35

LOW-ENERGY SCATTERING OF ELECTRON BY ATOMIC POTENTIAL WITH A LONG-RANGE r^{-4} TAIL

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ABSTRACT We have obtained, following Spector (1964), an expression for the S-matrix in the case of an attractive inverse fourth power potential An effective range formula for an atomic potential with r^{-4} tail has also been derived. A general expression for phase shifts η_l for different angular momenta l is given for an atomic potential which is represented by a screened Coulomb potential of Allis and Morse type when r is small and the long range r^{-4} potential when r is large. Numerical results are presented for η_0 for low energy e—He collision The effect of exchange has been neglected in our work.

INTRODUCTION

An exact analytical solution of the Schrodinger equation describing the scattering of an electron by a neutral atom is extremely difficult due to the complexity of the atomic potential. The potential surrounding the atom consists of an electrostatic screened field and a polarization field mainly of dipole nature induced by the incoming electron. The form of the latter field is usually taken as $\alpha(r)r^{-4}$ where $\alpha(r)$ for small values of r is a complicated function but for large values of r reduces to a constant α the electric polarizability. The Schrodinger equation with a central potential αr^{-4} can be solved by transforming it to a modified Mathieu equation. O'Malley, Spruch and Rosenberg (1961) have shown that in the case of a long range r^{-4} potential the expansion of k cot η_0 in the zero energy limit contains a number of terms not present in the usual effective range formula for short range potential. Spector (1964) has made a detailed study of the behaviour of the Mathieu function and its derivative at the transition point where the kinetic energy is equal to the magnitude of the potential energy due to the αr^{-4} term. He has worked out the scattering matrix for a repulsive potential but in most physical problems the attractive potential comes into play. So we have calculated the scattering relations with an attractive r^{-4} potential. Further in the zero energy limit $k = 0$ the expansion of k cot η_0 is influenced by the asymptotic form of the potential i.e. αr^{-4}, the expansion terms of k cot η_0 agree with those of O'Malley *et al* (1961).

In the potential term in the Schrodinger equation, we have taken for the screened coulomb part the form due to Allis and Morse (1931) when r is small and we assume that in this region the polarization potential is negligible We further maintain that for large values of r when the latter potential becomes predominant, $\alpha(r)$ reduces to the electric polarizability α of the atom. To simplify calculation the exchange effect due to the indistinguishibility of the incident and atomic electrons has been neglected. A general expression for low energy phase shift for different angular momenta has been deduced The phase shifts for zero angular momentum are calculated for e^-–He scattering in the low energy region 0–1 ev and a comparison with similar calculations of LaBahn and Callaway (1964) shows good agreement.

CALCULATION OF S-MATRIX

The Schrodinger equation describing low energy scattering of an electron in presence of an attractive polarization potential β^2r^{-4} is

$$\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}+\frac{\beta^2}{r^4}+k^2\right]\phi_l=0 \qquad \dots (1)$$

Here k is the wave number of the incident electron and $\beta^2=\alpha$.

(The atomic units are used throughout our calculations).

The equation (1) can easily be transormed into the Modified Mathieu equation (Spector 1964).

$$\left[\frac{d^2}{dz^2}-(l+\tfrac{1}{2})^2+2\beta k\cosh 2z\right]M(z)=0 \qquad \dots (2)$$

with the substitutions

$$\begin{aligned} r &= (\beta/k)^{\frac{1}{2}}e^{-z} && \text{when} \quad 0<r\leqslant(\beta/k)^{\frac{1}{2}} \\ &= (\beta/k)^{\frac{1}{2}}e^{z} && \text{when} \quad (\beta/l)^{\frac{1}{2}}<r<\infty \end{aligned} \qquad \dots (3)$$

and

$$\phi_l(r)=\sqrt{r}M(z)$$

The solutions of the equation (2) are the various forms of the modified Mathieu functions. It will be convenient to write down those which will be required in our work:

$$M_{e_{\pm\nu}}(z,k)=\sum_{p=-\infty}^{\infty}C^{\nu}{}_{2p}(k)e^{\pm(2p+\nu)Z} \qquad \dots (4)$$

