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On initial development of axisymmetric waves due to sources

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An initial value investigation into the linearised problem of axisymmetric wave motions in a fluid of finite and infinite depth generated by a harmonically oscillating three dimensional source is made in this paper. An asymptotic analysis of the problem is carried out in some detail for a clear understanding of the steady state and transient solutions. The limiting behaviour of the asymptotic solution as time tends to infinity is given due attention.

1. Introduction

In recent years, an initial value investigation into the linearised wave problems dealing with the generation of surface waves in a fluid with a free surface by harmonically oscillating pressure distributions on the free surface and sources beneath the free surface of the fluid, has received considerable attention by Stokes (1957), Miles (1962), Debnath (1967, 1969) and others. Debnath has explained the difficulties of the several methods developed independently by Lamb (1905, 1923, 1932), Lighthill (1960, 1964) and Thorne (1953) in connection with the steady state wave problems. He suggested various reasons in favour of the initial value approach with a special emphasis that the most rigorous way of deriving the unique solution of wave problems, without the need for any of the essentially physical assumptions of this methods stated above.

The primary aim of this paper is to investigate an initial value approach to the linearised problem of axisymmetric wave motions in a fluid of limited and unlimited depth produced by a harmonically oscillating point source situated at a finite depth below the undisturbed free surface of the fluid. An asymptotic analysis of the problem is carried out in some detail for a clear understanding of the steady state and transient solutions. The limiting behaviour of the asymptotic solution as time tends to infinity has also been examined.

Thorne has considered the corresponding steady state problem and obtained a solution of physical interest by imposing the radiation condition at infinity. This solution is obtained as a limiting case of the initial value problem conidered here.

2. Mathematical Formulation of the Problem

We consider linearised problem of axisymmetric wave propagation in inviscid, incompressible and homogeneous fluid with a free surface

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(initially at rest) due to a harmonically oscillating point source of fixed frequency ω.

We fix the origin of co-ordinates at the source at a depth $\bar{D}$, below the undisturbed free surface of the fluid and take $X-Z$ plane to be horizontal passing through the origin and Y-axis vertical positive upward. We choose the cylindrical polar co-ordinates (R, θ, Y) and assume the cylindrical symmetry about the Y-axis such that R is equal to $\sqrt{X^2+Y^2}$.

As the motion is irrotational, there exists a wave potential $\Phi(R, Y, T)$ which satisfies the Laplace equation

$$\Phi_{RR}+\frac{1}{R}\Phi_R+\Phi_Y=0 \qquad \text{.. (2.1)}$$

$$0 \leqslant R < \infty,\ \bar{D}\text{-}h \leqslant Y \leqslant \bar{D}$$

everywhere within the fluid of depth h except at the source at (0, 0).

At $R=0,\ Y=0,\ \Phi$ has the form

$$\left.\begin{aligned} &= m\,M(R_1)\,e^{i\omega T} \\ &= m\,M(R,Y)\,e^{i\omega T} \end{aligned}\right\}\ T \geqslant 0, \qquad \text{.. (2.2)}$$

where m denotes the strength of the source with the frequency ω and $R_1^2=R^2+Y^2$.

The boundary conditions are given by

$$\left.\begin{aligned} \Phi_T+gE&=0 \qquad (2.3) \\ \Phi_Y&=E_T \qquad (2.4) \end{aligned}\right\}\ \begin{aligned} Y&=\bar{D} \\ T&>0 \end{aligned}$$

where, $E=E(R, T)$ represents the free surface elevation at a distance R and at time T, and g the gravitational acceleration.

