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# Reduction of wave functions which transform as a complex antisymmetric tensor to the irreducible representation of Lorentz group

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The reduction of the wavefunction which transforms as a complex antisymmetric tensor to the irreducible representation of proper orthochronous inhomogenous Lorentz group, has been discussed by giving the proof of essential theorem which is used for the reduction. The effects of reality condition and wave equation are also discussed where the former reduces the expression to that of real wave function which transforms as a real antisymmetric tensor while the latter restricts the number of independent irreducible representations. By assuming the total energy to be positive, the constants of the expansion of wavefunction which satisfies wave equation, have been calculated to give the energies for our cases corresponding to positive and negative values of Hamiltonian density.

LIST OF SYMBOLS

$\Psi(\vec{x}, t)$ for wavefunction

$i$ for imaginary quantity $(-1)^{1/2}$

$i, j, k, \alpha, \beta, \gamma$ for integers (1, 2, 3)

$\vec{\beta}$ for pure Lorentz transformation.

$\vec{\theta}$ for rotation.

$\vec{a}$ for translation.

$F^{ij}$ for components of complex tensor.

0 for null matrix.

$\epsilon_{\alpha\beta\gamma}$ for antisymmetric three index symbol.

$\nabla$ for Laplace operator.

$r$ for index denoting real and complex parts of wavefunction.

$\mu$ for eigen values of mass operator.

$\epsilon$ for eigen values of operator $H$ ( ±1 )

$\lambda$ for integer which lies between 1 and 2

$s$ for spin corresponding to $S_i$

$S'_i$ for spin matrix

$I$ for unit matrix

$\xi$ for collective representation of the variables upon which the function in given representation depends.

$\omega(\mu p)$ for $(\mu^2+p^2)^{1/2}$

$\Sigma$ for summation

$M(\mu, \epsilon)$ for measure function

$C$, $D$ for constants

$\delta_{ij}$ for Kronecker symbol.

$\delta(\mu\text{-}m)$ for Dirac Delta function.

## 1. Introduction

All the relativistic particles were classified corresponding to the irreducible representations of the proper, orthochronous inhomogeneous Lorentz group by Wigner (1939) who showed how the wavefunctions for these particles in the momentum representation transform under the transformation of the group. Moses (1966) showed how electromagnetic vector potential can be reduced to photon wave function in a linear momentum basis. Here photon is defined as corresponding to a massless particle of spin 1 in Wigner's classification. Using the result of Moses (1965, 1967 a) reduction in linear momentum basis can be transformed to that in angular momentum basis. Moses (1966) showed how the wavefunction of photon is contained in the vector potential while in other paper by Moses (1959) the way that Maxwell's equations contain the photon wave-function is given. As discussed by Moses (1967b) the recipe of Lomont and Moses (1967) enables one to reduce any unitary ray representation of the proper, orthochronous, inhomogeneous Lorentz group. The methods of reduction have been discussed for both non-zero and zero-mass systems where for the former one obtains the Foldy (1956)—Shirokov (1958, 1959) relations and for the latter one is led to the Lomont-Moses (1964) realization. These methods are applied by Moses (1967b) to reduce the wavefunctions $\Psi(\vec{x}, t)$ which transform by means of unitary transformations to another function $\Psi(\vec{x}, t)$ when the space time coordinates undergo any transformation of the proper orthocronous homogeneous Loretz group, to the irreducible representation of the group with the restriction to the cases where only nonzero-mass irreducible representation appear. It is shown in those discussions that to reduce the wavefunctions only the transformation properties are necessary while the requirement that $\Psi(\vec{x})$

satisfies the wave equation, restricts the number of independent irreducible representations which appear. The reduction of the wavefunction which transforms as a real antisymmetric tensor has been discussed by Moses (1967b). We reduce here the wavefunction which transforms as a complex antisymmetric tensor by giving the proof of the essential theorem which is used in this case. The effects of reality condition and wave equation are also discussed.

