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Entry-length flow in a vertical cooled pipe

By S. N. Singh and R. N. Pandey

Department of Mathematics, Banaras Hindu University, Varanasi-5

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A treatment of the flow and heat transfer due to free convection in the entry region of a cooled vertical pipe, which is open at both ends, has been given. The analysis is based on the Lighthill method Further in the analysis, velocity and temperature profiles have been assumed, which satisfy all the boundary conditions Parameters involved in the profiles have been calculated by assuming series solution In the analysis, we have defined a new non-dimensional number M, which happens to be a function of boundary layer nondimensional thickness δ This number has an influence over the fluid flow in the boundary layer region. A value of M has been obtained for which vertical displacement Q is the maximum, on taking only first two terms of Q It has been observed that there is a deceleration of flow in the vicinity of the wall in boundary layer region due to cooling of the wall and increasing the Rayleigh number. For $R = 0(10^3)$, there is a reversal of flow in the vicinity of the wall

Introduction

Ostrach (1954) obtained an expression for buoyancy forces on the fluid within the pipe relative to the cooled fluid at the same level outside the pipe. In this it will be better to regard the fluid as moving only under the influence of a pressure gradient and relative buoyancy forces within. Ostroumov (1958) has given an extensive treatment of the natural convection in cylindrical channels in term of Bessel & Neumann functions. Lighthill (1953) has given an analysis of the flow through a cylindrical pipe in which one end is closed and the wall of the pipe maintained at a constant temperature, the body forces acting in the direction of the closed end. It has been considered here that at the open end there is an orifice which supplies fluid. The flow of fluid depends upon the parameter l/R, for given Prandtl and Rayleigh numbers When l/R is small, the flow is like free convection about a flat plate, but when l/R is large, the flow is not like free convection. In his treatment, he has used the integral method and in case of similarity regime he finds that the flow fills the whole of the tube for a particular value of l/R. Different authors have adopted this technique in the case of free convection in combined flow to slightly varied physical situations. Martin (1967) has performed experiments of heat transfer due to natural convection in a long extremely cooled vertical cylinder with uniform wall temperature, containing heat generating fluid in a laminar flow (with Prandtl number equal to or greater than unity). Takhar (1967) has given a treatment for the entry length flow in a vertical heated open pipe. He finds that at Rayleigh numbers greater than 10^3, the flow in the middle of the pipe becomes stagnant. This analysis cannot produce satisfactory results before the boundary layer fills the whole of the tube.

In the present paper, an attempt has been made to study the flow in entrance region of a vertical pipe which has both its ends open and is being cooled with (i) a constant temperature at the wall and (ii) the wall temperature decreasing exponentially as a function of vertical height It has been taken for granted that; (*a*) kinematic viscosity and thermal conductivity are approximately constant and Boussinesque approximation holds; (*b*) velocity and temperature profiles are assumed so as to satisfy the initial and boundary conditions; (*c*) the equations of motion, continuity and heat conduction have been integrated to find out the various parameters involved in the analysis with the help of the equations at the axis and at the walls; (*d*) the momentum and thermal boundary layer thicknesses are assumed to be equal; (*e*) The parameter Q in the assumed profiles gives the vertical displacement outside the boundary layer thickness

It is seen that boundary layer fills the entire tube so as to give the fully developed flow through the pipe, and a reversal of the flow occurs at the cooling Rayleigh number greater than 10^3. Graph has been plotted between

$$M = \frac{\beta_1 \nu^2}{\sigma \beta^2 g a^3}$$

and the boundary layer thickness δ.

It seems that the analysis may prove useful to engineering problems on free convection in the entrance region of the pipe.