The condition at the bottom boundary is given by

$$\Phi_Y=0 \quad \text{at} \quad Y=-(h-\bar{D}) \qquad \text{...(2.5)}$$

The initial conditions are given by

$$E(R, T)=0, \qquad \text{everywhere at } T=0 \qquad \text{...(2.6)}$$

$$\left.\begin{aligned} &\Phi=0, \qquad \text{everywhere except at (0, 0)} \\ &\qquad\qquad\ \text{at time } T=0 \\ &\Phi=m\,M\,(R_1)\,e^{i\omega T} \quad \text{at} \quad (0, 0),\ T \geqslant 0 \end{aligned}\right\} \qquad \text{.. (2.7)}$$

which are equivalent to

$$\Phi=m\,M(R_1)\,\delta(R)\,e^{i\omega T},\quad T \geqslant 0 \qquad \text{...(2.7)}$$

We complete the formulation of the initial value problem with a three dimensional source together with the further assumption that the functions Φ and E possess the Hankel transform with respect to R.

Remarks. The formulation of the correponding axisymmetric wave problem as a steady state considered by Thorne can be obtained from that of the initial value problem stated above, just by omitting the initial conditions (2.6) — (2.7). Thorne investigated the steady state problem and obtained a solution of physical interest by imposing the radiation condition at infinity

3. Formal Solution of the Problem

For simplicity, we introduce non-dimensional variables r, y, $\bar{d}$, t, r_1, η and $\tilde{\phi}$ defined by the relations

$$(r, y; \bar{d}, r_1) = \frac{\omega^2}{g}(R, Y, \bar{D}, R_1)$$

$$t = \omega T, \quad \phi = \frac{g}{m\omega^3}\Phi, \quad \eta = \frac{g^2 E}{m\omega^3},$$

and we introduce a non-dimensional parameter D by the relation.

$$D = \frac{\omega^2 h}{g}.$$

These relations enable us to rewrite the fundamental equations (2.1)—(2.7) into the form

$$\tilde{\phi}_{rr} + \frac{1}{r}\tilde{\phi}_r + \tilde{\phi}_{yy} = 0 \qquad \ldots(3.1)$$

$$0 \leqslant r < \infty, \qquad (\bar{d} - D) \leqslant y \leqslant \bar{d}$$

everywhere within the fluid except at (0, 0). At $r = 0$, $y = 0$, $\tilde{\phi}$ has the form

$$\tilde{\phi} = M(r_1)e^{it} \qquad t \geqslant 0 \qquad \ldots(3.2)$$

The boundary conditions reduce to

$$\left.\begin{array}{ll} \tilde{\phi}_t + \eta = 0 & (3.3) \\ \tilde{\phi}_y = \eta_t & (3.4) \end{array}\right\} \quad y = \bar{d}, \qquad t > 0$$

The condition at the bottom is then

$$\tilde{\phi}_y = 0 \quad \text{at} \quad y = \bar{d} - D \qquad \ldots(3.3)$$

The initial conditions are given by

$$\eta(r, t) = 0, \quad \text{everywhere at} \quad t = 0 \qquad \ldots(3.6)$$

$$\left.\begin{array}{ll} \tilde{\phi} = 0 & \text{everywhere except at } (0, 0) \\ & \text{at time } t = 0 \\ \tilde{\phi} = e^{it} M(r_1) & \text{at } (o, o),\ t \geqslant 0 \end{array}\right\} \qquad \ldots(3.7)$$

which are equivalent to

$$\tilde{\phi} = M(r_1)\delta(r)e^{it} \quad \text{at} \quad t \geqslant 0 \qquad \ldots(3.8)$$

Now we introduce a bounded expression

$$\phi = \tilde{\phi} - e^{it}M(r_1) \qquad \ldots(3\,9)$$

for all r, y and t.

