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DYNAMICS OF THE EXTENSIONAL VIBRATION OF A FREE-FREE BAR

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ABSTRACT. The paper discusses the Dynamics of Vibration of a Free-Free Bar excited by an inelastic longitudinal Impact, taking account of the Inertia of Lateral Motion The problem is worked out using the powerful Operational method Unlike other theories, this method is free from any assumption, and gives results of higher accuracy. The present problem gives the extension and pressure at the struck-end as functions of time, the other end ef the rod being free.

In Section I the nature of vibration and displacement of the struck-end is discussed and in Section II the pressure at the struck-end at different epoch is found out as functions of time.

INTRODUCTION

The expression for pressure and displacement at the struck-end of a thin rod hammered by elastic load with different end conditions has been worked out by Ghosh (1951). But the present paper proposes to consider all the above phenomena taking account of inertia of lateral motion in the case of a thin rod hammered by an inelastic load at one end the other end remaining free. The equation of motion in such a case is given by,

$$\rho\left(\frac{d^2\omega}{dt^2} - \sigma^2k^2\frac{d^4\omega}{ds^2dt^2}\right) = E\left(\frac{d^2\omega}{ds^2}\right) \quad .. \quad (1)$$

The second term of the L.H.S. of (1) i.e. $\rho\sigma^2k^2\frac{d^4\omega}{ds^2dt^2}$ is due to the Inertia of lateral motion and is the most general equation of vibration of a thin rod. An important contribution due to the second term is that it gives the velocity of wave propagation in the rod with higher accuracy.

Explanation of the symbols used :

E = Modulus of elasticity of the bar.

σ = Poisson's ratio.

γ = Area of cross-section of the bar.

k = Radius of Gyration of a cross-section of the rod about its central line.

ω = Displacement of any section at any time.

t = Variable time.

l = Length of the rod used.

s = Distance of the particle on the central line from the free-end.
ρ = Volume density of the rod.
ρ_0 = Linear density of the rod.
ω_l = Displacement of a particle at the struck-end.
m = Mass of the load.
c = Velocity of longitudinal wave along the rod.
$t_n = t - n\theta$, where $n = 1, 2, 3$, etc.
θ = Period of free vibration of the bar. $= 2l/c$
v_0 = velocity of impact.
J = Momentum of impact $= mv_0$
P = Pressure exerted by the load.
D = Operator $\dfrac{d}{dt}$, $\eta = \dfrac{\sigma^2 k^2}{c^2}$

The differential equation (1) for the extensional vibration of the rod is solved by using operational method of Heaviside instead of using St. Venant's 'Variational method' which is long and laborious.

Now Equation (1) in the operational notations,

$$\frac{d^2\omega}{ds^2} = \frac{D^2(1+\eta D^2)^{-1}}{c^2} \cdot \omega \qquad \ldots \quad (1\cdot1)$$

The solution of this equation is given by.

$$\omega = A \cosh \frac{D(1+\eta D^2)^{-\frac{1}{2}}}{c} s + B \sinh \frac{D(1+\eta D^2)^{-\frac{1}{2}}}{c} s \qquad \ldots \quad (2)$$

The end-conditions are at $s = 0$,

$$\frac{d\omega}{ds} = 0 \qquad \ldots \quad (3\cdot1)$$

and

at $\qquad s = l, \quad \omega = \omega_l \qquad \ldots \quad (3\cdot2)$

From (2), (3·1) and (3·2),

$$\omega = \omega_l \frac{\cosh \dfrac{D(1+\eta D^2)^{-\frac{1}{2}}}{c} \cdot s}{\cosh \dfrac{D(1+\eta D^2)^{-\frac{1}{2}}}{c} \cdot l} \qquad \ldots \quad (4)$$

The equation of motion for the striking body is,

$$m \frac{d^2\omega_l}{dt^2} = -\gamma E \left(\frac{d\omega}{ds} \right)_{s=l} \qquad \ldots \quad (5)$$

