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On the unsteady hydromagnetic free convection flow past a vertical infinite flat plate

BY IOAN POP

University of Cluj, Cluj Roumania

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This paper considers the unsteady laminar free convection flow of viscous incompressible and conducting fluid past a vertical infinite flat plate whose temperature varies as some powers of time in the presence of a constant horizontal magnetic field.

INTRODUCTION

Gupta (1960) using the method of characteristic has studied the effect of a constant horizontal magnetic field on two dimensional unsteady laminar free convection flow past a vertical infinite flat plate for a stepwise change in the surface temperature. The purpose of this paper is to present analytical solutions for the same problem when the wall temperature varies as some powers of time.

BASIC EQUATIONS

Let us assume that the origin of the coordinate system is at the lowest point of a flat plate, the $\bar{x}$-axis being along the plate vertically upwards and the $\bar{y}$—axis perpendicular to it. A uniform magnetic field of strength $\bar{H}_o$ is applied perpendicular to the plate. Under this condition considering an infinite vertical plate, the unsteady hydromagnetic laminar free convection boundary-layer equations may be written as :

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial y^2} = \theta - mu, \qquad \ldots(1)$$

$$\frac{\partial \theta}{\partial t} = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial y^2}, \qquad \ldots(2)$$

where

$$\left.\begin{aligned} &y = \bar{y}/L\,,\ t = \nu\bar{t}/L\,,\ u=\bar{u}L/\nu\,, \\ &\theta = \beta g L^3\,(\,\bar{T} - \bar{T}_\infty\,)/\nu^2,\ m = \sigma \bar{B}_o^2 L^2/\rho\nu, \end{aligned}\right\} \qquad \ldots(3)$$

where $\bar{t}$ is the time variable, $\bar{u}$, the velocity component along the plate, $\bar{T}$, the temperature variable, $\bar{T}_\infty$, the ambient temperature, L, the characteristic length, $\bar{B}_o=\mu_o\bar{H}_o$, the magnetic induction, ν, the kinematic viscosity μ_o, the magnetic permeability, ρ, the density, g, the acceleration due to gravity, σ, the electrical conductivity, Pr, the Prandtle number, β, the coefficient of volumetric expansion and m, the hydromagnetic parameter.

Equations (1) and (2) are to be solved with the following initial and boundary conditions :

$$\left.\begin{aligned} &u\,(y,0) = \theta\,(y,0) = 0, \\ &u\,(0,t) = u\,(\infty,t) = \theta\,(\infty,t) = 0, \\ &\theta\,(0,t) = \theta_w\,(t) = a\,t^{\alpha}, \end{aligned}\right\} \qquad \ldots (4)$$

where θ_w is the prescribed value of θ at the surface, a, an arbitrary constant and α, any positive integer.

In (1) it is assumed that the magnetic Reynolds number is small, so that the induced magnetic field is negligible in comparison to the imposed magnetic field. Further, since no external electric field is applied, and the effect of polarization of ionized fluid is negligible, it can be assumed that the induced electric field is zero.

Solution of the governing equations

For small values of mt and for $Pr=1$ (it is a case frequently encountered in practical problems) we assume that the solution of the equations (1) and (2) is (Singh 1964) :

$$\theta = at^{\alpha}\zeta(\eta),\ u = ag\beta t^{\alpha+1} \sum_{n=0}^{\infty} fn(\eta)\,(mt)^n, \qquad \ldots (5)$$

where $\eta = y/2t^{1/2}$. Substituting (5) into (1) and (2) we get the equations :

$$\zeta'' + 2\eta\zeta' - 4\alpha\zeta = 0, \qquad \ldots (6)$$

$$f_0'' + 2\eta f_0' - 4(\alpha+1)\,f_0 = -4\zeta_0, \qquad \ldots (7)$$

$$f_1'' + 2\eta f_1' - 4(\alpha+2)\,f_1 = 4f_0, \qquad \ldots (8)$$

$$f_2'' + 2\eta f_2' - 4(\alpha+3)\,f_2 = 4f_1; \qquad \ldots (9)$$

where primes denote differentitation with respect to η. The corresponding boundary conditions of (6) — (9) are :

$$\left.\begin{aligned} &\zeta(0) = 1,\ \zeta(\infty) = 0, \\ &f_n(0) = f_n(\infty) = 0 \qquad (n \geqslant 0). \end{aligned}\right\} \qquad \ldots (10)$$

Case (*i*) : $\alpha = 0$ (step change in the surface temperature). The equation (6) becomes :

$$\zeta'' + 2\eta\zeta' = 0. \qquad \ldots (11)$$

Solution of (11) satisfying the boundary conditions (10) is given by (Carslaw & Jaeger 1948) :

$$\zeta(\eta) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-s^2}\,ds = erf_c\,\eta \qquad \ldots (12)$$

Now, the equation (7) becomes :

$$f_0'' + 2\eta f_0' - 4f_0 = -\,erf_c\,\eta. \qquad \ldots (13)$$

We now focus attention upon equation (13). We shall seek a particular solution of equation (13) by writing :

$$f_{op}(\eta) = X(\eta)\, erf_c\eta + Y(\eta), \qquad \ldots(14)$$

where the functions $X(\eta)$ and $Y(\eta)$ defined by equations :

$$\left.\begin{aligned} X'' + 2\eta X' - 4X &= -4, \\ Y'' + 2\eta Y' - 4Y &= \frac{4}{\sqrt{\pi}} X' e^{-\eta^2}, \end{aligned}\right\} \qquad \ldots(15)$$

are :

$$X(\eta) = -2\eta^2,\ Y(\eta) = \frac{2}{\sqrt{\pi}}\, \eta e^{-\eta^2}. \qquad \ldots(16)$$

