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Reduction of wavefunction which transforms as complex antisymmetric tensor to irreducible representation of Lorentz group (zero mass system)

By B. S. Rajput

Department of Physics, Kurukshetra University

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The reduction of the wavefunction which transforms as a complex antisymmetric tensor to the irreducible representation of proper, orthochronous, inhomogeneous Lorentz group for zero mass system has been discussed by giving the proofs of the essential theorems. The change of gauge is discussed as the unphysical change in the wavefunction under the pure Lorentz transformation. The effects of reality condition, wave equation and the canonical formalism as well as the second quantizations also have been discussed. By assuming the total energy to be positive the constants of the expansion of wavefunction which satisfies wave-equation, have been calculated to give the energies for four modes corresponding to positive and negative values of Hamiltonian density.

Introduction

The general ways of reduction of any unitary ray representation of the proper, orthochronous, inhomogeneous Lorentz group have been discussed by Lomont & Moses (1967) for both non-zero and zero mass systems where for the former one obtains the Foldy (1956)–Shirokov (1958, 1959) relations and for the latter one is led to the Lomont-Mose (1964) realization. These results were used by Moses (1967) to reduce the wavefunction which transforms as an anisymmetric (real) tensors for non-zero mass system to the irreducible representation of the proper orthochronous, inhomogeneous Lorentz group. We (Rajput 1969) extended this reduction for of the wavefunction which transforms as complex–antisymmetric tensor. Moses (1968) discussed the reduction of the wavefunction which transforms as real antisymmetric tensor for zero mass system. In the present paper we discuss the reduction of wavefunction which transforms as a complex anti-symmetric tensor for zero mass system to the irreducible representation of the inhomogeneous Lorentz group by giving the proofs of essential theorems which are used in this case. Here we calculate the change of gauge as the inessential change in the wavefunction due to the operations of three generators (corresponding to space time relations) of proper, orthochronous, inhomogeneous Lorentz group. We have also discussed the effect of reality condition and wave equation on the wavefunction and the second quantization in connection to canonical formalism. It is noted in these calculations that to reduce the wavefunction, only the transformation properties are necessary while the requirements of the wave equation and reality condition restrict the number of independent irreducible representations. The results of the present paper can be used to reduce the electromagnetic field wavefunction to the irreducible representations.

Transformation of the wavefunction

The components of a complex anti-symmetric tensor is given by

$$F^{ij} = F_R^{\ ij} + F_I^{\ ij}$$
$$F^{ij} = -F^{ji} \qquad \text{..(1)}$$

Where R denotes real part and I the imaginary part. To give the wavefunction field description of this tensor we define

$$\left.\begin{aligned} E_{iR} &= F_R^{\ 0i},\\ E_{iI} &= F_I^{\ 0i}\\ H_{iR} &= \epsilon_{ijk}\, F_R^{\ jk}\\ H_{iI} &= \epsilon_{ijk}\, F_I^{\ jk} \end{aligned}\right\} \qquad \text{...(2)}$$

Then the wave function ψ, which transforms as a complex anti-symmetric tensor, is the six components column vector given by

$$\psi = \begin{bmatrix} \psi_R \\ \psi_I \end{bmatrix} \qquad \text{...(3)}$$

where ψ_R and ψ_I are three components column vectors given by

$$\begin{aligned} \psi_R(\vec{x}, t, r) &= E_{rR}(\vec{x}, t) - i H_{rR}(\vec{x}, t)\\ \psi_I(\vec{x}, t, r) &= E_{rI}(\vec{x}, t) - i H_{rI}(\vec{x}, t), \; r = 1, 2, 3. \end{aligned} \qquad (4)$$

In terms of infinitesimal generators of proper, orthochronous, inhomogenous Lorentz group the wave function ψ transforms as (Rajput 1969)

$$\left.\begin{aligned} \psi'(x) &= \exp\,[\, i\sum_j a^j P_j\,]\, \psi(x)\\ \psi'(x) &= \exp\,[\, i\vec{\theta}.\vec{K}\,]\, \psi(x)\\ \psi'(x) &= \exp\,[\, i\vec{\beta}.\vec{Z}\,]\, \psi(x) \end{aligned}\right\} \qquad \text{...(5)}$$

where

$$\left.\begin{aligned} P_j &= P^j = -i\nabla_j I \;(j = 1, 2, 3)\\ P^0 &= -P_0 = H = -i\frac{\partial}{\partial t} I,\; i = (-1)^{1/2}\\ K_j &= M_j - i(\vec{x} \times \nabla) I,\\ Z_j &= N_j + i\left(x_j \frac{\partial}{\partial t} + t\,\nabla_j\right) I \end{aligned}\right\} \qquad (6)$$