Making reference to this relation (3.9), equations (3.1) – (3.7) can further be put into the form

$$\phi_{r\,r} + \frac{1}{r}\phi_r + \phi_{y\,y} = 0 \qquad \ldots(3.10)$$

$$0 \leqslant r < \propto, \bar{d} - D \leqslant y \leqslant \bar{d}$$

$$\phi_t + \eta = -\,ie^{it}M(r_1) \qquad (3.11)\ y = \bar{d}$$

$$\phi_y - \eta_t = -\,e^{it}\frac{\partial}{\partial y}M(r_1) \qquad (3.12)\ t \geqslant 0$$

$$\phi_y = -\,e^{it}\frac{\partial}{\partial y}M(r_1) \quad \text{at } y = (\bar{d} - D) \qquad \ldots(3.13)$$

$$\eta = 0 \qquad \text{everywhere at } t = 0 \qquad \ldots(3.14)$$

$$\left.\begin{array}{ll} \phi = -\,e^{it}M(r_1) & \text{everywhere except} \\ & \quad \text{at } (0, 0) \text{ at } t = 0 \\ \phi = 0 & \quad \text{at } (0, 0) \quad t \geqslant 0 \end{array}\right\} \qquad \ldots(3.15)$$

We introduce the Laplace thansforms $\bar{\phi}$, $\bar{\eta}$ of ϕ, η, respectively, with respect to t by the integral like

$$\bar{\phi} = \bar{\phi}(r,y\,;s) = \int_0^{\propto} e^{-st}\phi\,(r, y\,;t)\,dt$$

We next introduce the Hankel transforms $\bar{\bar{\phi}}$, $\bar{\bar{\eta}}$ of functions $\bar{\phi}$, $\bar{\eta}$, respectively, with respect to r by the integral like

$$\bar{\bar{\phi}} = \bar{\bar{\phi}}\,(k, y\,;s) = \int_0^{\infty} r\,J_0\,(kr)\,\bar{\phi}\,(r, y\,;s)\,dr$$

The joint Laplace and Hankel transforms enable us to transform equations (3.10)—(3.15) into their equivalent forms as

$$\bar{\bar{\phi}}_{y\,y} = k^2\,\bar{\bar{\phi}}\,;\ 0 \leqslant k < \propto,\ (\bar{d} - D) \leqslant y \leqslant \bar{d} \qquad \ldots(3.16)$$

$$\left.\begin{array}{ll} s\,\bar{\bar{\phi}} + \bar{\bar{\eta}} + \dfrac{s}{s-i}\,M_1 = 0 & \quad ..\ (3.17) \\[2ex] \bar{\bar{\phi}}y - s\,\bar{\bar{\eta}} + \dfrac{M_2}{s-i} = 0 & \quad ..\ (3.18) \end{array}\right\} \begin{array}{l} y = \bar{d} \\[2ex] s > 0 \end{array}$$

where M_1, M_2 are given by the integrals

$$\left.\begin{array}{l} M_1 = M_1\,(k,\bar{d}) = \displaystyle\int_0^\infty r\,J_0\,(kr)\,M\,(r,\bar{d})\,dr \\[3ex] M_2 = M_2\,(k,\bar{d}) = \displaystyle\int_0^\infty r\,J_0\,(kr)\,\frac{\partial M}{\partial y}\,(r,\bar{d})\,dr \end{array}\right\} \quad ..\ (3.19)$$

$$\bar{\bar{\phi}}_y = -\frac{M_2(k, -y)}{(s-i)} \quad \text{at } y = (\bar{d} - D) \qquad (3.20)$$

$$\bar{\bar{\eta}} = 0 \text{ everywhere at } s = 0 \qquad ..\ (3.21)$$

$$\left.\begin{array}{ll} \bar{\bar{\phi}} = -\dfrac{M_1}{(s-i)} & \text{except at } k = 0,\ y = 0,\ s = 0 \\[2ex] \bar{\bar{\phi}} = 0 & \text{at } (0,0) \quad s \geqslant 0 \end{array}\right\} \quad ..\ (3.22)$$