$$M_\nu^{(1)}(z,k)=\left[\sum_{p=-\infty}^{\infty}C^{\nu}{}_{2p}(k)\right]^{-1}\sum_{p=-\infty}^{\infty}(-1)^pC^{\nu}{}_{2p}(k)J_{\nu+2p}(2\sqrt{\beta k}\cosh z) \qquad (5)$$

$$M_\nu^{(2)}(z,k)=\left[\sum_{p=-\infty}^{\infty}C^{\nu}{}_{2p}(k)\right]^{-1}\sum_{p=-\infty}^{\infty}(-1)^pC^{\nu}{}_{2p}(k)Y_{\nu+2p}(2\sqrt{\beta k}\cosh Z) \qquad (6)$$

J_ν and Y_ν are the Bessel functions of the first and second kinds. The order ν of the Mathieu functions is a function of l and k.

When k is sufficiently small, ν is given by

$$\nu \approx l+\tfrac{1}{2}-\frac{\beta^2 k^2}{4(l+\frac{3}{2})(l+\frac{1}{2})(l-\frac{1}{2})}$$

the terms involving k^4 and higher powers of k being neglected. The same coefficients $C^{\nu}_{2p}(k)$ which are functions of k occur in $M_{e\pm\nu}$ or $M_{\nu}^{(1),(2)}$. These coefficients can be expressed as a continued fraction converging rapidly as $k\to 0$. In fact

$$C^{\nu}_{2p}\to C_0^{\nu}\frac{\Gamma(\nu+1)}{2^{2p}p!\,\Gamma(\nu+p+1)}(\beta k)^{2|p|} \text{ as } k\to 0 \qquad . \quad (8)$$

When p is negative $C^{\nu}_{2p}\to 0$ as $k\to 0$.
Lastly we may construct the modified Mathieu functions $M_{\nu}^{(3)}$ and $M_{\nu}^{(4)}$ from the equations (5) and (6) replacing the Bessel functions $J_{\nu+2p}$ and $Y_{\nu+2p}$ by the Hankel functions $H^{(1)}_{\nu+2p}$ and $H^{(2)}_{\nu+2p}$.

The Mathieu functions $M_{\nu}^{(1),(2)}$ and hence $M_{\nu}^{(3),(4)}$ are continuous functions of r but their derivativews with respect to r do not exist at $r=(\beta/k)^{\frac{1}{2}}$ whereas the functions $M_{e\pm\nu}$ and their derivatives are continuous everywhere. Because of this discontinuity of the derivatives, the general solution of the equation (1) which is a linear combination of the solutions $M_{\nu}^{(1)}$, $M_{\nu}^{(2)}$ or $M_{\nu}^{(3)}$, $M_{\nu}^{(4)}$ (each multiplied by $\sqrt{r}$) should have different coefficients for $r\leqslant(\beta/k)^{\frac{1}{2}}$ and $r\geqslant(\beta/k)^{\frac{1}{2}}$.

So the general solution of the equation (1) may be taken as

$$\phi_l(r)=A\sqrt{r}M_{\nu}^{(3)}\left(-\ln\left\{\left(\frac{k}{\beta}\right)^{\frac{1}{2}}r\right\},k\right)+B\sqrt{r}M_{\nu}^{(4)}\left(-\ln\left\{\left(\frac{k}{\beta}\right)^{\frac{1}{2}}r\right\},k\right),$$

$$r\leqslant\left(\frac{\beta}{k}\right)^{\frac{1}{2}} \quad \ldots \quad (9)$$

$$\phi_l(r)=A'\sqrt{r}M_{\nu}^{(3)}\left(\ln\left\{\left(\frac{k}{\beta}\right)^{\frac{1}{2}}r\right\},k\right)+B'\sqrt{r}M_{\nu}^{(4)}\left(\ln\left\{\left(\frac{k}{\beta}\right)^{\frac{1}{2}}r\right\},k\right)$$

$$r\geqslant\left(\frac{k}{\beta}\right)^{\frac{1}{2}} \quad \ldots \quad (10)$$

It is to be noted that when r is small,

$$\phi_l\to A\sqrt{\frac{2}{\pi\beta}}\,re^{i\left(\frac{\beta}{r}-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}+B\sqrt{\frac{2}{\pi\beta}}\,re^{-i\left(\frac{\beta}{r}-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)} \quad \ldots \quad (9a)$$

and when r is large

$$\phi_l\to A'\sqrt{\frac{2}{\pi k}}\,e^{i\left(kr-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}+B'\sqrt{\frac{2}{\pi k}}\,e^{-i\left(kr-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)} \quad \ldots \quad (10a)$$

