)^n \operatorname{erfc} \left\{ \frac{kx+(2n+1)l}{2(h_1t)^{\frac{1}{2}}} \right\} \quad \ldots (16)$$

where $$k = \left(\frac{h_1}{h_2} \right)^{\frac{1}{2}}$$

Again, the temperature gradient at the surface is found to be

$$\left(\frac{\partial v_1}{\partial x} \right)_x = -l_1 = \frac{-V}{(\pi h_1 t)^{\frac{1}{2}}} \left[1+2 \Sigma (m)^n e^{-\frac{n^2l^2}{h_1t}} \right] \quad \ldots (17)$$

For large value of time the exponential may be replaced by unity and we have

$$\left(\frac{\partial v_1}{\partial x} \right)_x = -\frac{V}{(\pi h_1 t)^{\frac{1}{2}}} \{1+2m(1+m+m^2+\ldots)\}$$

$$= -\frac{V}{(\pi h_1 t)^{\frac{1}{2}}} \left(1+ \frac{2m}{1-m} \right)$$

$$= -\frac{V}{(\pi h_1 t)^{\frac{1}{2}}} \left(\frac{k_2\rho_2c_2}{k_1\rho_1c_1} \right)^{\frac{1}{2}} \quad \ldots (18)$$

The equation (18) is in agreement with that obtained by Carslaw & Jaegar (1959) who used this equation for a correct estimate of the age of the earth. Taking the case of granite and air as the composition of earth and the surrounding thin film of air, the quantity $\left(\frac{k_2\rho_2c_2}{k_1\rho_1c_1} \right)^{\frac{1}{2}}$ comes out to be nearly 450. A similar observation was made by Carslaw & Jaeger

Special Cases

Case 1. When l is small, that is, when a thin film of another substance is attached to the semi-infinite medium, expanding the hyperbolic functions and retaining only upto the first power of l we have from equation (10·1)

$$v_2 = \frac{V}{1+\sigma q_1 l} \cdot e^{-q_2x} = \frac{hV}{h+q_2} \cdot e^{-q_2x}, \qquad \text{where} \qquad h = \frac{k_1}{k_2l}.$$

The Operational solution of the above equation will be

$$v_2 = V\left[\operatorname{erfc} \frac{x}{2(h_2t)^{\frac{1}{2}}} - e^{hx+h_2th^2} \times \operatorname{erfc}\left\{ \frac{x}{2(h_2t)^{\frac{1}{2}}} + h(h_2t)^{\frac{1}{2}} \right\} \right] \quad \ldots (19$$

The equation (19) is computed by using the following data :

The material of the film is cork of conductivity $k_1 = 0{\cdot}0001$ and diffusivity $h_1 = 0{\cdot}0014$. The second material is taken to be copper whose conductivity $k_2 = 0{\cdot}93$ and diffusivity $h_2 = 1{\cdot}14$. The temperature v_2 is calculated in the specimen at different distances $x = 10$ cm, $x = 50$ cm, $x = 100$ cm, after one hour when the temperature is assumed to be steady.

A theoretical graph is drawn between the film thickness *vs.* temperature on the infinite region at different distances. This is shown in figure 1. At a

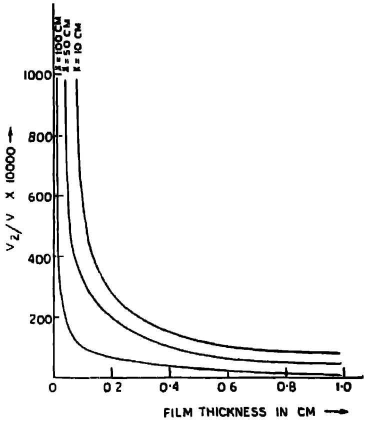

Figure 1

given value of x increasing the film thickness decreases the value of temperature At low value of thickness, in all the three cases, the temperature rapidly falls to a lower value.

Case 2. Retaining the terms upto l^2, we have from equation (10·1).

$$v_2 = V \frac{e^{-q_2 x}}{\frac{l^2}{2}\frac{D}{h_1} + \frac{\sigma l D^{\frac{1}{2}}}{(h_1)^{\frac{1}{2}}} + 1} = V \frac{e^{-q_2 x}}{\frac{l^2 h_2}{2h_1} \cdot q_2^2 + \frac{1}{h} q_2 + 1} = hV \frac{e^{-q_2 x}}{h' q_2^2 + q_2 + h}$$

where $$h' = \frac{l^2 h\, h_2}{2h_1}.$$

$$v_2 = \frac{hV}{h'} \cdot \frac{e^{-q_2 x}}{\left[q_2^2 + \frac{1}{h'} q_2 + \frac{h}{h'}\right]} \qquad \dots \quad (20)$$

Now two case may arise :