For tensor field we have

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$N_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus M_i are Hermitian matrices while N_i are not so. The mass-less representation of the infinitesimal generators of the inhomogeneous Lorentz group is given in terms of the representation of the infinitesimal generators of the two dimensional Euclidean group. For this we consider the three operators T_1, T_2, J which satisfy the commutation rules for the infinitesimal generators of the two dimensional Euclidean group given by

$$[T_1, \; T_2] = 0,$$
$$[T_1, \; J\,] = -iT_2,$$
$$[T_2, \; J\,] = iT_1$$

The matrices or kernels : $T_1(\lambda \mid \lambda')$, $T_2(\lambda \mid \lambda')$ and $J(\lambda \mid \lambda')$ defined as :

$$T_1 F(\lambda) = \sum_{\lambda'} T_1(\lambda \mid \lambda') F(\lambda')$$

$$T_2 F(\lambda) = \sum_{\lambda'} T_2(\lambda \mid \lambda') F(\lambda')$$

$$J F(\lambda) = \sum_{\lambda'} J(\lambda \mid \lambda') F(\lambda')$$

Constitute a representation of the infinitesimal generator of two dimensional Euclidean group. $F(\lambda)$ here is the function of the real variable λ which can be continuous, discrete or finite dimensional and represents the eigen values of the matrix J. The operators T_1, T_2 and J are given in terms of M_i and N_i as follows

$$\left.\begin{aligned} T_1 &= -M_2 - \epsilon N_1 \\ T_2 &= M_1 - \epsilon N_2 \\ J &= M_3 \end{aligned}\right\} \qquad \ldots(6)$$

Then the realization of the infinitesimal generators of inhomogeneous Lorentz group given by Lomont & Moses (1967) can be taken as

$$\hat{P}_0 f(\vec{p}) = H f(\vec{p}) = \epsilon p f(\vec{p})$$

$$\hat{P}_i f(\vec{p}) = p_i f(\vec{p})$$

$$\hat{K}_1 f(\vec{p}) = \left[L_1 + \frac{p_1 J}{p + p_3} \right] f(p)$$

$$\hat{K}_2 f(\vec{p}) = \left[L_2 + \frac{p_2 J}{p + p_3} \right] f(p).$$

$$\hat{K}_3 f(\vec{p}) = [L_3 + J] f(p)$$

$$\hat{Z}_1 f(\vec{p}) = \epsilon \left\{ i p \frac{\partial}{\partial p_1} + \frac{p_2}{p + p_3} J + \left[\frac{p_1^2}{p^2(p + p_3)} - \frac{1}{p} \right] T_1 + \frac{p_1 p_2}{p^2 (p + p_3)} T_2 \right\} f(\vec{p}).$$

$$\hat{Z}_2 f(\vec{p}) = \epsilon \left\{ i p \frac{\partial}{\partial p_2} - \frac{p_1}{p + p_3} J + \frac{p_1 p_2}{p^2 (p + p_3)} T_1 + \left[\frac{p_2^2}{p^2 (p + p_3)} - \frac{1}{p} \right] T_2 \right\} f(\vec{p})$$

$$\hat{Z}_3 f(\vec{p}) = \epsilon \left\{ i p \frac{\partial}{\partial p_3} + \frac{1}{p^2} [p_1 T_1 + p_2 T_2] \right\} f(\vec{p}) \quad (7)$$

Where L_1, L_2 and L_3 are the components of orbital angular momentum given by

$$L_i f(\vec{p}) = - i \sum_{ij} \epsilon_{ijk} p_j \frac{\partial}{\partial p_k} f(\vec{p}) \quad ...(8)$$

and $\epsilon = \pm 1.$

Here $\hat{P}^a$, $\hat{K}$, and $\hat{Z}$ are the infinitesimal generators of the unitary ray reprerentation of Lorentz group. They satisfy the same comunation rules as those of the generators P^a, K and Z. Hence the required reduction requires the expressi n of the wavefunction $\psi(x)$ in terms of $f(\vec{p})$.

Reduction of the wavefunction

In equations (6) the matrix M_3 is Hermitian and hence diagonalised by a untary matrix U :

$$U^{-1} M_3 U = d$$

where d is diagonal matrix. Then λ, the eigenvalues of M_3 are given by :

$$| M_3 - \lambda I | = 0$$

which gives

$$\lambda = 1,\ 0,\ -1.$$

Hence,
$$d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad ...(10)$$

By equations (9) and (10) we get

$$U = \begin{bmatrix} -(2)^{-1/2} & 0 & (2)^{-1/2} \\ -i(2)^{-1/2} & 0 & -i(2)^{-1/2} \\ 0 & 1 & 0 \end{bmatrix} \qquad ...(11)$$