Analysis

The equations of motion here are similar to differential equations of free convection except that the pressure no longer takes the hydrostatic value The flow is assumed to be of boundary layer type, which means that gradient of a quantity along the pipe is small as compared to the pressure gradient in the radial direction Keeping this in view, we have the equations of :

conservation of mass

$$\frac{\partial u}{\partial z} + \frac{1}{r} \frac{\partial (rv)}{\partial r} = 0,$$

conservation of momentum

$$\left(u \frac{\partial u}{\partial z} + v \frac{\partial u}{\partial r} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) - \rho g,$$

$$0 = -\frac{1}{\rho} \frac{\partial p}{\partial r}$$

and conservation of heat

$$u\frac{\partial T}{\partial z}+v\frac{\partial T}{\partial r}=k\left[\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right]$$

where u, v are the axial and radial velocities k and $\mu/\rho(=\nu)$ are the thermal diffusivity and the coefficient of kinematic viscosity, respectively. The initial and boundary conditions are

(i) $r=a,\ u=0=v,\ T=T_w$ for all $z>0$

(ii) $z=0,\ u=u_0,\ v=0,\ T=T_0$ for all $r>0$.

Let T_w and T_0 be the temperature of the wall and that of the fluid at the entry, respectively. Let us assume that the wall temperature is of the form

$$T_w=T_0-\Delta T f(z),\quad \text{where}\quad \Delta T=T_w-T_0.$$

Further, the variation of all physical properties are ignored except the density involved in the buoyancy term Also the viscous dissipation and work done against gravity field are neglected Thus we have for the density in the buoyancy term

$$\rho=\rho_{T_w}+\left(\frac{\partial\rho}{\partial T}\right)_{T_w}(T-T_w)+\frac{1}{2!}\left(\frac{\partial^2\rho}{\partial T^2}\right)_{T_w}(T-T_w)^2+\dots \qquad \dots\ (2)$$

which can be regarded as Taylor equation of state.

Introducing coefficient of volume expansion at T_w as

$$\beta=-\frac{1}{\rho}\left(\frac{d\rho}{dT}\right)_{T_w}\ \text{and the quantity}\ \beta_1=-\frac{1}{\rho}\left(\frac{\partial^2\rho}{\partial T^2}\right)_{T_w},$$

we can write (2) as

$$-\rho=-\rho_w-\beta\rho(T_w-T)-\frac{\beta_1\rho}{2}(T_w-T)^2.$$

Substituting this in (1), we get

$$u\frac{\partial u}{\partial z}+v\frac{\partial u}{\partial r}=-\left(\frac{1}{\rho}\frac{\partial p}{\partial z}+g\right)+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)-g\beta(T_w-T)-\frac{g\beta_1}{2}(T_w-T)^2.$$

The temperature $\Delta T+T_w-T_0$ defines a cooling Rayleigh number

$$Ra=\frac{\beta g\,a^3\Delta T}{k\nu}.$$

If we introduce the non-dimensional quantities as $r=aR,\ z=aZ$,

$$u=\frac{k}{a}U,\ v=\frac{kV}{a},\ L\frac{k\nu}{a^3}=-\left(\frac{1}{\rho}\frac{dp}{dZ}+g\right),\ T=T_w+\frac{\Delta T}{Ra}\Theta,$$

$$M=\frac{\beta_1\nu^2}{\sigma\beta^2 g a^3},$$

We have the equations of mass, momentum and heat as

$$\frac{\partial U}{\partial Z}+\frac{1}{R}\frac{\partial (RV)}{\partial R}=0 \qquad \dots \quad (3)$$

$$\frac{1}{\sigma}\left[\frac{1}{2}\frac{\partial}{\partial Z}U^2+V\frac{\partial U}{\partial R}\right]=L+\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{\partial}{\partial R}UR\right)-\tfrac{1}{2}M\Theta^2+\Theta \qquad \dots \quad (4)$$

$$\frac{\partial P}{\partial R}=0$$

$$-Ra\,U\frac{df}{dZ}+\left(U\frac{\partial \Theta}{\partial Z}+V\frac{\partial \Theta}{\partial R}\right)=\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Theta}{\partial R}\right) \qquad \dots \quad (5)$$

subject to the conditions

$$\left.\begin{array}{l} R=1,\ Z>0,\ U=V=\Theta=0, \quad \text{at the wall} \\ R>0,\quad Z=0,\quad U=U_0,\quad V=0,\ \Theta=Ra, \text{ at the entry} \end{array}\right\} \qquad \dots \quad (6)$$

where U_0 is the non-dimensional velocity at the entry along Z-direction.