The solution of equation (3.16) with the boundary conditions (3.17), (3.18) and (3.20) can be obtained in the form

$$\bar{\bar{\phi}}(k, y\,;s) - \frac{M_2(k, D-\bar{d})}{k(s-i)} e^{k(\bar{d}-y-D)}$$

$$+\left[M_2(k,\ D-\bar{d})e^{-kD}\left(1-\frac{s^2}{k}\right) - \left\{M_2(k,\bar{d}) + s^2M_1(k,\bar{d})\right\}\right]$$

$$+\frac{\cosh k(y-\bar{d}+D)}{(s-i)(s^2+\alpha^2)\cosh kD} \qquad ...(3.23)$$

And the expression for $\bar{\bar{\eta}}(k, s)$ is given by

$$\bar{\bar{\eta}}(k,s) = \frac{s\left[M_2(k,d) - \alpha^2 M_1(k,\bar{d}) - \left(1+\frac{\alpha^2}{k}\right)M_2(k, D-\bar{d})e^{-kD}\right]}{(s-i)(s+\alpha^2)} \qquad ...(3.24)$$

$$\alpha^2 = \alpha^2(k) = k \tanh kD \qquad ...(3.95)$$

Using the inversion theorem for the Laplace and Hankel transforms combined with the convolution theorem for the Laplace transform, we obtain the wave potential $\phi(r, y\,; t)$ in the form

$$\phi(r, y\,;t) = \int_0^\infty \Big[M_2(k, D - \bar{d}) e^{k(d-y-D)+it}$$

$$+ \frac{\cosh k(y - \bar{d} + D)}{(a^2 - 1)\cosh kD} \Big\{ (k + a^2) \frac{e^{-kD}}{k} M_2(k, D - \bar{d})$$

$$+ a^2 M_1(k, \bar{d}) - M_2(k, \bar{d}) \Big\} k e^{it}$$

$$- \frac{\cosh k(y - \bar{d} + D)}{\cosh kD} \Big\{ \frac{e^{-kD}}{k} M_2(k, D + \bar{d}) + M_1(k, \bar{d}) \Big\} k e^{it}$$

$$+ \frac{\cosh k(y - \bar{d} + D)}{(a^2 - 1)\cosh kD} \Big\{ \Big(\frac{1}{a} + \frac{a}{k} \Big) e^{-kD} M_2(k, D - \bar{d})$$

$$+ a M_1(k, \bar{d}) - \frac{M_2(k, \bar{d})}{a} \Big\} (i \sin at + a \cos at) \Big] J_0(kr)\, dk \qquad \ldots(3.26)$$

This is a general expression for the wave potential $\phi(r, y\,;t)$.

Similarly, we can derive the expression for the surface elevation $\eta(r, t)$ as

$$\eta(r, t) = \int_0^\infty \Big[\Big[\Big\{ M_2(k, \bar{d}) - a^3 M_1(k, \bar{d})$$

$$- \Big((1 + \frac{a^2}{k} \Big) e^{-kD} M_2(k, D - \bar{d}) \Big\} i e^{it}$$

$$+ \Big\{ M_2(k, \bar{d}) - a^2 M_1(k, \bar{d}) - \Big(1 + \frac{a^2}{k} \Big) e^{-kD} M_2(k, D - \bar{d}) \Big\}$$

$$(a \sin at - i \cos at) \Big] k (a^2 - 1)^{-1} J_0(kr)\, dk \qquad \ldots(3.27)$$

This is a general integral representation for the surface elevation $\eta(r, t)$.

We next derive the integral form of the wave potential $\phi(r, y\,;t)$ as well as the surface elevation $\eta(r, t)$ in case of a fluid of *infinite* depth (*i.e.* when $D \to \infty$) as

$$\phi(r, y\,;t) = \int_0^\infty k(k - 1)^{-1} e^{k(y-d)} J_0(kr)$$

$$(M_1 - M_2) e^{it} + (M_2 - k M_1) (\cos \sqrt{k}t + \frac{1}{\sqrt{k}} \sin \sqrt{k}\, t) \qquad \ldots(3.28)$$

$$\eta(r, t) = \int_0^\infty (M_2 - k M_1)(i e^{it} - i \cos \sqrt{k}\, t + \sqrt{k} \sin \sqrt{k}\, t)$$

$$\times\, k(k - 1)^{-1} J_0(kr)\, dk \qquad \ldots(3.29)$$

Remarks : The integral representation of the solution for the wave potential $\phi(r, y\,; t)$ and the surface elevation $\eta(r, t)$ in a fluid of finite and infinite depth cannot, in general, be worked out exactly except for simple cases of interest. Hence one needs asymptotic methods (Copson 1965) to evaluate them for a clear understanding of the wave motions. We propose to do it in the next section.