It is evident from (10a) that the S-matrix for the scattering of a charged particle in the presence of an attractive long range $\beta^2 r^{-4}$ potential is

$$S(k, l) = i\, \frac{A'}{B'}\, e^{-i\nu\pi}\, e^{il\pi} \qquad \dots \quad (11)$$

with ν defined in the equation (7)

To evaluate A'/B' we shall follow the procedures enunciated by Spector (1964) in his development of the S-matrix for a repulsive r^{-4} potential. In order to connect the solutions (9) and (10) at the point $r = (\beta/k)^{\frac{1}{2}}$ where their derivatives do not exist. We shall make use of the Mathieu functions $Me_{\pm\nu}$ given in equation (4). These latter functions and their derivatives are continuous every where. For some r_1 such that $0 < r_1 < (\beta/k)^{\frac{1}{2}}$ we may write

$$AM_\nu^{(3)} + BM_\nu^{(4)} = \alpha Me_\nu + \beta Me_{-\nu}$$

and determine the constants α and β in terms of A and B by solving above equation together with the equation

$$AM_\nu^{(3)\prime} + BM_\nu^{(4)\prime} = \alpha M'_{e\nu} + \beta M'_{e_{-\nu}}$$

Similarly for some $r_2 > (\beta/k)^{\frac{1}{2}}$ we take

$$A'M_\nu^{(3)} + B'M_\nu^{(4)} = \gamma Me_\nu + \delta Me_{-\nu}$$

and determine γ and δ in terms of A' and B'. The constants γ and δ can now be expressed in terms of α and β by using the caonditions for the continuity of the solution and its derivative with respect to r at $r = (\beta/k)^{\frac{1}{2}}$ i.e. at $z = 0$. With z as defined in (3) we find that $\frac{dz}{dr}$ discontinuously changes its sign at this point. With the help of relation (4) we finally get

$$\gamma = \beta \quad \text{and} \quad \delta = \alpha.$$

Utilizing the various properties of the Mathieu functions as reported in Spector (1964) it is easy to show that

$$\frac{A'}{B'} = \frac{1 - R_l^2(A - Be^{2i\nu\pi})/(A-B)}{1 - R_l^2(Ae^{-2i\nu\pi} - B)/(A-B)} \qquad (12)$$

where

$$R_l = \frac{M_\nu^{(1)}(0)}{M_{-\nu}^{(1)}(0)}$$

$$\simeq (\tfrac{1}{4}\beta k)[\Gamma(1-\nu)/\Gamma(1+\nu)] \times (1 - \tfrac{1}{4}\nu\beta^2 k^2/(1-\nu^2)^2) \qquad \dots \quad (13)$$

neglecting higher powers of k

For $l = 0$

$$R_0 \simeq \beta/k \left(1 + \frac{4}{3}\, \beta^2 k^2 \ln \frac{\beta k}{4} - \frac{8}{3}\, \beta^2 k^2 \psi(3/2) + \frac{20}{9}\, \beta^2 k^2\right) \qquad \dots \quad (13a)$$

$$\psi(3/2 = 0.0365$$

Substituting the value of A'/B' in (11) one gets an expression for the S-matrix. If A/B is known, then in principle one can determine the phaseshifts η_l by virtue of the relation $S(k, l) = e^{2i\eta l}$. The formula (12) is of great importance in the development of the present work. It will be worth-while to note that since B is complex conjugate of A, B' is the complex conjugate of A'.