Now substituting the values of $(d\omega/ds)_{s=l}$ in (5) and imposing the boundary condition, the motion being started by impulse J, we get

$$mD^2 + \frac{E\gamma}{c} D(1+\eta D^2)^{-\frac{1}{2}} \tanh \frac{D(1+\eta D^2)^{-\frac{1}{2}}}{c} \cdot l = DJ \qquad \ldots \quad (6)$$

The Pressure exerted by the load is,

$$P = m\frac{d^2\omega_l}{dt^2} \qquad \dots (7)$$

From (4), (5) and (7) the expression for P is,

$$P = -\frac{E\gamma}{c}\,\omega_l D(1+\eta D^2)^{-\frac{1}{2}}\tanh\frac{D(1+\eta D^2)^{-\frac{1}{2}}}{c}\,.\,l \qquad \dots (8)$$

$$= -\frac{E\gamma}{c}\,(1+\eta D^2)^{-\frac{1}{2}}\tanh\frac{D(1+\eta D^2)^{-\frac{1}{2}}}{c}\,.\,l\,.\,\omega_l \qquad \dots (9)$$

Now putting mv_0 for J in (6) it is found that,

$$\omega_l = \frac{1}{F(D)}\,.\,v_0 \qquad \dots (10)$$

where,

$$F(D) = D+\frac{E\gamma}{mc}\,(1+\eta D^2)^{-\frac{1}{2}}\tanh\frac{D(1+\eta D^2)^{-\frac{1}{2}}}{c}\,l \qquad \dots (11)$$

On substituting the exponential values for hyperbolic tangents in equation (11), neglecting terms containing η^2 (η being very small) in the binomial expansion of $(1+\eta D^2)^{-\frac{1}{2}}$ and writing $D_1 = D+\alpha$, $D_2 = D+\beta$ we have the final form $F(D)$ to be,

$$F(D) = \frac{D_1D_2}{(\alpha+\beta)[1+\exp\{-D(1-\frac{1}{2}\eta D^2)\theta\}]}$$

$$\times\left[1-\frac{(D-\alpha)(D-\beta)}{D_1D_2}\exp\left\{-D\left(1-\frac{1}{2}\eta D^2\right)\theta\right\}\right] \qquad \dots (12)$$

where, $$D_1D_2 \equiv (D+\alpha)(D+\beta) \equiv D^2-\frac{2mc}{\gamma E\eta}D-\frac{2}{\eta} \qquad \dots (13)$$

and $-\alpha$, $-\beta$ are the roots of, $$D^2-\frac{2mc}{\gamma E\eta}D-\frac{2}{\eta} = 0 \qquad \dots (14)$$

given by,

$$[\alpha,\beta] = -\frac{1}{2}\left[\frac{2mc}{\gamma E\eta}\mp\left(\frac{4m^2c^2}{\gamma^2E^2\eta^2}+\frac{8}{\eta}\right)^{\frac{1}{2}}\right] \qquad \dots (15)$$

Expanding terms under the radical sign binomially and neglecting higher powers of η other than the first we have from (15),

$$[\alpha,\beta] = \frac{E\gamma}{mc},\ -\frac{mc}{\gamma E\eta}\left(2+\frac{\gamma^2E^2\eta}{m^2c^2}\right) \qquad \dots (16)$$

$$-\frac{\rho_0 c}{m},\ -\frac{mc}{\rho_0\sigma^2k^2}\left(2+\frac{\sigma^2k^2\rho_0^2}{m^2}\right) \qquad \ldots \quad (16.a)$$

where,

$$E=\rho c^2,\quad \rho_0=\rho\gamma,\quad \eta=\frac{\sigma^2k^2}{c^2}.$$

DISPLACEMENT AT THE IMPACT-END

The displacement at the impact-end can now be obtained by the help of equations (10) and (12) as follows :

$$w_i=\frac{(\alpha+\beta)[1+\exp\{-D(1-\frac{1}{2}\eta D^2)\theta\}]}{D_1D_2}$$

$$\times\left[1-\frac{(D-\alpha)(D-\beta)}{D_1D_2}\exp\left\{-D\left(1-\frac{1}{2}\eta D^2\right)\theta\right\}\right]^{-1}\cdot v_0$$