4. Asymptotic Treatment of the Problem for Sources of Physical Interest.

An important three dimensional source of independent interest related to a particular form $M(r_1)$ as

$$M(r_1) = M(r, y) = \frac{1}{r_1} = \frac{1}{\sqrt{r^2 + y^2}}$$

would be considered.

Then

$$M_1 = \int_0^\infty \frac{rJ_0(kr)dr}{\sqrt{r^2 + \bar{d}^2}} = \frac{e^{-k\bar{d}}}{k}$$

$$M_2 = -\int_0^\infty rJ_0(kr)\,\frac{\bar{d}}{(r^2 + \bar{d}^2)^{3/2}}\,dr = -\,e^{-k\bar{d}}\,,$$

where $\bar{d} > 0$.

These lead us to obtain the wave potential $\phi(r, y : t)$ in the form

$$\phi(r, y;\, t) = \int_0^\infty \Big[e^{it}\, e^{k(2\bar{d} - 2D - y)}$$

$$+ \frac{\cosh k(y - \bar{d} + D)}{\cosh kD} \left\{ e^{-k\bar{d}} - e^{-k(2D - \bar{d})} \right\} \left(\frac{k+1}{a^2 - 1} \right) e^{it}$$

$$+ \frac{k \cosh k(y - \bar{d} + D)}{(a^2 - 1)\cosh kD} \left\{ e^{-k(2D - \bar{d})} - e^{-k\bar{d}} \right\} \left(\frac{1}{a} + \frac{a}{k} \right)$$

$$\times k\,(i \sin at + a \cos at) \Big] J_0(kr)\,dk \qquad \ldots(4.1)$$

Similarly, we find

$$\eta(r, t) = \int_0^\infty \left\{ e^{-k(2D - \bar{d})} - e^{-k\bar{d}} \right\}$$

$$\times \{i\, e^{it} + (a \sin at - i \cos at)\}(a^2 + k)(a^2 - 1)^{-1} J_0(kr)\,dk \qquad \ldots(4.2)$$

Making $D \to \infty$, the corresponding result for $\phi(r, y\,; t)$ and $\eta(r, t)$ in the case of infinite depth can be obtained as

$$\phi(r, y ; t) = \int_0^\infty e^{k(y-i\bar{d})} \left[(1+k)e^{it} - 2(k \cos \sqrt{k}t + i \sqrt{k} \sin \sqrt{k}t) \right] \times (k-1)^{-1} J_0(kr)dk \quad \ldots(4.3)$$

$$\eta(r, t) = 2 \int_0^\infty ke^{-k\bar{d}}(ie^{it} - i \cos \sqrt{k}t + \sqrt{k} \sin \sqrt{k}t) \times (1-k)^{-1} J_0(kr)dk \quad \ldots(4.4)$$

Remarks : We thus obtain

$$\tilde{\phi}(r, y ; t) = M(r_1)e^{it} + \phi(r, y ; t) \qquad . (4.5)$$

where $\phi(r, y ; t)$ is given by the integral (4.1) or (4.3) according as the depth of the fluid is finite or infinite.

This integral expression for $\phi(r, y ; t)$ contains a transient term in addition to a steady state term which is the solution of the corresponding stationary problem considered by Thorne. In order to compare Thorne's steady state solution, one has to treat the integral for $\phi(r, y ; t)$ as the Cauchy principal value, which is permissible.