## 2. Transformations of wavefunctions

We consider a complex antisymmetric tensor ;

$$F^{ij} = F_R^{\,ij} + F_I^{\,ij} \qquad \ldots(2.1).$$

Where $F_R^{ij}$ and $F_I^{ij}$ are real and imaginary parts of the tensor $F^{ij}$ with $F^{ij} = -F^{ji}$ (antisymmetric) which transform as a tensor under the transformation of proper homogeneous Lorentz group. It is proper to introduce the wavefunction field discription of this tensor. For this we define :

$$\left.\begin{array}{llll} E_{iR} = F^{oi} & ; & E_{iI} = F_I^{\,oi} & , \\ H_{1R} = F_R^{\,23} & , & H_{2R} = F_R^{\,31} & , \\ H_{3R} = F_R^{\,12} & , & H_{1I} = F_I^{\,23} & , \\ H_{2I} = F_I^{\,31} & , & H_{3I} = F_I^{\,12} & , \end{array}\right\} \qquad \ldots(2.2)$$

We then construct the two 3-components column vectors $\Psi_R(\vec{x})$ and $\Psi_I(\vec{x})$ from $F_R^{\,i}$ and $F_I^{\,i}$ respectively as :

$$\left.\begin{array}{l} \Psi_R(\vec{x}, t, r) = E_{rR}(\vec{x}, t) - i\, H_{rR}(\vec{x}, t) \\ \Psi_I(\vec{x}, t, r) = E_{rI}(\vec{x}, t) - i\, H_{rI}(\vec{x}, t) \end{array}\right\} r=1, 2, 3. \qquad \ldots(2.3).$$

The wavefunction here is the six components column vector formed by placing the three components column vectors $\Psi_R$ and $\Psi_I$

$$\Psi = \begin{bmatrix} \Psi_R \\ \Psi_I \end{bmatrix} \qquad \ldots(2.4).$$

where

$$\Psi_R = \Psi_R(\vec{x}, t) = \begin{bmatrix} F_R^{\,01} \\ F_R^{\,02} \\ F_R^{\,02} \end{bmatrix} - i \begin{bmatrix} F_R^{\,23} \\ F_R^{\,31} \\ F_R^{\,12} \end{bmatrix}$$

$$\Psi_I = \Psi_I(\vec{x}, t) = \begin{bmatrix} F_I^{\,01} \\ F_I^{\,02} \\ F_I^{\,03} \end{bmatrix} - i \begin{bmatrix} F_I^{\,23} \\ F_I^{\,31} \\ F_I^{\,12} \end{bmatrix}. \qquad \ldots(2.5).$$

Generally we consider the set of the functions $\Psi(x, t, r)$ given by

$$\Psi(\vec{x}, t, r) = \Psi_R(\vec{x}, t, r) + i\Psi_I(\vec{x}, t, r) \qquad \ldots(2.6)$$

where the variable $r$ runs through the set of discrete or continuous values. It is also useful sometimes to regard $\Psi(\vec{x}, t)$ as being a column vector with components $\Psi(\vec{x}, t, r)$ where every value of $r$ has two signs one for real components and the other for imaginary components.

Here $\vec{x}$ denotes the space vector : $\vec{x} = \vec{x}_1 + \vec{x}_2 + \vec{x}_3$.

Let $x^\alpha (\alpha = 0, 1, 2, 3)$ denotes the components of the space-time four component vector with $x^0 = -x_0 = t$, $x^1 = x_1$, $x^2 = x_2$ and $x^3 = x_3$ with the units $\hbar = c = 1$ then

$$\Psi(\vec{x}^\alpha) = \Psi(\vec{x}, t)$$

Any transformation of the proper, orthochronous, inhomogeneous Lorentz group can be regarded as the product of three particular transformations, i. e. translation $\tau(a^\alpha)$, rotation $R(\vec{\theta})$ and pure Lorentz transformation $L(\vec{\beta})$, where the direction of $\vec{\beta}$ is in the opposite direction of moving frame of reference and the magnitude $\beta = |\vec{\beta}|$ is given by $\cosh \beta - (1 - v^2)^{1/2}$. Under these transformations the components of $\vec{x}^\alpha$ in the new frames are given (Moses 1967) by

$$\left.\begin{aligned}
\vec{x}' &= T(\vec{a})\vec{x} = \vec{x} - \vec{a}, \\
\vec{x}' &= R(\vec{\theta})\,\vec{x} - \vec{x}\cos\theta + \frac{1-\cos\theta}{\theta^2}(\vec{\theta}.\vec{x})\,\vec{\theta} - \frac{\sin\theta}{\theta}(\vec{\theta} \times \vec{x}) = \exp(i\vec{\theta}.\vec{M}')\vec{x}, \\
\vec{x}' &= L(\vec{\beta})\,\vec{x} = \vec{x} + \vec{\beta}(\vec{\beta}.\vec{x})\left(\frac{\cosh\beta - 1}{\beta^2}\right) + \vec{\beta}\,x^0\left(\frac{\sinh\beta}{\beta}\right) = \exp(i\vec{\beta}.\vec{N}')\vec{x}
\end{aligned}\right\} (2.7)$$