Two particular solutions of homogeneous equation (13) are :

$$f_{oh}(\eta) = 1 + 2\eta^2,\ f_{oh}(\eta) = \frac{1}{4}(1 + 2\eta^2)\, erf_c\eta - \frac{2}{\sqrt{\pi}}\, \eta e^{-\eta^2}. \qquad \ldots(17)$$

The general solution of (13) is :

$$f_o(\eta) = C_1(1 + 2\eta^2) + C_2\left[\frac{1}{4}(1 + 2\eta^2)\, erf_c\eta - \frac{2}{\sqrt{\pi}}\, \eta e^{-\eta^2}\right] - 2\eta^2\, erf_c\eta + \frac{2}{\sqrt{\pi}}\, \eta e^{-\eta^2}. \qquad \ldots(18)$$

To determine the constants C_1 and C_2 we shall make use of the boundary conditions (10). We have $C_1 = C_2 = 0$ and therefore (18) becomes :

$$f_o(\eta) = -2\eta^2\, erf_c\eta + \frac{2}{\sqrt{\pi}}\, \eta e^{-\eta^2}. \qquad \ldots(19)$$

The solutions of the equations (8) and (9) will be determined in the same way as the solution of equation (13) and have the following form :

$$\left.\begin{aligned} f_1(\eta) &= -\frac{2}{3}\eta^4\, erf_c\eta + \frac{1}{3\sqrt{\pi}}(2\eta^3 - \eta)\, e^{-\eta^2}, \\ f_2(\eta) &= -\frac{4}{45}\eta^6\, erf_c\eta + \frac{2}{45\sqrt{\pi}}\left(2\eta^5 - \eta^3 + \frac{3}{2}\eta\right) e^{-\eta^2} \end{aligned}\right\} \qquad \ldots(20)$$

Case (ii) : α is any positive integer. In this case the solutions of the equations (6) — (9) satisfying the boundary conditions (10) are (Pop 1967, 1968) :

$$\left.\begin{aligned} \zeta(\eta) &= 2^{2\alpha}\Gamma(\alpha+1)g_\alpha(\eta), \\ f_o(\eta) &= 2^{2\alpha}\Gamma(\alpha+1)g_\alpha(\eta) - 2^{2(\alpha+1)}\Gamma(\alpha+2)g_{\alpha+1}(\eta), \\ f_1(\eta) &= 2^{2(\alpha+1)}\Gamma(\alpha+2)g_{\alpha+1}(\eta) - 2^{2\alpha-1}\Gamma(\alpha+1)g_\alpha(\eta) \\ &\quad - 2^{2\alpha+3}\Gamma(\alpha+3)g_{\alpha+2}(\eta), \\ f_2(\eta) &= 1/3\,[2^{2\alpha-1}\Gamma(\alpha+1)g_\alpha(\eta) - 2^{2\alpha+5}\Gamma(\alpha+4)g_{\alpha+3}(\eta)] \\ &\quad + 2^{2\alpha+3}\Gamma(\alpha+3)g_{\alpha+2}(\eta) - 2^{2\alpha+1}\Gamma(\alpha+2)g_{\alpha+1}(\eta), \end{aligned}\right\} \qquad \ldots(21)$$

where Γ is the symbol of the gamma function and

$$g_\alpha(\eta) = \frac{2^{1/2-\alpha}}{\sqrt{\pi}} e^{-1/2\eta^2} D_{-1-2\alpha}(\eta\sqrt{2}), \qquad \ldots(22)$$

$D_{-1-2\alpha}(\eta\sqrt{2})$ being the parabolic-cylinder function of order $-1-2\alpha$, for $\alpha \geqslant 0$ (Whittaker & Watson 1927). The properties of the function $g_\alpha(\eta)$ have already been discussed in detail by Watson (1955).

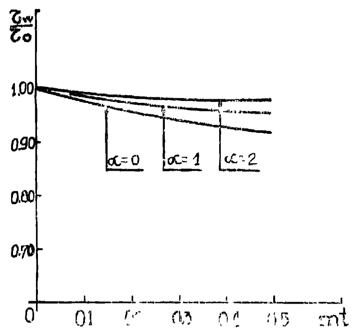

Figure 1. The variation of the ratio τ_w/τ_o with mt.

Now, we can calculate the skin friction at the plate. Thus,

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{ag\beta\rho}{2} t^{\alpha+1/2}\nu^{1/2}\Big[f_0'(0)+f_1'(0)(mt) + f_2'(0)(mt)^2+\ldots\Big]. \qquad \ldots(23)$$

In the absence of magnetic field the local skin friction at the plate is given by :

$$\tau_o = \frac{ag\beta\rho}{2} t^{\alpha+1/2}\nu^{1/2} f_0'(0). \qquad \ldots(24)$$

Considering (21) the ratio of the skin friction with and without the hydromagnetic interactions is given by :

$$\frac{\tau_w}{\tau_o} = 1 - \frac{1}{2(\alpha+3/2)}\left(\frac{mt}{2}\right) + \frac{1}{2(\alpha+3/2)(\alpha+5/2)}\left(\frac{mt}{2}\right)^2 + \ldots. \qquad \ldots(25)$$

The skin friction ratio τ_w/τ_o, is plotted against mt in figure 1 for $\alpha=0$, 1 and 2. It is clear that the effect of magnetic field is to decrease the skin friction. It is further noted that the skin friction increases when α increases.

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