Let us define the column vector $\chi(\epsilon, p, \lambda)$ as

$$\chi(r \mid \epsilon, p, \lambda) = [\exp(i\vec{\omega}.M)\exp(i\nu N_3)U]_{r\lambda} \qquad (12)$$

where $\vec{p}$ is in one to one correspondence with $\vec{\omega}$ and scalar ν through the expressions

$$p = e^{\epsilon\nu},$$

$$p_1 = -p\left(\frac{\sin\omega}{\omega}\right)\omega_2$$

$$p_2 = p\left(\frac{\sin\omega}{\omega}\right)\omega_1 \qquad ...(13)$$

$$p_3 = \cos\omega, \quad \omega_3 = 0, \quad \omega = |\vec{\omega}|$$

Using the values of $\exp(i\vec{\omega}.M)$ and $\exp(i\nu N_3)$ from our previous paper (Rajput 1969) the vector $\chi(\epsilon, p, \lambda)$ is written as follows

$$\chi(\epsilon, p, 0) = \begin{bmatrix} p_1 | p \\ p_2 | p \\ p_3 | p \end{bmatrix} \text{ for } \lambda = 0 \qquad ..(14)$$

$$\chi(\epsilon, p, \lambda) = [\lambda p^{-\epsilon\lambda}/(2)^{1/2}]\,\sigma(p, \lambda)$$

$$= [\lambda p^{-\epsilon\lambda} | (2)^{1/2}] \begin{bmatrix} \dfrac{p_1(p_1+i\lambda p_2)}{p(p+p_3)} - 1 \\ \dfrac{p_2(p_1+i\lambda p_2)}{p(p+p_3)} - i\lambda \\ \dfrac{p_1+i\lambda p_2}{p} \end{bmatrix} \text{ for } \lambda = \pm 1 \qquad ...(15)$$

Now we consider the function $f(\xi)$ as the representation of the vector ψ in the basis which is characterised by the space of wavefunctions in Hilbert space upon which the generators of Lorentz group operate. ξ

collectively denotes all the variables upon which the function in the given representation depends. Then as the result of Lomont & Moses (1967) we have

$$f(\xi)=\sum_{\epsilon}\sum_{\lambda}\int d\mu\int\frac{dp}{\omega(\mu,p)}\times<\xi\mid\mu,\epsilon,p,\lambda>f(\mu,\epsilon,p,\lambda)\quad\ldots(16)$$

In our case $\mu=0,\ \lambda=0$ or ±1 and hence equation (16) reduces to :

$$f(\xi)=\sum_{\epsilon=\pm1}\sum_{\lambda=0,\pm1}\int\frac{dp}{p}\times<\xi\mid 0,\epsilon,p,\lambda>f(\epsilon,p,\lambda)\quad\ldots(17)$$

with the assumption that all the generators of the inhomogeneous Lorentz group are Hermitian. The value of the transformation function $<\xi\mid 0,\epsilon,p,\lambda)$ for the present case is given by

$$<\xi\mid 0,\epsilon,p,\lambda>=<x,t,r,\mid\epsilon,p,\lambda>$$

$$=\{\exp[i\,\vec{\omega}.\,K\,)].\ \exp.\ [i\,\nu\,Z_3]\,U\}\,f(x,t,r;\epsilon,\lambda)$$

$$=\{\exp[i\,\vec{\omega}.\,M].\ \exp[i\,\nu\,N_3]\,U\}\,\exp[\vec{\omega}.\,(\vec{x}\times\nabla\,]\,I\,]$$

$$\times\exp\left[[i\,\nu\left(x_3\frac{\partial}{\partial t}+t\frac{\partial}{\partial x_3}\right)I\right]f(x,t,r;\epsilon,\lambda)\quad\ldots(18)$$

The function $f(x,t,r;\epsilon,\lambda)$ satisfies the following equation

$$P_1f(x,t,r;\epsilon,\lambda)=P_2f(x,t,r;\epsilon,\lambda)=0$$

$$P_3f(x,t,r;\epsilon,\lambda)=f(x,t,r;\epsilon,\lambda)\quad\ldots(19)$$

$$P^0f(x,t,r;\epsilon,\lambda)=Hf(x,t,r;\epsilon,\lambda)=\epsilon f(x,t,r;\epsilon,\lambda).$$

Using equations (5) in (19) we get

$$\frac{\partial}{\partial x_1}f(x,t,r;\epsilon,\lambda)=\frac{\partial}{\partial x_2}f(x,t,r;\epsilon,\lambda)=0$$

$$\frac{\partial}{\partial x_3}f(x,t,r;\epsilon,\lambda)=i\,f(x,t,r;\epsilon,\lambda)$$

$$\frac{\partial}{\partial t}f(x,t,r;\epsilon,\lambda)=-\,i\,\epsilon f(x,t,r;\epsilon,\lambda)$$

the general solution of which is given by

$$f(x,t,r;\epsilon,\lambda)=\exp[\,i\,(x_3-\epsilon t)\,]\,C(\epsilon,\lambda)\quad\ldots(20)$$

where $C(\epsilon,\lambda)$ is the constant of integration. Using equation (20), (5) and (12) in equation (18) we get

$$<x, t, r ; \epsilon, \lambda> = \text{Exp}\,[\,i\,(x_3 - \epsilon t)\,]\,\chi\,(r \mid \epsilon, p, \lambda\,).$$

$$\psi(x) = \sum_{\epsilon}\sum_{\lambda}\Big[C(\epsilon,\lambda)\int\frac{d\vec{p}}{p}\chi(\epsilon,p,\lambda)\,f(\epsilon,p,\lambda)\exp\{\,i\,(\vec{p}.\vec{x} - \epsilon pt)\,\}$$