Where $M'$ and $N'$ are introduced by Moses (1966) and they satisfy the commutation rules of the infinitesimal generators of the proper orthochronous, homogeneous Lorentz group :

$$[M'_\alpha, M'_\beta] = i\sum_\gamma \epsilon_{\alpha\beta\gamma} M'_\gamma$$

$$[M'_\alpha, N'_\beta] = -i\sum_\gamma \epsilon_{\alpha\beta\gamma} N'_\gamma. \qquad \ldots(2{\cdot}8)$$

$$[N'_\alpha, N'_\beta] = i\sum_\gamma \epsilon_{\alpha\beta\gamma} M_\gamma.$$

The matrices $M'$ appear in reduced form and can be written as,

$$M'_i = \begin{bmatrix} S_i & O \\ O & S'_i \end{bmatrix}, \qquad \ldots(2.9)$$

where, $\overleftarrow{S}=O$ and matrices $S'_i$ constitute the irreducible representations of the generators of rotation group corresponding to the vector rotations (Moses 1967b). Under these transformations the wavefunction $\Psi(\vec{x})$ transforms as follows,

$$\Psi'(\vec{x}) = \psi(\vec{x} + \vec{\alpha})$$

$$\Psi'(\vec{x}) = \exp(i\vec{\theta}.\vec{M})\,\Psi\,[R(-\vec{\theta})\,\vec{x}\,] \qquad \ldots(2.10)$$

$$\Psi'(\vec{x}) = \exp(i\beta.\vec{N})\,\Psi[L(-\beta)\,\vec{x}\,].$$

Here the matrices $M'_i$ and $N'_i$ satisfy the commutation rules for the infinitesimal generators of the Lorentz group and can be used to generate a ray representation of homogeneous Lorentz group. For this case these matrices are given by $M_i = S_i$ and $N_i = -iS'$ where $S'_i$, which are used here in equation (2.9), are given by :

$$S'_1 = \begin{pmatrix} o & o & o \\ o & o & -i \\ o & i & o \end{pmatrix}, \; S'_2 = \begin{pmatrix} o & o & i \\ o & o & o \\ -i & o & o \end{pmatrix}, \; S'_3 = \begin{pmatrix} o & -i & o \\ i & o & o \\ o & o & o \end{pmatrix} \ldots(2.11)$$

If,

$$(\vec{\theta}.\,S')^3 = \theta^2\,(\vec{\theta}.\,S'),$$

then,

$$\exp\,(i\,\vec{\theta}.\,S') = 1 + i\,(\vec{\theta}.\,S')\left(\frac{\sin\theta}{\theta}\right) + (\vec{\theta}.\,S')^2 \times \left(\frac{\cos\theta - 1}{\theta^2}\right) = \hat{R}(\vec{\theta})$$

where, 1 is unit matrix. Hence matrix elements of $\hat{R}\,(\vec{\theta})$ are given by,

$$[\,\hat{R}(\theta)\,]_{\alpha\beta} = \delta_{\alpha\beta}\cos\theta + \frac{\theta_\alpha\theta_\beta}{\theta^2}(\cos\theta - 1) + \sum_\gamma \epsilon_{\beta\alpha\gamma}\theta_\gamma\left(\frac{\sin\theta}{\theta}\right). \quad (2.12)$$

Similarly if :

$$L(\vec{\beta}) = \exp\,(i\vec{\beta}.N)$$

$$= 1 + i(\vec{\beta}.N)\left(\frac{\sinh\beta}{\beta}\right) + (\vec{\beta}.N)^2\left(\frac{1-\cosh\beta}{\beta^2}\right),$$

where $(\beta.N)^3 = -\beta^2 (\vec{\beta}.N)$.