EFFECTIVE RANGE FORMULA

We shall now utilize the formula (12) to develop in a straight forward manner an effective range formula for the scattering of an electron by a central field potential which is assumed to vanish as r^{-4} with no other long range components For this purpose it will be convenient to take the solution of the wave equation (1) as a linear combination of $\sqrt{r}M_\nu^{(1)}$ and $\sqrt{r}M_\nu^{(2)}$:

$$\phi_l = c\sqrt{r}(M_\nu^{(1)}+DM_\nu^{(2)}) \qquad r \leqslant \left(\frac{\beta}{k}\right)^{\frac{1}{2}} \quad .. \quad (14)$$

$$\phi_l = c'\sqrt{r}(M_\nu^{(1)}+D'M_\nu^{(2)}) \qquad r \geqslant \left(\frac{\beta}{k}\right)^{\frac{1}{2}} \quad ... \quad (15)$$

Where C and C' are the normalization constants and D and D' are arbitrary constants. It follows from the known properties of $M_\nu^{(1)}$ and $M_\nu^{(2)}$ (Spector 1964; Meixner and Schafke 1954) that ϕ_l behaves near the origin as

$$\phi_l \sim \frac{r}{\beta}\left[\sin\left(\frac{\beta}{r}-\frac{l\pi}{2}+\delta_l\right)-D\cos\left(\frac{\beta}{r}-\frac{l\pi}{2}+\delta_l\right)\right] \text{ for } r << \left(\frac{\beta}{k}\right)^{\frac{1}{2}} \quad .. \quad (16)$$

$$\delta_l = \frac{\pi\beta^2k^2}{8(l+3/2)(l+1/2)(l-1/2)} \quad ... \quad (17)$$

In (17), the powers of k higher than two are neglected.

Now we consider an electron scattered by an atomic potential $U(r)$ which is supposed to be central. The scattering wave functions $u_l(r)$ satisfy the radial wave equation

$$\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-U(r)+k^2\right]u_l(r) = 0 \quad ... \quad (18)$$

We assume here that $U(r)$ tends to $-\beta^2r^{-4}$ where r is sufficiently large, and that $u_l(0) = 0$. It is to be noticed that when r is large, $u_l(r)$ tends to the solutions $\phi_l(r)$ of the equation (1) to which the equation (18) is reduced when $r \to \infty$.

Taking $u_l^{(1)}$, $u_l^{(2)}$ and $\phi_l^{(1)}$, $\phi_l^{(2)}$ as the solutions of the equations (18) and (1) respectively for the wave numbers k_1 and k_2 it is easy to show that (c.f. Bethe 1959)

$$\lim_{r\to 0}\left|\phi_l^{(1)}\frac{d}{dr}\phi_l^{(2)}-\phi_l^{(2)}\frac{d}{dr}\phi_l^{(1)}\right|_r = (k_2^2-k_1^2)\int_0^\infty(\phi_l^{(1)}\phi_l^{(2)}-u_l^{(1)}u_l^{(2)})dr \quad ... \quad (19)$$

If D_1 and D_2 be the values of D in the equation (14) and $\delta_l^{(1)}$ and $\delta_l^{(2)}$ the values of δ_l corresponding to the wave numbers k_1 and k_2, the equation (19) leads to

$$\left(\frac{D_1}{\beta}-\frac{D_2}{\beta}\right) \cos(\delta_l^{(1)}-\delta_l^{(1)})-\frac{1+D_1D_2}{\beta} \sin(\delta_l^{(1)}-\delta_l^{(2)})$$

$$=(k_2^2-k_1^2)\int_0^{\infty}(\phi_l^{(1)}\phi_l^{(2)}-u_l^{(1)}u_l^{(2)})\,dr$$

so that when $k_1=0$ and $k_2=k$ one has

$$\left(\frac{D_0}{\beta}-\frac{D}{\beta}\right)\cos\delta_l+\frac{1+D_0D}{\beta}\sin\delta_l=k^2\int_0^{\infty}(\phi_l^{(0)}\phi_l-u_l^{(0)}u_l)dr \qquad \ldots (20)$$

Therefore following Bethe (1949) we have for the effective range fomula

$$\frac{D_0}{\beta}-\frac{D}{\beta}+\frac{1+D_0D}{\beta}\sin\delta_l\approx\tfrac{1}{2}r_ek^2 \qquad \ldots (21)$$

where the effective range

$$r_e=2\cdot\int[\{\phi_0^{0}\}^2-\{u_0^{0}\}^2]dr \qquad \ldots (22)$$

Now we propose to determine a formula from which D/β and D_0/β can be calculated.