$$+C'(\epsilon,\lambda)\int\frac{d\vec{p}}{p}\chi'(\epsilon,p,\lambda)\,f'(\epsilon,p,\lambda)\quad\exp\{\,i\,(\vec{p}.\vec{x} - \epsilon pt\,\}\Big]$$

$$= \sum_{\epsilon} C(\epsilon,0)\int\frac{d\vec{p}}{p}\chi(\epsilon,p,0)\,f(\epsilon,p,0)\quad\exp\{\,i(\vec{p}.\vec{x} - \epsilon pt)\,\}$$

$$+C'(\epsilon,0)\int\frac{d\vec{p}}{p}\chi'(\epsilon,p,0)\,f'(\epsilon,p,0)\quad\exp\{\,i\,(\vec{p}.\vec{x} - \epsilon pt\,)\}$$

$$+\sum_{\lambda=\pm1}\frac{\lambda}{(2)^{1/2}}\Big\{C(\epsilon,\lambda)\int\frac{d\vec{p}}{p}p^{-\epsilon\lambda}\sigma(p,\lambda)\,f(\epsilon,p,\lambda)\exp\{i(\vec{p}.\vec{x} - \epsilon pt)\Big\}$$

$$+ C'(\epsilon,\lambda)\int\frac{d\vec{p}}{p}p^{-\epsilon\lambda}\sigma'(p,\lambda)\,f'(\epsilon,p,\lambda)\exp\{\,i\,(\vec{p}.\vec{x} - \epsilon pt\,)\} \qquad ...(21)$$

where unprimed functions represent ψ_R and primed functions repesent ω_T. No. let,

$$h(\epsilon,p,\lambda) = \exp(-2i\lambda\phi)\,f^*(-\epsilon,-p,\lambda)$$

where, $\tan\phi = p_2 \mid p_1$

$$C(\lambda) = C(+1,\lambda),\ D(\lambda) = C(-1,\lambda)$$

$$f(p,\lambda) = f(+1,p,\lambda).$$

then

$$\psi(x) = \int\frac{d\vec{p}}{p}\chi(p,0)[C(0)\,f(\vec{p},0)\exp\{i\,(\vec{p}\,\vec{x} - pt)\}$$

$$-D(0)h^*(\vec{p},0)\exp\{-i(\vec{p}.\vec{x} - pt)\}$$

$$+\int\frac{d\vec{p}}{p}\chi'(p,0)[C'(0)f'(p,o)\exp\{i(\vec{p}.\vec{x} - pt)\}$$

$$-D'(0)h^{*\prime}(p,0)\exp\{-i\,(\vec{p}.\vec{x} - pt)\}]$$

$$+\sum_{\lambda=\pm1}\frac{\lambda}{(2)^{1/2}}\Big[C(\lambda)\int\frac{d\vec{p}}{p}p^{-\lambda}\sigma(\vec{p},\lambda)\,f(\vec{p},\lambda)\exp\{i(\vec{p}.\vec{x} - pt)\}$$

$$-D(\lambda)\int\frac{d\vec{p}}{p}p^{\lambda}\sigma^*(p,\lambda)h^*(p,\lambda)\exp\{-i(\vec{p}.\vec{x} - pt)\}$$

$$+C'(\lambda)\int\frac{d\vec{p}}{p}\,p^{-\lambda}\sigma'(\vec{p},\lambda)f'(\vec{p},\lambda)\exp\{i(\vec{p}.\vec{x}-pt)\}$$

$$-D'(\lambda)\int\frac{d\vec{p}}{p}\,p^{\lambda}\sigma^{*\prime}(\vec{p},\lambda)h^{*\prime}(\vec{p},\lambda)\exp\{-(i\vec{p}.\vec{x}-pt)\}\Bigg] \quad ...(22)$$

Gauge Change

If A is any of the infinitesimal generators P^i, H, J_l then $A\ \psi(x)$ has the same expansion (22) on replacing $f(\epsilon,\vec{p},\lambda)$ by $\hat{A}f\ (\epsilon,\vec{p},\lambda)$, where $\hat{A}$ is the corresponding finite spin generator given by equation (7). If A stands for any of Z_i, then $\hat{A}$ is not Hermitian operator and $\hat{A}f(\epsilon,\vec{p},\lambda)$ consists of two parts, one of which corresponds to a true physical change of wavefunction and other gives unphysical change (change of gauge)

$$\hat{Z}_i f(\vec{p}) = g_i(\vec{p}) + \hat{Z}_i' f(\vec{p}) \quad ...(23)$$

where $\hat{Z}_i$ is finits spin operator for which $T_i = 0$, and $g_i(\vec{p})$ is the nonessential change in the wavefunction or the gauge change given by

$$g_1(\vec{p}) = \epsilon\left[\left\{\frac{p_1^2}{p^2(p+p_3)} - \frac{1}{p}\right\}T_1 + \frac{p_1p_2}{p^2(p+p_3)}T_2\right]f(\vec{p}) = \epsilon B_1 f(\vec{p}),$$

$$g_2(\vec{p}) = \epsilon\left[\frac{p_1p_2}{p^2(p+p_3)}T_1 + \left\{\frac{p_2^2}{p^2(p+p_3)} - \frac{1}{p}\right\}T_2\right]f(\vec{p}) = \epsilon B_2 f(\vec{p}),$$

$$g_3(\vec{p}) = \epsilon\left[\frac{p_1T_1+p_2T_2}{p^2}\right]f(\vec{p}) = \epsilon B_3 f(\vec{p}) \quad ...(24)$$