Then the matrix elements of $\hat{L}(\vec{\beta})$ are given by,

$$[L(\vec{\beta})]_{\alpha\beta} = \delta_{\alpha\beta} \cosh\beta - \frac{\beta_\alpha\beta_\beta}{\beta^2}(\cosh\beta - 1) + i\sum_\gamma \epsilon_{\alpha\beta\gamma}\beta_\gamma \frac{\sinh\beta}{\beta}. \qquad ...(2\cdot13)$$

## 3. Reduction of wavefunctions

The ten infinitesimal generators of the proper, orthochronous, inhomogeneous Lorentz groups are the energy $H$, components of momentum $P_i$ ($i = 1, 2, 3$), the three components of angular momentum $K_i$ and the three generators corresponding to space-time relations $Z_i$. As operators they are defined as,

$$\left.\begin{aligned} &P_i = P^i, \; i = 1, 2, 3, \\ &P^\circ = -P_\circ = H. = -i\frac{\partial}{\partial t}\, 1. \\ &P_j = -i\nabla_j\, l = -i\frac{\partial}{\partial x_j}\, 1. \\ &K_j = M_j - i[\vec{x}\times\nabla]\, 1. \\ &Z_j = N_j + i[x_j\frac{\partial}{\partial t} + t\nabla_j]\, 1. \end{aligned}\right\} \qquad ...(3\cdot1)$$

These generators satisfy the following commutation relations :

$$\left.\begin{aligned} &[P^\alpha, P^\beta] = 0, \\ &[K_\alpha, K_\beta] = i\sum_\gamma \epsilon_{\alpha\beta\gamma} K_\gamma, \\ &[K_\alpha, P_\beta] = i\sum_\gamma \epsilon_{\alpha\beta\gamma} P_\gamma, \\ &[Z_\alpha, Z_\beta] - = i\Sigma\epsilon_{\alpha\beta\gamma}K_\gamma, \\ &[Z_\alpha, P^\circ] = iP_\alpha, \\ &[Z_\alpha\, P_\beta] = i\delta_{\alpha\beta}P_\circ. \end{aligned}\right\} \qquad ...(3\cdot2)$$

Hence equations ( 2.10 ) can now be written as :

$$\left.\begin{aligned} &\Psi'(\vec{x}') = \exp\,[i\sum_j a^j P_j]\Psi(\vec{x}). \\ &\Psi'(\vec{x}') = \exp\,[i\vec{\theta}.\vec{K}]\Psi(\vec{x}). \\ &\Psi'(\vec{x}') = \exp\,[i\vec{\beta}.Z]\Psi(\vec{x}). \end{aligned}\right\} \qquad ...(3\cdot3)$$

We introduce a complex function $f(\mu, \epsilon, p, \lambda)$, where, the vector $\vec{p}$ has components $p_i$ ( $i = 1, 2, 3$ ) each of which takes on every value from $-\infty$ to $\infty$. The variable $\mu$ takes on all the eigenvalues of the mass operator $M$, where,

$$M = [\,H^2 - P^2\,], \quad \text{with} \quad P^2 = \Sigma_i\, P_i^2,$$

The variable $\epsilon$ takes on the values which occur in the spectrum of the operator H. $\lambda$ may have any value from 1 to $2s+1$ where $s$ is the spin corresponding to matrices $S'_i$.

If $\hat{P}_\alpha$, $\hat{K}_i$, $\hat{Z}_i$ are the infinitesimal generators of the unitary ray represntation of Lorentz group then the function $f(\mu,\epsilon,p,\lambda)$ transforms under these operators as,

$$\left.\begin{aligned}
&\hat{P}^0 f(\mu,\epsilon,p,\lambda)=\hat{H}f(\mu,\epsilon,p,\lambda)=\epsilon\omega(\mu,p)\, f(\mu,\epsilon,p,\lambda).\\
&\hat{P}^i f(\mu,\epsilon,p,\lambda)=p_i f(\mu,\epsilon,p,\lambda)\\
&\hat{K}^i f(\mu,\epsilon,p\lambda)=\left[-i\sum_{jk}\epsilon_{ijk}\, p_j\frac{\partial}{\partial p_k}+S_i\right]f(\mu,\epsilon,p,\lambda).\\
&\hat{Z}_i f(\mu,\epsilon,p,\lambda)=\epsilon\left[i\,\omega(\mu p)\frac{\partial}{\partial p_i}+\frac{1}{\omega(\mu,p)+\mu}\sum_{jk}\epsilon_{ijk}p_j S_k\right]f(\mu,\epsilon,p,\lambda).
\end{aligned}\right\} \quad ...(3\cdot4)$$

$$\text{where,} \qquad \omega(\mu, p) = [\mu^2+p^2]^{1/2}. \quad ...(3\cdot5)$$

These generators satisfy the commutation rules as those of the generators $P^\alpha$, $K$, $Z$ given by equations (3·2). Hence the required reduction of the wavefunction $\Psi(\vec{x}, t)$ is done if we express it in terms of $f(\mu, \epsilon, p, \lambda)$. This can be done by using following theorem.