Integrating (3), (4), (5) over a cross-section subject to the conditions (6), we get

$$\left.\begin{array}{l} \int_0^1 RUdR=\tfrac{1}{2}U_0 \\ \frac{1}{2\sigma}\frac{d}{dZ}\int_0^1 RU^2dR=\frac{L}{2}+\left(\frac{\partial U}{\partial R}\right)_{R=1}+\int_0^1 R\Theta dR-\tfrac{1}{2}M\int_0^1 \Theta^2 RdR \\ \text{and}\quad -\tfrac{1}{2}RaU_0\frac{df}{dZ}+\frac{d}{dZ}\int_0^1 RU\Theta dR=\left(\frac{\partial \Theta}{\partial R}\right)_{R=1} \end{array}\right\} \qquad \dots \quad (7)$$

and the equations

$$\left.\begin{array}{l} \frac{1}{2\sigma}\frac{\partial}{\partial Z}U^2=L+\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{\partial}{\partial R}UR\right)+\Theta-\tfrac{1}{2}M\Theta^2, \\ U\frac{\partial \Theta}{\partial Z}=\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{\partial}{\partial R}R\Theta\right) \end{array}\right\} \qquad \dots \quad (8$$

at the axis of the pipe, and

$$\left.\begin{array}{l} 0=L+\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial U}{\partial R}\right) \\ 0=\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Theta}{\partial R}\right) \end{array}\right\} \qquad \dots \quad (9)$$

at the wall.

Here, we shall assume the velocity and temperature profiles of Kerman—Pohlhausen type, which satisfies the conditions given in (6) These profiles are :

$$U = \begin{bmatrix} PU_0 & (\delta < X < 1) \\ PU_0 \left\{1-\left(1-\frac{X}{\delta}\right)^2 \left(1-\frac{OX}{\delta}\right\}\right) & (0 < X < \delta) \end{bmatrix}$$

$$\Theta = \begin{cases} Ra & (\delta < X < 1) \\ Ra\left\{1-\left(1-\frac{X}{\delta}\right)^2\right\} & (0 < x < \delta) \end{cases} \quad \ldots \quad (10)$$

where δ corresponds to a boundary layer of non-dimensional thickness, enclosing a potential core of radius $\delta_1 = 1-\delta$ and $X = 1-R$. Here, we have two cases depending upon the way the pipe is cooled.

Case I.

When the temperature of the wall is constant the equation (5) becomes

$$U\frac{\partial\Theta}{\partial Z} + V\frac{\partial\Theta}{\partial R} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Theta}{\partial R}\right)$$

Case II.

When the wall temperature decreases exponentially, *i.e*, when $T_w = T_0 - \Delta T \exp(\alpha X)$, where α is a small quantity.

On using (8) and (10) we can find the value of L. Hence substituting the values given in (10) and the value of L in (7), we obtain for case I

$$P[5(6-4\delta+\delta^2)+Q\delta(5-2\delta)] = 30 \quad \ldots \quad (11)$$

$$\frac{U_0^2}{840\sigma}\frac{d}{d\delta}[P^2\{14(15-28\delta+8\delta^2)+4Q\delta(21-10\delta)+Q^2\delta(8-3\delta)\}]$$

$$= \left\{\frac{-\delta(4-\delta)}{12}Ra - \frac{Z+Q}{\delta}PU_0 - \frac{M}{60}(4\delta^2-14\delta+15)\right\}\frac{d\delta}{dZ} \quad \ldots \quad (12)$$

and

$$\frac{d}{d\delta}[P\{14(15-14\delta+4\delta^2)+Q\delta(21-10\delta)\}] = \left(\frac{-840}{U_0\delta}\right)\frac{dZ}{d\delta} \quad \ldots \quad (13)$$

We see that for $\delta = 0$, $P = 1$, $Q = 0$ at $E = 0$.