Hence,

$$Z_i\psi(x) = \sum_{\epsilon}\sum_{\lambda}\Big[C(\epsilon,\lambda)\int\frac{d\vec{p}}{p}\,\chi(\epsilon,\vec{p},\lambda)\hat{Z}_i f(\epsilon,\vec{p},\lambda) \times \exp\left\{i(\vec{p}.\vec{x}-\epsilon pt)\right\}\Big]$$

$$+\sum_{\epsilon}\sum_{\lambda}\Big[C'(\epsilon,\lambda)\int\frac{d\vec{p}}{p}\,\chi'(\epsilon,\vec{p},\lambda)\hat{Z}_i f'(\epsilon,\vec{p},\lambda) \times \exp\left\{i(\vec{p}.\vec{x}-\epsilon pt)\right\}\Big] + G_i(x) \quad ...(25)$$

where $G_i(x)$ is the gauge change and should be added to ψ when the frame of reference is changed by an infinitesimal space time transformation.

Hence,

$$G_i = \sum_\epsilon \sum_\lambda \left[C(\epsilon,\lambda) \int \frac{d\vec{p}}{p} \chi(\epsilon,\vec{p},\lambda)\, g_i(\vec{p}) \exp\left\{ i(\vec{p}.\vec{x} - \epsilon pt) \right\} \right]$$

$$+ \sum_\epsilon \sum_\lambda \left[C'(\epsilon,\lambda) \int \frac{d\vec{p}}{p} \chi'(\epsilon,\vec{p},\lambda)\, g_i'(\vec{p}). \times \exp\left\{ i(\vec{p}.\vec{x} - \epsilon pt) \right\} \right].$$

$$= \sum_\epsilon \sum_\lambda \left[C(\epsilon,\lambda) \int \frac{d\vec{p}}{p} \chi(\epsilon,\vec{p},\lambda)\, \epsilon B_i f(\epsilon,\vec{p},\lambda) \exp \left\{ i(\vec{p}.\vec{x} - \epsilon pt) \right\} \right]$$

$$+ \sum_\epsilon \sum_\lambda \left[C'(\epsilon,\lambda) \int \frac{d\vec{p}}{p} \chi'(\epsilon,\vec{p},\lambda)\, \epsilon B_i f'(\epsilon,\vec{p},\lambda) \times \exp\left\{ i(\vec{p}.\vec{x} - \epsilon pt) \right\} \right]$$

$$= \sum_\epsilon \sum_\lambda \left[C(\epsilon,\lambda)\, \epsilon \int \frac{d\vec{p}}{p} \Gamma_i(\epsilon,\vec{p},\lambda) f(\epsilon,\vec{p},\lambda) \times \exp\left\{ i(\vec{p}.\vec{x} - \epsilon pt) \right\} \right]$$

$$+ \sum_\epsilon \sum_\lambda \left[C'(\epsilon,\lambda)\, \epsilon \int \frac{d\vec{p}}{p} \Gamma_i(\epsilon,\vec{p},\lambda) f'(\epsilon,\vec{p},\lambda) \times \exp\left\{ i(\vec{p}.\vec{x} - \epsilon pt) \right\} \right] \quad ...(26)$$

where the column vector $\Gamma_i(\epsilon,\vec{p},\lambda)$ is given by

$$\Gamma_i(r \mid \epsilon,\vec{p},\lambda) = \left\{ \exp(i\vec{\omega}.M) \exp(i\nu N_3)\, B_i\, U \right\}_{r\lambda} \quad .. (27)$$

If we define a matrix A_i as

$$A_i = \exp\left[i\vec{\omega}.M \right] \exp\left[i\nu N_3 \right] B_i \quad ...(28)$$

then as discussed by Moses (1967) we have

$$e^{-A} B\, e^{A} = \sum_n \left\{ B, A \right\}^n \frac{1}{n!} \quad ...(29)$$

Using equations (29) and (6) and commutation rules for M_i and N_i we get

$$\exp(i\nu N_3)\,T_i \exp(-i\nu N_3) = pT_i$$

or

$$\exp(i\nu N_3)\,T_i = p\,T_i \exp(i\nu N_3) \qquad \ldots(30)$$

Using equations (24) in (30) we have

$$\exp(i\nu N_3)\,B_i = pB_i \exp(i\nu N_3)$$

Similarly,

$$\exp\left[i\vec{\omega}.M\right]B_i = \frac{1}{p^2}\left[-(\vec{p}\times M)_i - \epsilon\left(\frac{p_i}{p}\right)\left(\vec{p}.N\right) + \epsilon p N_i\right]\times \exp\left[i\vec{\omega}.M\right]$$