"The expansion of the function $\Psi(\vec{x})$ which transforms as complex tensor, in terms of irreducible unitary ray representation of the proper, orthochronous inhomogeneous Lorentz group is given by,

$$\Psi(\vec{x})=\sum_\epsilon\sum_r d\,M^r(\mu,\epsilon)\int\frac{d\,p}{\omega(\mu,p)}\ \exp\,[i\{\vec{p}.\vec{x}-\epsilon\,\omega(\mu,p)\,t\}]$$

$$\times[\omega(\mu,p)\,f(\mu,\epsilon,p)-\frac{\vec{p}.\{\vec{p}.f(\mu,\epsilon,p)\}}{w(\mu,p)+\mu}-i\,\epsilon\,\{\vec{p}\times f(\mu,\epsilon,p)\}] \quad ...(3.6)$$

where, $r$ has two values, one for $\Psi_R$ wavefunction and the other for $\Psi_I$ wavefunction. $dM(\mu,\epsilon)$ is the measure function of masses and energies which appear. It is an arbitrary measure in Stieltzes sense."

Proof : As discussed by Lomont & Moses (1967) we have ,

$$f(\xi)=\Psi(\vec{x})=\sum_\epsilon\int d\mu\sum_\lambda\int\frac{d\,p}{\omega(\mu,p)}\times\langle\xi\mid\mu,p,\lambda\rangle f(\mu,\epsilon,p,\lambda), \quad (3.7)$$

where, $f(\xi)$ is the representation of vector $\Psi$ in the basis, being characterised by the space of wavefunctions in Hilbert space upon which the generators operate. $\xi$, collectively denotes all the variables upon which the functions in the given representation depend. The transformation function

$<\xi \mid \mu,\epsilon,p,\lambda>$ may be considered to be the inner product of the bra $<\xi \mid$ and the ket $\mid \mu,\epsilon,p,\lambda >$ and it is given by,

$$<\xi \mid \mu,\epsilon,p,\lambda > = \exp [-i\beta.Z]\, g\, ( \xi,\mu,\epsilon,\lambda ), \qquad ...(3.8)$$

where $g\, ( \xi,\mu,\epsilon,\lambda )$ is the solution of the equations :

$$P_i g\, ( \xi,\mu,\epsilon,\lambda ) = 0$$

$$Hg\, ( \xi,\mu,\epsilon,\lambda ) = \epsilon\omega g\, ( \xi,\mu,\epsilon,\lambda )\,. \qquad ...(3.9)$$

This may also be written as :

$$g\, (\xi,\mu,\epsilon,\lambda) = <\xi \mid \mu,\epsilon,0,\lambda >$$

with,

$$P_i \mid \mu,\epsilon,p,\lambda> = p_i \mid \mu,\epsilon,p,\lambda>, \qquad ...(3\cdot10)$$

$$H \mid \mu,\epsilon,p,\lambda> = \epsilon\omega\, (\mu,p) \mid \mu,\epsilon,p,\lambda>.$$

Using equations (3.1) we can write (3.8) as,

$$\xi \mid \mu,\epsilon,p,\lambda > = \exp\left[ -i\vec{\beta}.N + (\vec{\beta}.\vec{x}\frac{\partial}{\partial t} + \vec{\beta}.\nabla t)1 \right] g\, ( \xi,\mu,\epsilon,\lambda ) \quad ...(3.11)$$

where,

$$\left.\begin{array}{l} \vec{p} = -\epsilon\,\mu\,\frac{\vec{\beta}}{\beta}\,\sinh\beta, \\ \omega\,(\mu,p) = \mu\cosh\beta, \\ p = \mid \vec{p} \mid = \mu\sinh\beta. \end{array}\right| \qquad ...(3.12)$$