$$=\left[\frac{(\alpha+\beta)}{D_1D_2}-\left\{\frac{2(\alpha+\beta)^2D}{D_1^2D_2^2}-\frac{2(\alpha+\beta)}{D_1D_2}\right\}\exp\left\{-D\left(1-\frac{1}{2}\eta D^2\right)\theta\right\}\right]$$

$$+\left\{\frac{4(\alpha+\beta)^3D^2}{D_1^3D_2^3}-\frac{6(\alpha+\beta)^2D}{D_1^2D_2^2}+\frac{2(\alpha+\beta)}{D_1D_2}\right\}\exp\left\{-2D\left(1-\frac{1}{2}\eta D^2\right)\theta\right\}$$

$$-\left\{\frac{8(\alpha+\beta)^4D^3}{D_1^4D_2^4}-\frac{16(\alpha+\beta)^3D^2}{D_1^3D_2^3}+\frac{10(\alpha+\beta)^2D}{D_1^2D_2^2}-\frac{2(\alpha+\beta)}{D_1D_2}\right\}$$

$$+\ldots\ldots\ldots\ldots\qquad \exp\left\{-3D\left(1-\frac{1}{2}\eta D^2\right)\theta\right\}$$

$$+(-1)^{n-1}\left\{\frac{2^n(\alpha+\beta)^{n+1}D^n}{D_1^{n+1}D_2^{n+1}}-\ldots-+(1)^n\frac{2(\alpha+\beta)}{D_1D_2}\right\}$$

$$\exp\left\{-nD\left(1-\frac{1}{2}\eta D^2\right)\theta\right\}+\ldots\Big]v_0 \qquad \ldots \quad (17)$$

$$\eta \text{ being }=\frac{\sigma^2k^2}{c^2}$$

Now writing,

$$f_1(t)=\frac{(\alpha+\beta)}{D_1D_2}\,v_0 \qquad \ldots \quad (17.1)$$

$$f_2(t)=\frac{(\alpha+\beta)^2D}{D_1^2D_2^2}\cdot v_0 \qquad \ldots \quad (17.2)$$

$$f_3(t)=\frac{(\alpha+\beta)^3D^2}{D_1^3D_2^3}\,v_0 \qquad \ldots \quad (17.3)$$

and so on,

$$f_n(t) = \frac{(\alpha+\beta)^n D^{n-1}}{D_1^n D_2^n} v_0 \qquad (17)$$

we get,

$$\begin{aligned}\omega_l = [f_1(t) &- \{2f_2(t)-2f_1(t)\} \exp\{-D(1-\tfrac{1}{2}\eta D^2)\theta\} \\ &+ \{4f_3(t)-6f_2(t)+2f_1(t)\} \exp\{-2D(1-\tfrac{1}{2}\eta D^2)\theta\} \\ &- \{8f_4(t)-16f_3(t)+10f_2(t)-2f_1(t)\} \exp\{-3D(1-\tfrac{1}{2}\eta D^2)\theta\}+\ldots \\ &+ \{2f_1(t)-(4n-2)f_2(t)+(2n-2)^2 f_3(t)+\ldots+(-1)^n 2^n f_{n+1}(t)\} \\ &\quad \exp\{-nD(1-\tfrac{1}{2}\eta D^2)\theta\}+\ldots] \qquad \ldots \quad (18)\end{aligned}$$

Now since,

$$\begin{aligned}\exp\{-nD(1-\tfrac{1}{2}\eta D^2)\theta\} f_n(t) &= f_n(t-n\theta)+\tfrac{1}{2} n\theta\eta f_n'''(t-n\theta) \\ &= f_n(t_n)+\tfrac{1}{2} n\theta\eta f_n'''(t_n) \qquad \ldots \quad (18.1)\end{aligned}$$