I. When the roots of the equation $q_2^2 + \frac{1}{h'} q_2 + \frac{h}{h'} = 0$ are real and unequal, *i.e.* when $\left(\frac{1}{h'}\right)^2 > \frac{4h}{h'}$ and given by

$$[-\alpha, -\beta] = \frac{-\frac{1}{h'} \pm \left\{ \left(\frac{1}{h'}\right)^2 - 4\frac{h}{h'} \right\}^{\frac{1}{2}}}{2}$$

then by the method of partial fraction, equation (20) becomes

$$v_2 = \frac{hV}{h'(\beta-\alpha)} \left\{ \frac{e^{-q_2 x}}{q_2+\alpha} - \frac{e^{-q_2 x}}{q_2+\beta} \right\} \tag{20·1}$$

The Operational solution of the equation (20·1) will be

$$v_2 = \frac{hV}{h'(\beta-\alpha)} \left[\frac{1}{\alpha} \operatorname{erfc} \frac{x}{2(h_2 t)^{\frac{1}{2}}} - \frac{1}{\alpha} e^{\alpha x + h_2 t \alpha^2} \right.$$

$$\times \operatorname{erfc} \left\{ \frac{x}{2(h_2 t)^{\frac{1}{2}}} + \alpha (h_2 t)^{\frac{1}{2}} \right\} - \frac{1}{\beta} \operatorname{erfc} \frac{x}{2(h_2 t)^{\frac{1}{2}}}$$

$$\left. + \frac{1}{\beta} \cdot e^{\beta x + h_2 t \beta^2} \times \operatorname{erfc} \left\{ \frac{x}{2(h_2 t)^{\frac{1}{2}}} + \beta (h_2 t)^{\frac{1}{2}} \right\} \right] \tag{20·2}$$

II When the roots of the equation $q_2^2 + \frac{1}{h'} q_2 + \frac{h}{h'} = 0$ are real and equal

i.e., when $$\left(\frac{1}{h'}\right)^2 = \frac{4h}{h'} \text{ or } 4hh' = 1.$$

$$-\alpha = -\beta = -\frac{1}{2h'}$$

then the equation (20) can be written as

$$v_2 = \frac{hV}{h'} \frac{e^{-q_2 x}}{\left(q_2 + \frac{1}{2h'}\right)^2} \quad \ldots \tag{20·3}$$

The Operational solution of v_2 in equation (20·3)

$$v_2 = \frac{hV}{h'} \left[4h'^2 \operatorname{erfc} \frac{x}{2(h_2 t)^{\frac{1}{2}}} - 4h' \left(\frac{h_2 t}{\pi} \right)^{\frac{1}{2}} \cdot e^{-\frac{x^2}{4h_2}} \right.$$

$$\left. -4h'^2 \left(1 - \frac{x}{2h'} - \frac{2h_2 t}{4h'^2} \right) e^{\frac{x}{2h'} + \frac{h_2 t}{4h'^2}} \times \operatorname{erfc} \left\{ \frac{x}{2(h_2 t)^{\frac{1}{2}}} + \frac{(h_2 t)^{\frac{1}{2}}}{2h'} \right\} \right]$$

$$v_2 = \frac{hV}{h'} \quad 4h'^2 \left[\operatorname{erfc} \frac{x}{2(h_2t)^{\frac{1}{2}}} - \frac{1}{h'} \left(\frac{h_2t}{\pi}\right)^{\frac{1}{2}} . \, e^{-\frac{x^2}{4h_2t}} \right.$$

$$\left. - \left(1 - \frac{x}{2h'} - \frac{2h_2t}{4h'^2}\right) . \, e^{\left(\frac{x}{2h'} + \frac{h_2t}{4h'^2}\right)} \times \operatorname{erfc}\left\{ \frac{x}{2(h_2t)^{\frac{1}{2}}} + \frac{(h_2t)^{\frac{1}{2}}}{2h'} \right\} \right]$$

Since $4hh' = 1$.

$$v_2 = V \left[\operatorname{erfc} \frac{x}{2(h_2t)^{\frac{1}{2}}} - \left(\frac{h_2t}{\pi}\right)^{\frac{1}{2}} . \, e^{-\frac{x^2}{4h_2t}} \right.$$

$$\left. - \left(1 - \frac{x}{2h'} - \frac{h_2t}{2h'^2}\right) e^{\left(\frac{x}{2h'} + \frac{h_2t}{4h'^2}\right)} \times \operatorname{erfc}\left\{ \frac{x}{2(h_2t)^{\frac{1}{2}}} + \frac{(h_2t)^{\frac{1}{2}}}{2h'} \right\} \right] \quad \ldots \quad (20{\cdot}4)$$

CONCLUSION

The equations (19) and (20) give approximately the temperature at any depth in the semi-infinite region bounded either by a thin film or a film of finite thickness having definite thermal capacity. The equation (20·4) shows that the temperature v_2 is independent of the conductivity of the thin film. This clearly indicates the development of new thermoplastic device satisfying the condition $\Delta hh' = 1$, *i.e.* $k_2\rho_2c_2 = 2k_1\rho_1c_1$

Further work on the heat flow in composite solid in which the conductivities vary with distance and the rate of heat production also varies with depth is under consideration

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