If we define a column vector $\chi\, (\mu,\epsilon,p,\lambda)$ with componts $\chi\, (r_1\mu,\epsilon,p,\lambda)$ given by :

$$\chi\, ( r \mid \mu,\epsilon,p,\lambda ) = \exp [\, - i\vec{\beta}.N\, ]_{r,\lambda} \qquad ...(3.13)$$

then using the equations (3·9), (3·10), (3·11) and (3·1) and introducing an arbitrarily chosen measure function $M'\,(\mu,\epsilon)$ equation (3·7) can be written as :

$$\Psi(\vec{x}) = \sum_{\epsilon}\sum_{r}\sum_{\lambda}\int dM'\,(\mu,\epsilon)\int \frac{dp}{\omega(\mu,p)}\, \chi\,(\mu,\epsilon,p,\lambda)\, f'\,(\mu,\epsilon,p,\lambda)$$

$$\times \exp\,[i\{\vec{p}.\vec{x} - \epsilon\omega\,(\mu,p)t\}]. \qquad ...(3\cdot14)$$

Using equations (2·13), (3·11) into (3·12) and then putting the values of components of vector $\chi(\mu,\epsilon,p,\lambda)$ after labeling over $\lambda$ calculated in this way, in equation (3·13) gives the required result (on absorbing $\frac{1}{\mu}$ into the measure function ). Four irreducible representations of the inhomogeneous Lorentz group appear in the expansion (3·6) because index $\gamma$ can take two values with two signs for both the values of $r$ separately. In equation (3.6) $\epsilon$ has two values, *i. e.*, $+1$ and $-1$.

If we construct the vectors :

$$f^r(\mu, p,) = f^r(\mu, +1, p)$$
$$h^r(\mu, p) = f^{*r}(\mu, -1, p)$$
$$M^r(\mu) = M^r(\mu, +1)$$
$$N^r(\mu) = M^r(\mu, -1)$$

where,* denotes the complex conjugate, then equation (3.6) is written as :

$$\Psi(\vec{x}) = \sum_r \int d\, M^r(\mu) \int \frac{dp}{\omega(\mu, p)} \exp[\, i\{\vec{p}.\vec{x} - \omega(\mu, p)\, t\}\,]$$
$$\times [\,\omega(\mu, p) f^r(\mu, p) - \frac{\vec{p}\{\vec{p}.f^r(\mu, p)\}}{\omega(\mu, p) + \mu} - i\{\ \vec{p} \times f^r(\mu, p)\}\,]$$
$$+ \sum_r \int dN^r(\mu) \int \frac{dp}{\omega(\mu, p)} \exp[\, -i\{\vec{p}.\ \vec{x} - \omega(\mu, p)\, t\}\,]$$
$$\times [\,\omega(\mu, p)\, h^{*r}(\mu, p) - \frac{\vec{p}\{\vec{p}.\, h^{*r}(\mu, p)\}}{\omega(\mu, p) + \mu}$$
$$- i\{\vec{p} \times h^{*r}(\mu, p)\}\,]. \qquad \ldots(3.14)$$

Hence if we denote the arbitrary measures and functions by dashes as superfix for imaginary wavefunction, then,

$$\Psi(\vec{x}) = \int dM(\mu) \int \frac{dp}{\omega(\mu, p)} \exp[\, i\{\vec{p}.\ \vec{x} - \omega(\mu, p)\, t\}\,] \times [\omega(\mu, p)$$
$$f(\mu, p) - \frac{\vec{p}\{\vec{p}.f(\mu, p)\}}{\omega(\mu, p) + \mu} - i\{\vec{p} \times f(\mu, p)\}\,]$$
$$+ \int dM'(\mu) \int \frac{dp}{\omega(\mu, p)} \exp[\, i\{\vec{p}.\vec{x} - \omega(\mu, p)\, t\}\,] \times [\,\omega(\mu, p) f'(\mu, p)$$
$$- \frac{\vec{p}\{\vec{p}.f'(\mu, p)\}}{\omega(\mu, p) + \mu} - i\{\vec{p} \times f'(\mu, p)\}]$$
$$+ \int dN(\mu) \int \frac{dp}{\omega(\mu, p)} \exp[\, -i\{\vec{p}.\vec{x} - \omega(\mu, p)\, t\}\,]$$
$$\times [\omega(\mu, p)\ h^*(\mu p) - \frac{\vec{p}\{\vec{p}.\, h^*(\mu, p)\}}{\omega(\mu, p) + \mu} - i\{p \times h^*(\mu, p)\}\,]$$
$$+ \int d\ N'(\mu) \int \frac{dp}{\omega(\mu p,)} \exp[-i\{\vec{p}.\vec{x} - \omega(\mu, p)\, t\}] \times [\,\omega(\mu, p)\, h^{*\prime}(\mu, p)$$
$$- \frac{\vec{p}\{\vec{p}.\, h^{*\prime}(\mu, p)\}}{\omega(\mu, p) + \mu} - i\{\vec{p} \times h^{*\prime}(\mu, p)\}]. \qquad \ldots(3.15)$$