Therefore,

$$\begin{aligned}\omega_l = [f_1(t) &- \{2f_2(t_1)-2f_1(t_1)\}+\{4f_3(t_2)-f_2(t_2)+2f_1(t_2)\} \\ &- \{8f_4(t_3)-16f_3(t_3)+10f_2(t_3)-2f_1(t_3)\}+\ldots \\ &+ \{2f_1(t_n)-(4n-2)f_2(t_n)+(2n-2)f_3(t_n)+\ldots+(-1)^n 2^n f_{n+1}(t_n)\}+\ldots \\ &- \tfrac{1}{2}\theta\eta\{2f'''_2(t_1)-2f'''_1(t_1)\}+\tfrac{2}{2}\theta\eta\{4f_3'''(t_2)-6f_2'''(t_2)+2f_1'''(t_2)\} \\ &- \tfrac{3}{2}\theta\eta\{8f_4'''(t_3)-16f_3'''(t_3)+10f_2'''(t_3)-2f_1'''(t_3)\}+\ldots \\ &+ \tfrac{1}{2} n\theta\eta\{2f_1'''(t_n)-(4n-2)f_2'''(t_n)+(2n-2)^2 f_3'''(t_n)+\ldots \\ &\quad +(-1)^n 2^n f_{n+1}'''(t_n)\}+\ldots] \qquad \ldots \quad (19)\end{aligned}$$

Now the functions $f_1(t)$, $f_2(t)$ etc. can be obtained as follows.

$$f_1(t) = v_0 A\left[\frac{1}{\alpha}(1-e^{-\alpha t})-\frac{1}{\beta}(1-e^{-\beta t})\right] \qquad \ldots \quad (19.1)$$

$$f_2(t) = v_0 A^2\left[\frac{1}{\alpha}(1-A+\alpha t)e^{-\alpha t}+\frac{1}{\beta}(1+A+\beta t)e^{-\beta t}\right] \qquad \ldots \quad (19.2)$$

$$f_3(t) = v_0A^3\Big[\frac{1}{\alpha}\Big\{\frac{3}{2}(A-A^2)+\frac{1}{2}(3A-1)\alpha t-\frac{\alpha^2t^2}{2!}\Big\}e^{-\alpha t}$$

$$+\frac{1}{\beta}\Big\{\frac{3}{2}(A+A^2)+\frac{1}{2}(3A+1)\beta t+\frac{\beta^2t^2}{2!}\Big\}e^{-\beta t}\Big] \quad \ldots \quad (19.3)$$

etc.,

$$f_n(t) = v_0A^n\Big[\sum_{r=1}^{n}(-1)^{r-1}\frac{\Gamma(n+r-1)}{\Gamma(n)\,\Gamma(r)}B^{r-1}e^{-\alpha t}(D-\alpha)^{n-2}\frac{t^{n-r}}{(n-r)!}$$

$$+(-1)^n\sum_{r=1}^{n}\frac{\Gamma(n+r-1)}{\Gamma(n)\,\Gamma(r)}B^{r-1}e^{-\beta t}\,(D-\beta)^{n-2}\frac{t^{n-r}}{(n-r)!} \quad \ldots \quad (19.4)$$

and so on, where, $A = (\beta+\alpha)/(\beta-\alpha)$, $B = 1/(\beta-\alpha)$

If we now neglect the term containing the inertia of lateral motion in equation (1) i.e., if, $\eta = 0$ we must have $A = 1$, and $B = 0$.

and,

$$f_1(t) = \frac{v_0}{\alpha}(1-e^{-\alpha t}) = \frac{mv_0}{\rho_0 c}\Big(1-e^{-\frac{\rho_0 c}{m}t}\Big) \quad \ldots \quad (19.1a)$$

$$f_2(t) = \frac{v_0}{\alpha}\,.\,\alpha t\,.\,e^{-\alpha t} = \frac{mv_0}{\rho_0 c}\,.\,\frac{\rho_0 c}{m}\,t\,e^{-\frac{\rho_0 c}{m}t} \quad \ldots \quad (19.2a)$$

$$f_3(t) = \frac{v_0}{\alpha}e^{-\alpha t}\Big\{\alpha t-\frac{\alpha^2t^2}{2!}\Big\} = \frac{mv_0}{\rho_0 c}\Big\{\frac{\rho_0 c}{m}t-\frac{\rho_0^2c^2}{2m^2}t^2\Big\}e^{-\frac{\rho_0 c}{m}t} \quad \ldots \quad (19.3a)$$

and so on. These results of $f_1(t)$, $f_2(t)$ etc. are found similar to those obtained by Ghosh (1953).