Obviously, $\delta = 0$ is a singularity for the above equations, for which we have to find the values of P and Q in the neighbourhood of $\delta = 0$ by considering series solution in terms of δ. Thus

$$P = 1+a_1\delta+a_2\delta^2+\ldots$$
$$Q = d_1\delta+d_2\delta^2+\ldots$$

On substituting these values in (11), (12) and (13) and comparing the coefficients of various powers of δ, we obtain

$$P = 1+.666\delta+(.016M-.033)\delta^2$$

$$+\frac{(6M^2-548.6M+.24M^3-747\ 3Ra+4705.4+60\ 6MRa)}{1500+157.5M}\delta^3+\ldots$$

$$Q=(1\ 86-M)\delta+(20\ 07-.17M-.01M^2-3\ 3Ra)\delta^2+\ldots$$

Now to find Z, substituting the values of P and Q in either of (12) ad (13), we have

$$Z=\frac{1}{30}[\delta^2+(1\ 01-033M)\delta^3+\ldots]$$

Case II.

When the temperature of the wall decreases exponentially (with α as a small number), we have the treatment exactly similar to the case *I*. Now we obtain after simplification

$$-[P\{14(15-14\delta+4\delta^2)+Q\delta(21-10\delta)\}$$

$$=\left[-\frac{840}{U_0\delta}+210\alpha\right]\frac{dZ}{d\delta}$$

the equation (11) and (12), and

$$Z=.033\delta^2+(101+.005\alpha-003M)\delta^3+.$$

Thus, we have P, Q and Z in terms of the boundary layer thickness δ.

Now from the analysis of the problem, we have,

$$\text{Nusselt number} = \frac{4}{\delta}$$

$$\text{Heat flux at the wall}\left(\frac{\partial\Theta}{\partial Z}\right)_{Z=0}=\frac{2Ra}{\delta},$$

$$\text{Skin friction at the wall}\left(\frac{\partial U}{\partial Z}\right)_{Z=0}=PU_0\left(\frac{2+Q}{\delta}\right).$$

Discussion

We observe that due to cooling of the wall, the fluid in the vicinity of the wall in the boundary layer region becomes decelerated. This deceleration of the fluid also depends upon the cooling Rayleigh number Ra. As Ra increases fluid goes on decelerating and ultimately when it becomes greater than 10^3, the reversal of flow occurs in the fluid in the vicinity of wall.

Further, in the case of the pipe the flow is due to its continuity. It is noted that in free convection, the adverse pressure gradient is confined to the boundary layer produced by buoyancy forces, but in the case of forced convection, this takes places in the main stream also

We see that for $M = 0$, $Ra = 0$, the boundary layer fills the whole of the tube for $Z = 0.045$, but at increasing Ra this value decreases, we easily see that at

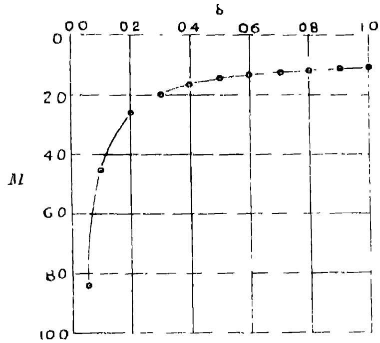

Figure 1

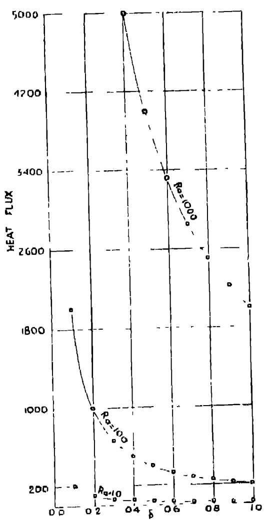

Figure 2

$Ra = 0(10^3)$, the value of Z is approximately 0.004. The value of M, for which Q is maximum upto the first two terms, comes out to be

$$M = \frac{-6.79\delta - 3.88}{\delta},$$

which shows that this depends upon δ The graph of M *vs* δ is shown in figure 1.

Also, we see that Nusselt number is a function of δ; heat flux at the wall and the skin friction at the wall are functions of Ra and δ.

The graph of heat flux *vs* δ for fixed Ra is shown in figure 2.

References

Lighthill M J 1953 *Q. J. Mech Appl. Math* **6**, 398

Martin B W. 1967 *Proc Roy Soc.* **A301**, 327

Ostrach S. 1954 *NACA Tech. Note* 3141

Ostroumov G. A. 1958 *NACA Tech. Mem.* (Translation) 1407

Takhar H. S. 1967 *Proc. Jap Soc Mech. Engs*, Semi International Symposium, Paper No. 201, 1-8.