### 4. Reality condition

If the wavefunction contains only the real part then the measure function $M$ need not be labeld by the index $r$,

$$\Psi(x)=\int dM\,(\mu)\int\frac{dp}{\omega(\mu,p)}\exp\,[i\{\vec{p}.\vec{x}-\omega(\mu,p)\,t\}]\times[\omega(\mu,p)\,f(\mu,p)-\frac{\vec{p}\{\vec{p}.f(\mu,p)\}}{\omega(\mu,p)+\mu}$$

$$[-i\{\vec{p}\times f(\mu,p)\}]+\int dN(\mu)\int\frac{dp}{\omega(\mu,p)}\exp\,[-i\{p.x+\omega(\mu,p)t\}]\times[\omega(\mu,p)h^*(\mu,p)$$

$$-\frac{\vec{p}\{\vec{p}.h^*(\mu,p)\}}{\omega(\mu,p)+\mu}-i\{\vec{p}\times h^*(\mu,p)\}\qquad\ldots(4{\cdot}1)$$

Moreover, $\psi^*=\psi$, so

$$dM(\mu)[\omega(\mu,p)f(\mu,p)-\frac{\vec{p}}{\omega(\mu,p)+\mu}\{\vec{p}.f(\mu,p)\}]=dN(\mu)[\omega(\mu,p)h(\mu,p)-\frac{\vec{p}\{\vec{p}.h(\mu,p)\}]}{\omega(\mu p)+\mu}\qquad\ldots(4{\cdot}2)$$

and
$$dM(\mu)\{\vec{p}f(\mu,\ p)\}=dN(\mu)\{p\times h(\mu,\ \vec{p})\}.\qquad\ldots(4.3)$$

Putting these values in (4.1) we get the expansion for real wavefunction $\Psi$ which transforms as a real antisymmetric tensor given by Moses (1967b).

### 5. Wave equation

Let us assume that the wavefunction which transforms as a complex tensor satisfy the wave equation,

$$\left(\frac{\partial^2}{\partial t^2}-\nabla^2+m^2\right)\Psi\,(\vec{x})=0\qquad\ldots(5.1)$$

where, $m$ is the mass of the complex tensor particle. In terms of the infinitesimal generator equation (5.1) can be written as,

$$[H^2-P^2]\,\Psi\,(x)\;=\;m^2\;\Psi\,(x)\qquad\ldots(5.2)$$

For any infinitesimal generator $M$, $M\Psi(x)$ is obtained from equation (3.15) by replacing $f(\mu, p)$ and $h(\mu, p)$ by $M f(\mu, p)$ and $M h(\mu, p)$, where M's are used in equations (3.4). Then equation (3.15) can be written as,

$$\int dM(\mu)\int \frac{dp}{\omega(\mu,p)}(\mu^2-m^2)[\omega(\mu, p)\, f(\mu, p) - \frac{\vec{p}\{\vec{p}.f(\mu, p)\}}{\omega(\mu, p)+\mu} - i\,\{\vec{p}\times f(\mu, p)\}]$$

$$\times \exp[i\{\vec{p}\vec{x}-\omega(\mu, p)\,t\}] + \int dM'(\mu)\int \frac{dp}{\omega(\mu, p)}(\mu^2-m^2)[\omega(\mu, p)\, f'(\mu, p)$$

$$- \frac{\vec{p}\{\vec{p}\, f'(\mu, p)\}}{\omega(\mu, p+\mu} - i\,\{\vec{p}\times f'(\mu, p)\}] \times \exp[i\{\vec{p}.\vec{x}-\omega(\mu, p)\,t\}]$$

$$+\int dN(\mu)\int \frac{dp}{\omega(\mu, p)}(\mu^2-m^2)[\omega(\mu, p)h^*(\mu, p) - \frac{\vec{p}\{\vec{p}.h^*(\mu, p)\}}{\omega(\mu, p)+\mu} - i\,\{\vec{p}\times h^*(\mu, p)\}]$$