Substituting the relations

$$M_\nu^{(1)}=\tfrac{1}{2}[M_\nu^{(3)}+M_\nu^{(4)}]$$

and

$$M_\nu^{(2)}=\frac{1}{2i}[M_\nu^{(3)}-M_\nu^{(4)}]$$

in the equations (14) and (15) we get from the equation (12) on simplification

$$D'=\frac{R_l^2(1+\cos 4\delta_l+D\sin 4\delta_l)}{2D-R_l^2(D-D\cos 4\delta_l+\sin 4\delta_l)} \qquad \ldots (23)$$

Again comparing the asymptotic form of ϕ_l in (14) :

$$C'\sqrt{\frac{2}{\pi k}}\left[\sin\left(kr-\frac{l\pi}{2}+\delta_l\right)-D'\cos\left(kr-\frac{l\pi}{2}+\delta_l\right)\right]$$

with the aysmptotic form of the actual wave function u_l :

$$u_l\rightarrow\text{const.}\times\sin\left(kr-\frac{l\pi}{2}+\eta_l\right),$$

we obtain a relation connecting D' with the phase shift η_l :

$$\tan\eta_l=\frac{\tan\delta_l-D'}{1+D'\tan\delta_l}$$

On substitution of the value for D', this formula yields

$$\tan \eta_l = \frac{\tan \delta_l \{2D - R_l^2(D - D\cos 4\delta_l + \sin 4\delta_l) - R_l^2(1+\cos 4\delta_l + D\sin 4\delta_l)}{2D - R_l^2(D - D\cos 4\delta_l + \sin 4\delta_l) + R_l^2 \tan \delta_l(1+\cos 4\delta_l + D\sin 4\delta_l)} \quad \ldots (24)$$

Retaining only the relevant terms, we have

$$D \approx -\frac{R_l^2(\cot \eta_l + \delta_l)}{1 - \delta_l \cot \eta_l + 2R_l^2\delta_l \cot \eta_l} \quad \ldots \quad (25)$$

When $l = 0$ one obtains

$$-\frac{D}{\beta} = k \cot \eta_0 \Big(1 + \frac{4}{3}\beta^2 k^2 l_n \frac{\beta k}{4} - \frac{8}{3}\beta^2 k^2 \psi(3/2) + \frac{20}{9}\beta^2 k^2$$

$$- \frac{\pi \beta^2 k^2}{3}\cos \eta_0 + \frac{2}{3}\pi\beta^3 k^3 \cot \eta_0 + \frac{1}{9}\pi^2\beta^4 k^4 \cot^2 \eta_0 + \cdots\cdots \Big) \quad (26)$$

and

$$-\frac{D_0}{\beta} = \lim_{k \to 0} k \cot \eta_0 = -\frac{1}{A_0}, \quad \ldots \quad (27)$$

A_0 being the scattering length. Then from the equations (21), (26) and (27) we get the expansion of $k \cot \eta_0$ in the low energy limit

$$k \cot \eta_0 = -\frac{1}{A_0} + \frac{\pi\beta^2 k}{3A_0^2} + \frac{4\beta^2 k^2}{3A_0} l_n \frac{\beta k}{4} + \Big(\tfrac{1}{2} r_e - \frac{8\beta^2}{3A_0}\psi(3/2)$$

$$+ \frac{20\beta^2}{9A_0} - \frac{\pi\beta^3}{3A_0^2} - \frac{\pi^2\beta^4}{9A_0^3} + \frac{\pi\beta}{3} \Big) k^2 + \ldots \quad \ldots \quad (28)$$

This expansion is identical with that of O'Malley *et al* (1961). Finally in order to obtain an expression for $\tan \eta_l$ we retain only the leading terms involving k^{2l+1} in the series for R_l^2 as given by the equation (13) It is not difficult to show from the equation (24) that

$$\tan \eta_l \approx \tan \delta_l - \frac{(2l+1)^2 A_0(\beta k)^{2l+1}}{[(2l+1)!!]^4 \beta} \quad \ldots \quad (29)$$

which is again the same as that obtained by O'Malley *et al*(1961).