It may be noticed that the integral for $\phi(r, y ; t)$ has no singularities in $(0, \infty)$. Hence the path of integration can be deformed into a path M (say) in the $s = k + i\mu$ plane, which coincides with the range $(0, \infty)$ except that it is diverted round the zero of the denominator. We then break up the integral into a sum of components where the integrals do become singular at the zero of the denominator.

Then it is possible to work out each component asymptotically by asymptotic methods combined with calculus of residues. Unfortunately, Thorne did not evaluate the steady state wave integral obtained as a solution. We propose to evaluate the solution for $\eta(r, t)$ asymptotically in a considerable detail.

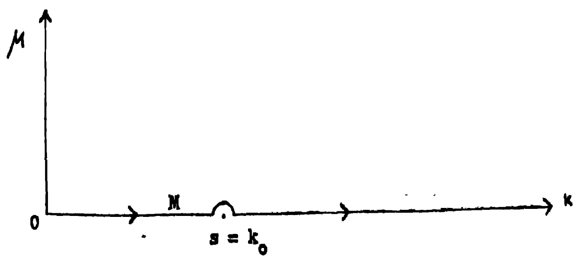

Figure 1. The $s = k + i$ plane.

An argument similar to the wave potential $\phi(r, y\,; t)$ enables us to obtain $\eta(x, t)$ in the form

$$\eta(r, t) = I_1 + I_2$$

where I_1 and I_2 are given by the integrals

$$I_1 = ie^{it} \int_M \left\{ e^{-s(2D-\bar{d})} - e^{-s\bar{d}} \right\} \left(\frac{a^2 + s}{a^2 - 1} \right) J_0(sr)\,ds$$

$$I_2 = \int_M \left\{ c^{-s(2D-\bar{d})} - e^{-s\bar{d}} \right\} (a \sin \alpha t - i \cos \alpha t)$$

$$\times \left(\frac{a^2 + s}{a^2 - 1} \right) J_0(sr)\,ds.$$

With $a = a(s) = \sqrt{s \tanh sD}$, $s = k_0$ is the only real root of the equation

$$a^2(s) = 1$$

in $(0, \infty)$ and $-\pi < \arg s \leqslant \pi$.

To evaluate the steady state integral I_1, we replace $J_0(sr)$ by a pair of Hankel functions (Whittaker & Watson 1920), $H_0^{(1)}(sr)$ and $H_0^{(2)}(sr)$. As a consequence, we obtain

$$I_1 = \tfrac{1}{2} e^{it} (I_1' + I_1'')$$

where I_1' and I_1'' are given by

$$I_1' = \int_M \left\{ e^{-s(2D-\bar{d})} - e^{-s\bar{d}} \right\} \left(\frac{a^2 + s}{a^2 - 1} \right) H_0^{(1)}(sr)\,ds$$

$$I_1'' = \int_M \left\{ e^{-s(2D-\bar{d})} - e^{-s\bar{d}} \right\} \left(\frac{a^2 + s}{\alpha^2 - 1} \right) H_0^{(2)}(sr)\,ds$$

We take contours Γ_1 and Γ_2 for the integrals I_1' and I_1'', respectively. They are bounded by the path M, μ-axis and the circular arcs C_1, C_2 lying in the first and the fourth quadrants, respectively. We then make reference to Cauchy's theorem of residues, and it follows from partial integration that the integrals along the μ axis are $0\left(\frac{1}{r}\right)$

For evaluating the integrals along the arcs C_1, C_2, we replace the Hankel functions by their asymptotic value for large sr and it can be shown easily that the value of the integrals tend to zero as the radii of the arcs tend to infinity.