Hence equations (28) reduces to

$$A_i = \frac{1}{p}\left[-\left(\vec{p}\times M\right)_i - \epsilon\left(\frac{p_i}{p}\right)\left(\vec{p}.\vec{N}\right) + \epsilon p N_i\right] \times \exp\left(i\vec{\omega}.\vec{M}\right)\exp(i\nu N_3) \qquad (31)$$

If we define a column vactor $\phi_n(x)$ as

$$\phi_n(x) = \sum_{\epsilon}\sum_{\lambda}\Big[C(\epsilon,\lambda)\int \frac{d\vec{p}}{p^{n+1}}\, x(\epsilon,\vec{p},\lambda)\, f(\epsilon,\vec{p},\lambda)\exp\left\{i\left(\vec{p}.\vec{x}-\epsilon pt\right)\right\} + C'(\epsilon,\lambda)\int \frac{d\vec{p}}{p^{n+1}}\, x'(\epsilon,\vec{p},\lambda)\, f'(\epsilon,\vec{p},\lambda)\exp\left\{i\left(\vec{p}.\vec{x}-\epsilon pt\right)\right\}\Big] \qquad \ldots(32)$$

such that,

$$\frac{\partial^2}{\partial t^2}\phi_3(x) = -\phi_1(x).$$

Then putting $\nabla_i = \partial/\partial x$, and using the equations (27), (26), (32) and (31) we get

$$G_j(x) = i\left[-i\left(\vec{M}\times\vec{\nabla}\right)_j + \frac{\partial}{\partial t}N_j\right]\phi_1(x) + i\left(\vec{N}.\vec{\nabla}\right)\nabla_j\frac{\partial}{\partial t}\phi_3(x) \qquad \ldots(33)$$

If the unit vector in the direction of the i^{th} space axis is given by

$$\vec{e}_1 = (1, 0, 0).$$

$$G_j(x) = -\nabla \phi_{1,j}(x) + \vec{e}_j [\vec{\nabla}.\vec{\phi}_1(x)] - i \frac{\partial}{\partial t} [\vec{e}_j \times \vec{\phi}_1(x)] - i \frac{\partial^2}{\partial x_j \partial t} [\vec{\nabla} \times \vec{\phi}_2(x)] \quad \ldots(34)$$

where $\phi_{1,j}$ denotes the component of the vector.

REALITY CONDITION

If ψ transforms as a real antisymmetric tensor then we have

$$C'(0) = C'(\lambda) = D'(\lambda) = 0$$

and hence the equation (22) reduces to :

$$\psi(x) = \int \frac{d\vec{p}}{p} \chi(\vec{p},0) [C(0) f(\vec{p},0) \exp\{i(\vec{p}.\vec{x} - pt)\} - D(0) h^*(p,0) \exp\{-i(p.x - pt)\}]$$

$$+ \sum_{\lambda=\pm 1} \frac{\lambda}{(2)^{1/2}} \left[C(\lambda) \int \frac{d\vec{p}}{p} p^{-\lambda} \sigma(\vec{p},\lambda) f(\vec{p},\lambda) \exp\{i(\vec{p}.\vec{x} - pt)\} - D(\lambda) \int \frac{d\vec{p}}{p} p^{\lambda} \sigma^*(\vec{p}\lambda) h^*(\vec{p},\lambda) \exp(i(\vec{p}.\vec{x} - pt)) \right] \quad .. (35)$$

which is similar to the equation as derived by Moses (1968). If $\psi(x)$ given by equation (35) is real then,

$$\psi(x) = \psi^*(x) \quad \ldots(36)$$

so,

$$C(0) f(\vec{p},0) = -D(0) h(\vec{p},0).$$

and,

$$p^{-\lambda} C(\lambda) f(\vec{p},\lambda) = p^{\lambda} D(\lambda) h(\vec{p},\lambda)$$

then,

$$\psi(x) = A \int \frac{d\vec{p}}{p} \chi(\vec{p},0) f(\vec{p},0) \exp\{i(\vec{p}.\vec{x} - pt)\} + \sum_{\lambda=\pm 1} \frac{\lambda}{(2)^{1/2}} B(\lambda) \int \frac{d\vec{p}}{p} p^{-\lambda} \sigma(\vec{p},\lambda) \exp\{i(\vec{p}.\vec{x} - pt)\} \quad \ldots(37)$$

where A and $B(\lambda)$ are constants.