PHASE SHIFTS IN ELECTRON-ATOM COLLISION

We shall now deduce an expression for phase shifts for all angular moment a for low energy scattering of an electron by an atom. We shall take for the atomic potential $U(r)$ a screened coulomb potential of the Allis and Morse type joined smoothly with a long range r^{-4} potential at some distance r_0. The solution of the wave function with the Allis and Morse type potential is easily obtainable in terms of Whittaker's functions. Allis and Morse (1931) assumed that the incoming electron moved in a central attractive coulomb field of the nucleus and the average repulsive coulomb field due to the electrons of the atom. They obtained good results for low energy cross sections for elastic scattering of electrons by light atoms We have modified their potential with the intention of including the long range potential in the following manner :

$$U(r) = -2z\left(\frac{1}{r}-\frac{1}{a}\right), \quad r \leqslant r_0 \qquad \dots (30)$$

$$= -\frac{\beta^2}{r_4} \qquad r \geqslant r_0 \qquad \dots (31)$$

Z being the atomic number.

The potential $U(r)$ depends upon the two parameters a and r_0. The continuity condition at r_0 makes a dependent on r_0; there is thus arbitrariness of the single parameter r_0 The cut-off distance is so selected that the effect of screening due to the term $2z/a$ is maximum ;

That is, the selected value of r_0 is that value of r_0 for which a given by the equation

$$2z\left(\frac{1}{\ }-\frac{1}{\ }\right)=\frac{\beta^2}{\ } \text{ is minimum.}$$

Then we have $$, = \left(\frac{2\beta^2}{z}\right)^{\frac{1}{3}} \qquad \dots (32)$$

It will be seen later that the scattering length calculated from $U(r)$ defined in (30) and (31) with this value of r_0 is very nearly equal to the maximum scattering length obtainable by varying r_0.

We have completely ingored the effect of exchange of electrons, which is expected to play an important role in low energy scattering

The radial equations we have to solve are

$$\left[\frac{d^2}{dr^2} - K^2 + \frac{2\eta K}{r} - \frac{l(l+1)}{r^2}\right]u_l(r) = 0 \qquad r \leqslant r_0 \qquad \dots (33)$$

and $$\left[\frac{d^2}{dr^2} + k^2 + \frac{\beta^2}{r^4} - \frac{l(l+1)}{r^2}\right]u_l(r) = 0 \qquad r \geqslant r_0 \qquad \dots 34)$$

Here k is the wave number of the incident electron

$$K^2 = \frac{2Z}{a} - k^2 \quad \text{and} \quad \eta K = Z \quad \text{(Morse and Feshbach, 1953)}$$

To obtain phaseshift η_l we shall join at $r = r_0$ the solutions of the wave equations for the two regions. The regular solution of the equation (33) is well known (Morse and Feshbach 1953) :

$$u_l(r) = N(2Kr)^l r e^{-Kr} \; F(l+1-\eta;\; 2l+2,\; 2Kr) \qquad \ldots \quad (35)$$

where $F(l+1-\eta;\; 2l+2;\; 2Kr)$ is a confluent hypergeometric series and N is a constant.

Utilizing the properties of the confluent hyper-geometric series one readily obtains for the logarithmic derivative of the wave function (35) at $r = r_0$

$$\tan \Phi_l^k = \left(\frac{r}{U_l} \cdot \frac{d}{dr} u_l \right) r = 0 = \eta - 1 - Kr_0 + (l+1-\eta) \frac{F(l+2-\eta;\, 2l+2,\, 2Kr_0)}{F(l+1-\eta;\, 2l+2,\, 2Kr_0)} \qquad \ldots \quad (36)$$

The subscript l and the superscript k in $\Phi^k{}_l$ are used to indicate its dependence on l and k.

Now in the energy range considered by us, r_0 given by the equation (32) is less than $(\beta/k)^{\frac{1}{2}}$ so for the solution of the equation (34) in the region beyond r_0 we have to consider the ranges, $r_0 \leqslant r < (\beta/k)^{\frac{1}{2}}$ and $(\beta/k)^{\frac{1}{2}} \leqslant r < \infty$ separately.