Thus, it turns out that

$$I_1 \sim \frac{\{k_0 + \alpha^2 (k_0)\}}{W'(k_0)} \left\{ e^{-k_0(2D-\bar{d})} - e^{-k_0\bar{d}} \right\} \left(\frac{2\pi}{r k_0} \right)^{\frac{1}{2}}$$

$$\times\, e^{i(t - r k_0 + \frac{\pi}{4})} + 0\left(\frac{1}{r} \right) \qquad \text{.. (4.6)}$$

where the function $W(s)$ is given by

$$W(s) \equiv \alpha^2 (s) - 1. \qquad \text{..(4.7)}$$

In order to perform evaluation of the integral I_2, we first replace the Bessel function $J_0 (sr)$ by its integral formula (Whittaker & Watson, 1920).

$$J_0 (sr) = \frac{2}{\pi} \int_0^{\pi/2} \cos (sr \cos \theta)\, d\theta.$$

Then it follows, by a simple rearrangement of the integrand, that

$$I_2 = \frac{i}{2\pi} \int_0^{\pi/2} (L_1 + L_2 + L_3 + L_4)\, d\theta, \qquad \text{...(4.8)}$$

where L_1, L_2, L_3 and L_4 are given by the integrals

$$L_1 = \int_M \left\{ e^{-s(2D-\bar{d})} - e^{-s\bar{d}} \right\} \left(\frac{\alpha^2 + s}{\alpha - 1} \right) e^{-i(\alpha t - sr\cos\theta)}\, ds$$

$$L_2 = \int_M \left\{ e^{-s\bar{d}} - e^{-s(2D-\bar{d})} \right\} \left(\frac{\alpha^2 + s}{\alpha - 1} \right) e^{i(\alpha t + sr\cos\theta)}\, ds$$

$$L_3 = \int_M \left\{ e^{-s\bar{d}} - e^{-s(2D-\bar{d})} \right\} \left(\frac{\alpha^2 + s}{\alpha - 1} \right) e^{i(\alpha t + sr\cos\theta)}\, ds$$

$$L_4 = \int_M \left\{ e^{-s(2D-\bar{d})} - e^{-s\bar{d}} \right\} \left(\frac{\alpha^2 + s}{\alpha + 1} \right) e^{-i(\alpha t + sr\cos\theta)}\, ds$$

It may be observed that these transient integrals are very much similar to those already encountered before in Debnath (1967) and Debnath & Rosenblat (1969). Hence a similar asymptotic technique can be applied to evaluate them. Having done this, we work out the θ- integral involved in the integral I_2 by the method of stationary phase (Copson 1965) for large values of t.

Finally, we obtain the following asymptotic representation of the surface elevation $\eta (r, t)$

$$\eta(r,t) \sim \left(\frac{2\pi}{rk_0}\right)^{\frac{1}{2}} \frac{\{\alpha^2(k_0)+k_0\}}{W'(k_0)} \left\{ e^{-k_0(2D-\bar{d})} - e^{-k_0\bar{d}} \right\}$$

$$\times \left[\begin{array}{ll} e^{i(t-rk_0+\frac{\pi}{4})} & , k_1 > k_0 \\ e^{i(t-rk_0+\frac{\pi}{4})} - \frac{1}{2}\left\{1+\alpha(k_0)\right\} e^{i\{t\alpha(k_0)-rk_0+\frac{\pi}{4}\}} & , k_1 < k_0 \end{array} \right]$$

$$+ \frac{\{\alpha^2(k_1)+k_1\}\left\{ e^{-k_1\bar{d}} - e^{-k_1(2D-\bar{d})} \right\}}{\{4 rt\, k_1 \,|\, f_-''(k_1) \,|\,\}^{\frac{1}{2}}}$$

$$\times \frac{e^{-\{rk_1 - t\alpha(k_1)\}}}{\alpha(k_1)-1} - \frac{e^{i\{rk_1 - t\alpha(k_1)\}}}{\alpha(k_1)+1} \qquad \ldots(4.9)$$

In the case of a fluid of *unlimited* depth (i. e. when $D \to \infty$), the asymptotic representation of the surface elevation is given by

$$\eta(r,t) \sim \left[\begin{array}{ll} -2\,\frac{2\pi}{r}\, e^{i(t-r+\frac{\pi}{4})} & , t \gg 2r \\ 0 & , t \ll 2r \end{array} \right] .$$

$$- \frac{t^2 e^{-\frac{\bar{d}t^2}{4r^2}}}{\sqrt{2}\, r^3 \left(\frac{t^2}{4r^2}-1\right)} \left[\frac{t}{2r} \sin\left(\frac{t^2}{4r}\right) - i \cos\left(\frac{t^2}{4r}\right) \right] \qquad (4.10)$$