MAXWELL'S EQUATION

Using equations (2), (3) and (4) Maxwell's equations in vacuum for $\vec{E}$ and $\vec{H}$ become

$$\nabla . \psi = 0 \quad \ldots(38)$$

$$\nabla \times \psi = -i \frac{\partial}{\partial t} \psi \qquad \ldots(39)$$

From the equations (14), (15) we can prove that

$$\left.\begin{aligned} \nabla . \chi (\vec{p},0) \exp \{ i (\vec{p}.\vec{x}) \} &= ip \\ \nabla . \sigma (\vec{p},\lambda) \exp \{ i (\vec{p}.\vec{x}) \} &= 0 \end{aligned}\right\} \qquad \ldots(40)$$

Using equations (22) and (40) in (38) we get

$$C (0) f(\vec{p},0) + C' (0) f' (\vec{p},0) = 0$$

$$D (0) h(\vec{p},0) + D' (0) h' (\vec{p},0) = 0$$

The constants C (0), C' (0), D (0) and D' (0) are arbitrary, so

$$f (\vec{p},0) = f' (\vec{p},0) = h (\vec{p},0) = h' (\vec{p},0) = 0 \qquad \ldots(41)$$

Similarly,

$$\left.\begin{aligned} &\nabla \times \chi (\vec{p},0) \exp (i\vec{p}.\vec{x}) = 0 \\ \text{and,} \\ &\nabla \times \sigma (\vec{p},\lambda) \exp (i\vec{p}.\vec{x}) = p\lambda\sigma (\vec{p},\lambda) \end{aligned}\right\} \qquad \ldots(42)$$

Using equations (22) and (42) in (39) we get

$$\left.\begin{aligned} &\sum_{\lambda=\pm 1} \lambda (\lambda - 1) [C(\lambda) f (\vec{p},\lambda) + C' (\lambda) f'(\vec{p},\lambda)] = 0 \\ \text{and,} \\ &\sum_{\lambda=\pm 1} \lambda (1 - \lambda) [D(\lambda) h^* (\vec{p},\lambda) + D' (\lambda) h^{*\prime} (\vec{p},\lambda)] = 0 \end{aligned}\right\} \qquad \ldots(43)$$

first of equations (43) results into the following equations

$$C (+ 1) f (\vec{p}, + 1) + C' (+ 1) f' (\vec{p}, + 1) = 0$$

which gives

$$f (\vec{p}, + 1) = f' (\vec{p}, + 1) = 0 \qquad \ldots(44)$$

while second of equations (43) gives

$$h (\vec{p}, - 1) = h' (\vec{p}, - 1) = 0 \qquad \ldots(45)$$

Thus in the expansion (22) only the wavefunction $f (\vec{p}, -1)$ and $h (\vec{p}, -1)$ need not be identically zero for ψ to satisfy Maxwell's equation (38) and (39). Hence the wavefunction which transforms as a complex antisymmetric tensor and satisfies Maxwell's equation, is given by

$$\psi(x) = \frac{-1}{(2)^{1/2}}\,[\,D(+1)\int d\vec{p}\sigma^*(\vec{p},+1)\,h^*(\vec{p},+1)\exp\{-i(\vec{p}.\vec{x}-pt)\}$$
$$+ D'(+1)\int d\vec{p}\sigma^{*\prime}(\vec{p},+1)\,h^{*\prime}(p,+1)\exp\{-i(\vec{p}.\vec{x}-pt)\}$$
$$+ C(-1)\int d\vec{p}\sigma'(\vec{p},-1)\,f(\vec{p},-1)\exp\{i(\vec{p}.\vec{x}-pt)\}$$
$$+ C'(-1)\int d\vec{p}\sigma'(\vec{p},-1)\,f'(\vec{p},-1)\exp\{i(\vec{p}.\vec{x}-pt)\}]$$
$$= C\int d\vec{p}\sigma(\vec{p},-1)\,f(\vec{p})\exp\{i(\vec{p}.\vec{x}-pt)\}$$
$$+ C'\int d\vec{p}\sigma'(\vec{p},-1)\,f'(\vec{p})\exp\{i(\vec{p}.\vec{x}-pt)\}$$
$$+ D\int d\vec{p}\sigma^*(\vec{p},+1)\,h^*(\vec{p})\exp\{-i(\vec{p}.\vec{x}-pt)\}$$
$$+ D'\int d\vec{p}\sigma^{*\prime}(\vec{p},+1)\,h^*(\vec{p})\exp\{-i(\vec{p}.\vec{x}-pt)\} \quad ...(46)$$

where,

$$C = \frac{-1}{(2)^{1/2}}\,.\,C(-1),\; C' = \frac{-1}{(2)^{1/2}}\,.\,C'(-1)$$

$$D = \frac{-1}{(2)^{1/2}}\,.\,D(+1),\; D' = \frac{-1}{(2)^{1/2}}\,D'(+1)$$

and

$$f(\vec{p}) = f(\vec{p},-1),\; f'(\vec{p}) = f'(\vec{p},-1),$$

$$h(\vec{p}) = h(\vec{p},+1),\; h'(\vec{p}) = h'(\vec{p},+1).$$

CANONICAL FORMALISM

We choose the values of the constants Cs and Ds so that the usual canonical formalism in terms of Hamiltonian density agrees with the particle interpretation. Hamiltonian density of the field in the present case given by

$$H(x) = (8\pi)^{-1}\,\psi^{*}.\,\psi \quad ...(47)$$

and the energy of the field is given by

$$E = \int_{-a}^{a} H(x)dx \quad ...(48)$$

Now we consider following four modes

(i) $f'(\vec{p}) = h(\vec{p}) = h'(\vec{p}) = 0,\; f(\vec{p}) \neq 0$

then, $$\psi = C\int d\vec{p}\sigma(\vec{p},-1)\,f(\vec{p})\exp\{i(\vec{p}.\vec{x}-pt)\} \quad ...(49)$$