When $r_0 \leqslant r < (\beta/k)^{\frac{1}{2}}$ the solution of the equation (34) is

$$u_l(r) = \frac{A\sqrt{r}}{1+\dfrac{\beta^2k^2}{4(\nu+1)}} \left[H_\nu{}^{(1)} \left(kr + \frac{\beta}{r} \right) - \frac{\beta^2k^2}{4(\nu+1)} \, H_{\nu+2}{}^{(1)} \left(kr + \frac{\beta}{r} \right) \right]$$

$$+ \frac{B\sqrt{r}}{1+\dfrac{\beta^2k^2}{4(\nu+1)}} \left[H_\nu{}^{(2)} \left(kr + \frac{\beta}{r} \right) - \frac{\beta^2k^2}{4(\nu+1)} \, H_{\nu+2}{}^{(2)} \left(kr + \frac{\beta}{r} \right) \right] \ldots \qquad (37)$$

We have made use of the formulae (8) and (9) to obtain the equation (37). From the continuity of the logarithmic derivative of the wave function at $r = r_0$ we have

$$\frac{A}{B} = - \frac{\left(1-2\tan\Phi^k{}_l\right)\left(H_\nu{}^{(2)} - \dfrac{\beta^2k^2}{4(\nu+1)} H_{\nu+2}{}^{(2)} \right)}{\left(1-2\tan\Phi^k{}_l\right)\left(H_\nu{}^{(1)} - \dfrac{\beta^2k^2}{4(\nu+1)} H_{\nu+2}{}^{(1)} \right)}$$

$$\frac{+\,2\left(kr_0 - \dfrac{\beta}{r_0} \right) \left(H_\nu{}^{(1)\prime} - \dfrac{\beta^2k^2}{4(\nu+1)} H_{\nu+2}{}^{(2)\prime} \right)}{+\,2\left(kr_0 - \dfrac{\beta}{r_0} \right) \left(H_\nu{}^{(1)\prime} - \dfrac{\beta^2k^2}{4(\nu+1)} H_{\nu+2}{}^{(1)\prime} \right)} \qquad (38)$$

The argument $kr_0+\beta/r_0$ of the Hankel functions involved in (38) are suppressed.

On using the formulae connecting the Hankel functions with the Bessel functions (Morse and Feshbach 1953) we obtain

$$\frac{A}{B} = e^{-2i\gamma^{kl}} \qquad (39)$$

where $\cot \gamma^k_l$

$$2\left(kr_0-\frac{\beta}{r_0}\right)\left\{J'_\nu(z)\cos\nu\pi-J'_{-\nu}(z)-\frac{\beta^2k^2}{4(\nu+1)}\left(J'_{\nu+2}(z)\cos\nu\pi-J'_{-\nu-2}(z)\right)\right\}$$

$$\sin\nu\pi\left(1-2\tan\Phi^k_l\right)\left(J_\nu(z)-\frac{\beta^2k^2}{4(\nu+1)}J_{\nu+2}(z)\right)$$

$$=\frac{-\left((1-2\tan\Phi^k_l)\right)\left\{J_\nu(z)\cos\nu\pi-J_{-\nu}(z)-\frac{\beta^2k^2}{4(\nu+1)}\left(J_{\nu+2}(z)\cos\nu\pi-J_{-\nu-2}(z)\right)\right\}}{+2\left(kr_0-\frac{\beta}{r_0}\right)\left\{J'_\nu(z)-\frac{\beta^2k^2}{4(\nu+1)}J'_{\nu+2}(z)\right\}}\quad\ldots\quad(40)$$

z standing for $kr_0+\beta/r_0$

For the scattering of a slow electron by a He atom the argument $kr_0+\beta/r_0$ is small enough to justify power series expansions of the Bessel functions and their derivatives in (40). If for the scattering by other atoms the argument is large, asymtotic expansions may be used (Watson, 1958)

Now from the equations (12) and (39) one gets

$$\frac{A'}{B'}=e^{-2i\xi l}\quad\ldots\quad(41)$$

where $\tan\xi_l=\dfrac{R_l^2\operatorname{cosec}\gamma_l^k\sin\nu\pi\sin(\nu\pi+\gamma_l^k)}{1-R_l^2\operatorname{cosec}\gamma_l^k\cos\nu\pi\sin(\nu\pi+\gamma_l^k)}$.