Thus the displacement at the impact-end at any interval of time can be found as a function of time substituting all the values of $f_1(t)$, $f_2(t)$, etc. in the above equation (19).

During the interval, $0 < t < \theta$,

$$\omega_l = f_1(t) \quad \ldots \quad (19.A)$$

After time $t = \theta$, i.e. during, $\theta < t < 2\theta$,

$$\omega_l = f_1(t)-2\{f_2(t_1)-f_1(t_1)\}+\frac{\theta}{2}\eta\{2f_1'''(t_1)-2f_2'''(t_1)\} \quad \ldots \quad (19.B)$$

Similarly during, $2\theta < t < 3\theta$,

$$\omega_t = f_1(t)+\{2f_1(t_1)-2f_2(t_1)\}+\{2f_1(t_2)-6f_2(t_2)+4f_3(t_2)\}$$
$$+\tfrac{1}{2}\,\theta\eta\{2f_1'''(t_1)-2f_2'''(t_1)\}$$
$$+\frac{2}{2}\,\theta\eta\{2f_3'''(t_2)-6f_2'''(t_2)+4f_3'''(t_2)\} \qquad \dots \quad (19\cdot C)$$

and so on.

Equation (19) gives the most general form of displacement at the struck end of the rod. It is found that terms of the right hand side of (19) contain certain number of terms to be positive and some of them to be negative. By a negative term it is understood that waves formed are reflected from the respective ends of the rod. The displacement equation (19) of the struck-end is obtained in the functional form. By putting the functional values of $f_1(t)$ $f_2(t)$ etc. in (19) the displacement is obtained in terms of known quantities. The pressure at the struck-end is duscussed in Section II.

Further, the displacement equation shows, that the wave train does not return after reflection, as shown by the second term of equation (21) below .

Pressure at the struck end

The Pressure exerted by the load at the impact end can now be found out taking the help of equation (9) and equation (19) as follows :

$$P = -\frac{E\gamma}{C}\,(1+\eta D^2)^{-1/2}\tanh\frac{D(1+\eta D^2)^{-\frac{1}{2}}}{C}\,.\,l\quad \omega_t'$$

Expanding $(1+\eta D^2)^{-1}$ binomially as before, neglecting higher powers of η and writing exponential values for hyperbolic tangent the pressure equation becomes,

$$P = -\frac{E\gamma}{C}(1-\tfrac{1}{2}\eta D^2)[1-2e^{-D(1-\frac{1}{2}\eta D^2)\theta}+2e^{-2D(1-\frac{1}{2}\eta D^2)\theta}$$
$$-2e^{-3D(1-\frac{1}{2}\eta D^2)\theta}+2e^{-4D(1-\frac{1}{2}\eta D^2)\theta}$$
$$-\ \dots\ \dots+\ \dots\ \dots\ ]\ \omega'_t \qquad \dots \quad (20)$$

From Equations (19) and (20) the numerical value of pressure at the struck-end can be written as,

$$P = \frac{\gamma E}{C}\,[f'_1(t)-2f'_2(t_1)-\tfrac{1}{2}\,\eta\{f_1'''(t)-2f_2'''(t_1)\}$$
$$+\{4f'_3(t_2)-2f'_2(t_2)\}-\tfrac{1}{2}\eta\{4f'''_3(t_2)-2f'''_2(t_2)\}$$
$$+\eta\theta\{4f_3^{IV}(t_2)-2f_2^{IV}(t_2)-f_2^{IV}(t_1)\}+\ \dots\ ] \qquad \dots \quad (21)$$

Thus during the interval, $0 < t < \theta$ the pressure is

$$P_1 = \frac{E\gamma}{C}\left[\,f_1'(t)-\frac{1}{2}\,\eta f_1'''(t)\,\right] \qquad \dots \quad (21.1)$$