TRANSIENT HEAT CONDUCTION IN A FINITE WEDGE

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ABSTRACT. Inverse and direct problems of transient heat conduction through a cylindrical wedge have been solved with the help of integral transforms

INTRODUCTION

In most problems in the theory of heat conduction either temperature or heat transfer conditions are prescribed on the surface of a body, and conditions at interior points are to be determined Such problems are known as "Direct problems". There is another class of problems, (Inverse problems), in which either temperature or heat flux, on some part or whole of the surface of a body, is to be determined from the temperature distribution on suitable interior surfaces, and the remaining portion of the boundary surface. G. Stolz. Jr. (1960) obtained an integral equation, and out-lined a numerical method for solving inverse problems, with special reference to sphere. The problem occurred as a part of quenching programme (G. Stolz Jr. 1956). T. J. Mirsepassi solved the problem by a graphical method A V. Masket and A. C. Vastano (1962) solved similar problems of Mathematical Physics, using Laplace Transform and Separation of variables, and termed these as "Interior Value Problems". O. R. Burggraf (1964) has obtained the solution as a rapidly convergent series, with lumped capacitance approximation, as leading term. Burggraff has taken boundary conditions etc. as a function of time only. E. M. Sparrow, A. Haji Sheikh, and T. S. Lundgren (1964) have also tackled the inverse problems. Inverse problems arise in Quenching studies, the analysis of experimental data, and measurement of aerodynamic heating, etc.

I. *Inverse problem for a cylindrical wedge* $0 \leqslant r \leqslant a; o \leqslant \theta \leqslant \theta_0; o \leqslant z \leqslant h$. Temperature on the surface $\theta = \theta_0$ to be determined from the given temperature distribution on an interior plane $\theta = \alpha$ and zero temperature on the remaining boundary surfaces Initial temperature is zero. K represents the constant thermal diffusivity. $u(r, \theta, z, t)$ the temperature satisfies :

$$\frac{\partial u}{\partial t} = K \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \right);$$

$$0 < r < a;\ 0 < \theta < \theta_0;\ 0 < z < h;\ t > 0 \qquad \ldots (1)$$

$$u(r,\theta,z,0)=u(r,\theta,0,t)=u(r,\theta,h,t)=u(r,0,z,t)=u(a,\theta,z,t)=0 \quad \dots \quad (2)$$

$$u(r,\alpha,z,t)=f(r,z,t);\ 0<\alpha<\theta_0 \quad \text{(Known)}, \quad \dots \quad (3)$$

$$u(r,\theta_0,z,t)=g(r,z,t),\ \text{say} \quad \text{(unknown)}, \quad \dots \quad (4)$$

$u(r,\theta,z,t)$ tends to zero, as r tends to zero.

Applying to equations (1) through (4) the finite Fourier sine transform, with respect to z, and the Laplace transform with respect to t, defined successively as,

$$U(r,\theta,n,t)=\int_0^h u\ \text{Sin}\,\frac{n\tilde{\Lambda}z}{h}\,dz$$

and

$$\bar{U}(r,\theta,n,p)=\int_0^\infty U\exp\,(-pt)dt$$

We obtain ·

$$r^2\frac{\partial^2\bar{U}}{\partial r^2}+r\frac{\partial\bar{U}}{\partial r}-r^2q^2\bar{U}+\frac{\partial^2\bar{U}}{\partial\theta^2}=0;\quad \text{where}\quad q^2=\frac{n^2\Lambda^2}{h^2}+\frac{p}{K} \quad \dots \quad (5)$$

$$\bar{U}(r,0,n,p)=\bar{U}(a,\theta,n,p)=0 \quad \dots \quad (6)$$

$$\bar{U}(r,\alpha,n,p)=\bar{F}(r,n,p) \quad \dots \quad (7)$$

$$\bar{U}(r,\theta_0,n,p)=\bar{G}(r,n,p). \quad \dots \quad (8)$$

Further applying to equations (5) through (8) the finite Lebedev transform (Naylor, 1963; equation 9) defined as :

$$\bar{U}'(s,\theta,n,p)=\int_0^a[I_S(qa)K_S(qr)-I_S(qr)K_S(qa)]\bar{U}\,\frac{dr}{r}.$$

where $I_s(qr)$ and $K_s(qr)$ are the modified Bessel functions of the first and second kind of order s''.

$$\frac{d^2\bar{U}'}{d\theta^2}=-s^2\bar{U}' \quad \dots \quad (9)$$

$$\bar{U}'(s,0,n,p)=0$$

$$\bar{U}'(s,\alpha,n,p)=\bar{F}'(s,n,p)$$

$$\bar{U}'(s,\theta_0,n,p)=\bar{G}'(s,n,p).$$

Solution of equation (9), after some simplifications reduces to

$$\bar{G}'(s,n,p)=\bar{F}'(s,n,p)\,\frac{\sin\,(s\theta_0)}{\sin\,(s\alpha)} \quad \dots \quad (10)$$

Applying to equation (10) the inverse finite Lebedev transform (Naylor, 1963; equation 11), the inverse Laplace and the inverse finite Fourier Sine transform (Sneddon, 1951; *P.* 74, Th. 27), we obtain :

$$g(r,z,t) = -\frac{1}{h\pi^2}\sum_{n=1}^{\infty}\sin\frac{n\pi z}{h}\int_{c-i\infty}^{c+i\infty}\exp.(pt)\cdot dp\int_L \bar{F}'(s,n,p)$$

$$\times\frac{\sin(s\theta)}{\sin(s\theta_0)}\cdot\frac{I_S(qr)}{I_S(qa)}\,sds.$$

where L is the path $R(s) = C'$.

II. *A direct problem:* Cooke (1955), Craggs (1945) and Jaeger (1942) have solved direct heat conduction problems for a wedge, with constant surface temperature. Their results can be extended, if in article I, $g(r,z,t)$ the variable surface temperature is supposed to be known and $u(r,\theta,z,t)$ the temperature distribution in the wedge, is similarly determined to be :

$$u(r,\theta,z,t) = -\frac{1}{h\pi^2}\sum_{n=1}^{\infty}\sin\frac{n\pi z}{h}\int_{c-i\infty}^{c+i\infty}\exp(pt)\;dp\int_L \bar{G}'(s,n,p)$$

$$\frac{\sin(s\theta)}{\sin(s\theta_0)}\times\frac{I_S(qr)}{I_S(qa)}\cdot s.ds.$$

This problem can be used for the study of analogous problem of transient flow taking place in earth dams during drawdown.

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