Remarks : It may be remarked that the solutions (4.9) and (4.10) become invalid at $k_1 = k_0$ and $t = 2r$, respectively. We are particularly interested in the asymptotic solution for large values of $k_1 > k_0$ and $t \gg 2r$. So it appears to us that the computation of the solution for $\eta(r,t)$ valid at $k_1 = k_0$ and $t = 2r$ is not so important in the present analysis. However, it can be done by a method similar to Wurtele (1955).

5. Discussion of the Wave Motions

The above asymptotic analysis reveals an interesting conclusion that the transient term involved in the asymptotic solution for the surface elevation $\eta(r,t)$ *does* tend to zero as t tends to infinity for fixed values of r and $\bar{d}$ ($\neq 0$). As a consequence, an ultimate steady state is set up. In fact, the asymptotic value of $\eta(r,t)$ assumes the form

$$\eta(r,t) \sim -2\sqrt{\frac{2\pi}{r}}\; e^{i\left(t-r+\frac{\pi}{4}\right)-\bar{d}} .$$

This corresponds to progressive circular waves advancing with the phase velocity $\frac{\sigma}{\omega}$ and the group velocity $\frac{g}{2\omega}$, and the amplitude of the waves decays like $r^{-\frac{1}{2}}$.

On the other hand, when the source is on the free surface of the fluid (*i. e.* when $\bar{d} \to 0$), the asymptotic solution for $\eta(r, t)$ has the form

$$\eta(r,t) \sim \begin{bmatrix} -2\sqrt{\frac{2\pi}{r}}\, e^{i(t-r+\frac{\pi}{4})} & , \quad t \geqslant 2r \\ 0 & , \quad t \leqslant 2r \end{bmatrix}$$

$$-\frac{t^2\left[\frac{t}{2r}\sin\left(\frac{t^2}{4r}\right) - i\cos\left(\frac{t^2}{4r}\right)\right]}{\sqrt{2}r^3\left(\frac{t^2}{4r}-1\right)} + 0\left(\frac{1}{t^3}\right)$$

This solution suggests that the transient term is now free from the exponential factor $e^{-\bar{d}t^2/4r^2}$ and hence it does not tend to zero in the limit $t \to \infty$ for fixed r. In other words, the solution does not tend to the steady state in the limit when the source is situated on the undisturbed free surface of the fluid.

Furthermore, it may be observed that the nature of this asymptotic solution has a similarity with that obtained in Debnath (1967a) and Debnath (1967b) due to a harmonically oscillating pressure distribution with the forcing frequency ω in the form

$$P(R,T) = P\frac{\delta(R)}{R}\, e^{i\omega T} H(T)$$

acting on the undisturbed free surface of the fluid. To explain the strange character of the solution, physical and mathematical arguments similar to those suggested by Debnath (1968, 1967) in detail can also be advanced here. To avoid duplication of similar discussion, reference may be made to the above works of the author.

Next, proceeding to the limit $r \to \infty$, for fixed t, the solution for $\eta(r,t)$ given in (4.10) behaves as

$$\eta(r,t) \sim = \frac{t^2\, e^{-\bar{d}t^2/4r^2}}{\sqrt{2}\, r^3\left(\frac{t^2}{4r^2}-1\right)}\left[\frac{t}{2r}\sin\left(\frac{t}{4r}\right) - i\cos\left(\frac{t^2}{4r}\right)\right]$$

Finally, if the source is situated at an infinite depth (*i.e.* when $d \to \infty$) in an infinitely deep fluid, the solution for the surface elevation $\eta(r,t)$ becomes exponentially small as really expected,

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