The expectation energy for this mode is given by

$$\int_{-\infty}^{\infty} \frac{d\vec{p}}{p} f^*(\vec{p})p\vec{f}(\vec{p})=\int_{-\infty}^{\infty} d\vec{p}\,|\,f(\vec{p})\,|^2 \qquad ...(50)$$

Comparing equation (5) with (48) for ψ given by (49) we have

$$C = \frac{-1}{\pi(2)^{1/2}}.$$

(ii) $f(\vec{p}) = h(\vec{p}) = h'(\vec{p}) = 0, f'(\vec{p}) \neq 0.$

then, $$\psi = C'\int \frac{d\vec{p}}{p} \sigma'(\vec{p}, -1) f'(\vec{p}) \exp\{i(\vec{p}.\vec{x} - pt)\}.$$

and the expectation energy is

$$E'_f = \int_{-\infty}^{\infty} dp\,|\,f'(\vec{p})\,|^2 \qquad ...(51)$$

which leads to

$$C' = \frac{-1}{\pi(2)^{1/2}}$$

(iii) $f'(\vec{p}) = f(\vec{p}) = h'(\vec{p}) = 0, h(\vec{p}) \neq 0,$

then, $\psi(x) = hD\int d\vec{p}\sigma^*(\vec{p}, -1)\, h^*(\vec{p}) \exp\{-i(\vec{p}.\vec{x} - pt)\}$
and the expectation energy is

$$E_h = \int_{-\infty}^{\infty} d\vec{p}\,|\,h(\vec{p})\,|^2 \qquad ...(52)$$

which leads to, $D = -1/\pi(2)^{1/2}$

(iv) $f'(\vec{p}) = f(\vec{p}) = h(\vec{p}) = 0, h(\vec{p}) \neq 0$

Then, $$\psi(x) = D'\int d\vec{p}\sigma^{*\prime}(\vec{p}, -1)\, h^{*\prime}(\vec{p}) \exp\{-i(\vec{p}.\vec{x} - pt)\}$$

and, $$E_h' = \int_{-\infty}^{\infty} d\vec{p}\,|\,h'(\vec{p})\,|^2 \qquad ...(53)$$

which leads to $D' = \dfrac{-1}{\pi(2)^{1/2}}$

The expectation value of the total energy when the state is the superposition of all the modes is given by

$$E = E_f + E_f' + E_h + E_h' \qquad ...(54)$$

Second quantizations

To second quantize the theory, we consider $f(\vec{p}), f'(\vec{p}), h(\vec{p})$ and $h'(\vec{p})$ as creation operators and $f^*(\vec{p}), f'^*(\vec{p}), h^*(\vec{p})$ and $h'^*(\vec{p})$ as destruction operators. They satisfy the well known Boson-commutation rules. The operator $\psi(x)$ is then defined by replacing the amplitudes in the expansion (46) by these operators. Then for any operator $\hat{A}$, we define a second quantized operator $[A]$ by

$$[A] = \int \frac{d\vec{p}}{p} f^* (\vec{p},\lambda) \hat{A} f (\vec{p}\lambda)$$
$$+ \int \frac{d\vec{p}}{p} f^{*\prime} (\vec{p},\lambda) \hat{A} f' (\vec{p},\lambda) + \int \frac{d\vec{p}}{p} h^* (\vec{p},\lambda) \hat{A} h (\vec{p},\lambda)$$
$$+ \int \frac{d\vec{p}}{p} h^{*\prime} (\vec{p},\lambda) \hat{A} h' (\vec{p},\lambda) \quad . \quad (55)$$

The operators $[A]$ are the infinitesimal generators for the second quantized theory. Under the translation $T(\vec{a}^a)$, rotation $R(\vec{\theta})$ and pure Lorentz transformation $L(\vec{\beta})$ the set of operators $\psi(x)$ transforms to $\psi'(x)$ by

$$\psi'(x) = \psi(x+\vec{a}) = \exp\{-i\sum_a a^a[P_a]\} \psi(x) \exp\{i\sum_a a^a [P_a]\}$$
$$\psi'(x) = \exp\{-i\vec{\theta}.[K]\} \psi(x) \exp\{i\vec{\theta}.[K]\} \quad .. (56)$$
$$\psi'(x) = \exp\{-i\vec{\beta}.[Z]\} \psi(x) \exp\{i\vec{\beta}.[Z]\}.$$

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