$$\text{when } l=0,\ \tan\xi_0=\beta k\left(1+\tfrac{4}{3}\beta^2k^2\ln\frac{\beta k}{4}-\tfrac{8}{9}\beta^2k^2\psi(3/2)+\tfrac{20}{9}\beta^2k^2\right)\cot\gamma_0^k$$
$$-\tfrac{2}{3}\pi\beta^3k^3\quad\ldots\quad(42)$$

where terms containing k^4 and higher powers of k are neglected. Again using the equations (11) and (41) we get an expression for the phaseshifts for different angular momenta

$$\eta_1=m_l\pi-\xi_l+\frac{\pi\beta^2k^2}{8(l+3/2)(l+\frac{1}{2})(l-\frac{1}{2})}\quad\ldots\quad(43)$$

where $m_l\pi$ is the zero energy phaseshift for the scattering of an electron by an atom. The value of m_l can be determined from Swan's conjecture about an extension of Levinson's theorem (P Swan 1955; K Levinson 1949).

We get from the equations (42) and (43)

$$\cot\eta_0=-\frac{1-\frac{1}{3}\pi\beta^3k^3}{\beta k\left(1+\frac{4}{3}\beta^2k^2\ln\frac{\beta k}{4}-\frac{8}{9}\beta^2k^2\psi(3/2)+\frac{20}{9}\beta^2k^2\right)\cot\gamma_0^k-\frac{2}{3}\pi\beta^3k^3}$$

$$\text{whence}\quad\lim_{k\to0}k\cot\eta_0=-\frac{1}{\beta\cot\gamma_0^0}$$

Therefore the scattering length A_0 is given by

$$A_0 = \beta \cot \gamma^0{}_0 \qquad \ldots \quad (44)$$

NUMERICAL CALCULATION FOR PHASE SHIFTS OF THE ELECTRON HELIUM SCATTERING AT LOW ENERGIES

Though we can calculate phaseshifts η_l from the equation (43) for different angular momenta l and for various light atoms, we shall rest satisfied with the calculation of s-wave phawe shift η_0 for e^-He scattering Taking $\alpha = 1.376$ (in atomic unit) for helium (Wickner and Das 1957), one gets from the equation (32) the cut-off distance $r_0 = 1.112$ (a.u.) and the corresponding scattering length A_0 is .844, a result rather low compared with the recent results. The maximum value for the scattering length for the atomic potential as defined in the equations (30) and (31) is 854 corresponding to the cut off distance $r_0 = 1.200$ As these values of A_0 and r_0 do not improve S-wave phaseshifts and as one has to obtain this value of r_0 (i e. $r_0 = 1.3$) by trial, we shall accept the value $r_0 = 1.112$ (easily obtainable from the equation (32)) for the calculation of the phase shifts.

In the table below the s-wave phase shifts in $\bar{e}$-He collisions for various incident energies obtained in this work are compared with the corresponding values of the same calculated by LaBhan and Callaway taking into account both polarization and exchange effects These authors have followed the method of polarized orbitals used by Temkin and Lamkin (1961) for a similar calculation on phase shifts in e-H scattering; the resulting integro differential equations have been solved numerically. Disagreement between our results and their sincreases with k. The values of $\cot \gamma_0{}^k$ are also shown in the table The potential used in our calculation is not very exact but its advantage is that it allows a fully analytic treatment.

TABLE

K (a.u.)	Energy e.v.	$\cot\gamma_0{}^k$	η_0 (present work)	η_0 (LaBhan and callaway)
0	0	0.720	(0.844)[a]	(1.132)[a]
0.01	0.00136	0.701	3.1341	3 13016
0.05	0.034	0 6279	3 1015	3 0822
0 10	0 136	0 5409	3.0661	3.0186
0.1917	0 50	0.3986	3 0230	2.8972
0.25	0.85	0 3187	3.0060	2.8189
0.2712	1.00	0.2942	3 0030	2.7904

a scattering length in a.u.

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