$$\times \exp[-i\{\vec{p}.\vec{x}-\omega(\mu, p)\,t\}] + \int dN'(\mu)\int \frac{dp}{\omega(\mu, p)}(\mu^2-m^2)[\omega(\mu, p)\, h^{*\prime}(\mu, p)$$

$$-\frac{\vec{p}\{\vec{p}.\, h^{*1}(\mu, p)\}}{\omega(\mu, p)+\mu} - i\vec{p}.\{\vec{p}.\, h^{*\prime}(\mu,p)\}] \times \exp[\{\vec{p}\,\vec{x}-\omega(\mu p)\,t\}].$$

$$= 0 \qquad (5.3)$$

Hence for $f(\mu, p)$ and $h(\mu, p)$ not to be identically zero, $M(\mu)$ and $N(\mu)$ are constants for all the values of $\mu \neq m$. For $\mu = m$, they have a jump,

$$dN^i = D^i\delta(\mu-m)\, d\mu, \qquad dM^i = C^i\delta(\mu-m)\, d\mu. \qquad \ldots(5.4)$$

Where $C^i$ and $D^i$ are real positive constants. Then equation (3.15) can be written as :

$$\Psi(\vec{x}) = C\int \frac{dp}{\omega(p)} \exp[i\{(\vec{p}.\vec{x})-\omega(p)t\}][\omega(p)f(p) - \frac{\vec{p}\{\vec{p}.f(p)\}}{\omega(p)+\mu} - i\{\vec{p}\times f(p)\}]$$

$$+C'\int \frac{dp}{\omega(p)} \exp[i\{(\vec{p}.\vec{x})-\omega(p)t\}][\omega(p)f'(p) - \frac{\vec{p}\{\vec{p}.f'(p)\}}{\omega(p)+\mu} - i\{\vec{p}\times f'(p)\}]$$

$$+D\int \frac{dp}{\omega(p)} \exp[-i\{\vec{p}.\vec{x}-\omega(p)t\}][\omega(p)h^*(p) - \frac{\vec{p}\{\vec{p}.h^*(p)\}}{\omega(p)+\mu} - i\{\vec{p}\times h^*(p)\}]$$

$$+D'\int \frac{dp}{\omega(p)} \exp[-i\{\vec{p}.\vec{x}-\omega(p)t\}][\omega(p)h^{*\prime}(p) - \frac{\vec{p}\{\vec{p}.h^{*\prime}(p)\}}{\omega(p)+\mu} - i\{\vec{p}\times h^{*\prime}(p)\}]$$

$$(5.5)$$

where $f(p) = f(m, p)$, $h(p) = h(m, p)$, $\omega(p) = \omega(m, p)$. Equation (5.5) gives general solution of the equation (5.2).

## 6. Calculation of the constants

If $H(x)$ is the Hamiltonian density of the field which leads to the wavefunction (5.2) then $H'(x) = -H(x)$ is also the Hamiltonian density which leads to the same wavefunction, and then the energy $E$ of the field is given by $E = \int H(x)\, dx$. We choose the constants of the equation (5.5) in such a manner that $E$ is always positive. We choose Hamiltonian density $H(x)$ for the cases,

(1) $f(p) = h(p) = h'(p) = 0$

(2) $f(p) = h(p) = f'(p) = 0$

(3) $f'(p) = h(p) = h'(p) = 0$

(4) $f(p) = f'(p) = h'(p) = 0$

$H(x)$ is defined as usual,

$$H(x) = \Psi^* \Psi + \nabla \Psi^* \nabla \Psi + m^2 \Psi \Psi^* \quad \ldots\ldots (5.6).$$

Then for the requirement that $E$ is always positive we have,

$$C' = D' = (2)^{-1/2} [2\pi]^{-3/2}$$

Then energies for all the four cases above are given respectively as follows,

(1) $E_f' = \int f^{*\prime}(p) f'(p)\, dp$

(2) $E_h' = \int h^{*\prime}(p) h'(p)\, dp$

(3) $E_f = \int f^*(p) f(p)\, dp$

(4 $E_h = \int h^*(p) h(p)\, dp$

When the state is the superposition of all the four modes then the total energy $E$ of the field is given by :

$$E = E_f + E_h + E_h' + E_f' \quad \ldots \quad \ldots \quad \ldots \quad (5.8)$$

## Referances

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