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Mean scattering cross-section of radiation during diffusion

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In this paper, a general discussion for mean scattering cross-section for radiation during diffusion is presented. Necessary general formulas are derived for mean number of scatterings and application is made to one dimensional medium in which radiation of constant frequency undergoes diffusion.

Introduction

A knowledge of the mean time for which radiation is present in a medium is needed to determine the time for the establishment of radiative equilibrium in the medium. Determination of this mean time depends on a knowledge of the mean number of scatterings of the radiations in the medium. No work has been done towards the determination of the mean number of radiation scattered in a homogeneous medium during their diffusion. For this purpose in this paper necessary general formulae have been derived and application has been made to a one dimensional medium in which monoenergetic radiation undergo diffusion.

Let us consider (Sobolev 1956) a homogeneous medium which can absorb a radiation incident at any point on it and can scatter the incident radiation with a probability p, which is a constant throughout the medium under consideration, filling a volume v.

If Q is taken as the probability that the radiation absorbed at a particular point during diffusion will leave the medium. Then 1-Q is the probability that it will be absorbed. This probability Q is a function of the shape, optical dimension of the given volume v, parameter p, and of the coordinates of the point at which the original radiation-absorption took place. Let the mean number of scatterings of a radiation coming out of the medium be denoted by ${}^{\sigma}p_1$ and let the mean number of scatterings of an annihilated radiation during diffusion be denoted by ${}^{\sigma}p_2$.

Then the mean number of scatterings of radiation ${}^{\sigma}p = {}^{\sigma}p_1 Q + {}^{\sigma}p_2 (1-Q)$ ...(1)

An expansion of Q as a power series in p yields :

$$Q = pQ_1 + p^2Q_2 + p^3Q_3 + \qquad ...(2)$$

These terms on the right hand side successively give the probability of a radiation going out of the medium after first, second, third scatterings, etc. We can then write

$$ {}^{o}p_1 = \frac{1}{Q}\,(pQ_1 + 2p^2Q_2 + 3p^3Q_3 + 4p^4Q_4 + \,..\,) \qquad ...(3) $$

Equation (2) gives us ${}^{o}p_1 = p\,\dfrac{\partial \ln Q}{\partial p}$. ...(4)

Writing $\sigma p = 1 + p(1 - Q_1) + p^2(1 - Q_1 - Q_2) + \cdots,$ .. (5)
we get mathematical expectation of first, second, third scatterings, etc., given successively by the first, second, third, etc., terms on the right hand side of equation (5).

But $Q_1 + Q_2 + Q_3 + \ldots\ldots\ldots = 1$

Combining this with equation (2) we can write

$$ 1 - Q = (1-p)Q_1 + (1-p)^2Q_2 + (1-p)^3Q_3 + (1-p)^4Q_4 + \ldots.. $$

or $1-Q = (1-p)\,\{Q_1+(1+p)\,Q_2+(1+p+p^2)\,Q_3 + \ldots..\}$

$$ = (1-p)\,\{1 + p\,(1-Q_1) + p^2\,(1-Q_1-Q_2) + \cdots.\} \qquad (6) $$

A comparison between (5) and (6) gives

$$ 1-Q={}^{o}p\,(1-p) $$

Therefore

$$ {}^{o}p = \frac{1-Q}{1-p} \qquad ...(7) $$

Substituting (7) and (4) in (1) we have

$$ {}^{o}p_2 = \frac{1}{1-p} + p\,\frac{\partial \ln(1-Q)}{\partial p} \qquad ...(8) $$

Using (4), (7), and (8) we have

$$ {}^{o}p_1 = \frac{P\,{}^{o}p}{1-(1-p)^{o}p}\left[1-(1-p)\,\frac{\partial \ln {}^{o}p}{\partial p}\right] \qquad ...(9) $$

and

$$ {}^{o}p_2 = 1 + p\,\frac{\partial \ln {}^{o}p}{\partial p} \qquad ...(10) $$

Equation (9) and (10) show that a knowledge of ${}^{o}p$ gives us ${}^{o}p_1$ and ${}^{o}p_2$. An examination of (7) shows that for pure scattering ${}^{o}p$ is indeterminate for pure scattering $p = 1$; expansion gives therefore ${}^{o}p = \dfrac{\partial Q}{\partial p}$,

To consider various sources of radiation, let Q^+ stand for number of scatterings of photons in the entire medium due to any radiation source. Let β be the volume absorption co-efficient in the medium and $f\beta dv$ be the number of photons from the sources of radiation absorbed in the volume dv. f, here can be due to radiation sources outside and inside the medium.

Then total number of radiation absorbed in the medium $= \int f\beta dv$ and the fraction coming out of the medium is given by

$$\frac{\int Qf\beta dv}{\int f\beta dv} \quad \text{...(11)}$$

Let ${}^{o}p_1{}^{+}$ and ${}^{o}p_2{}^{+}$ stand for mean number of scatterings experimenced by photons which came out and those which are annihilated, respectively. Then the mean number of scatterings undergone by all the photons absorbed in the medium

$$^{o}p^{+} = \frac{\int (1-Q) f\beta dv}{(1-p)\int f\beta dv} \quad \text{...(12)}$$

or

$$^{o}p^{+} = \frac{1}{1-p}[1 - \int Qf\beta dv / \int f\beta dv]$$

Using $$\frac{Qf\beta dv}{f\beta dv} = Q^{+}$$

we can write

$$^{o}p^{+} = \frac{1-Q^{+}}{1-p} \quad \text{...(13)}$$

Then

$$^{o}p_1{}^{+} = \frac{p\partial \ln Q^{+}}{\partial p} \quad \text{...(14)}$$

$$^{o}p_2{}^{+} = \frac{1}{1-p} + p\,\frac{\partial \ln(1-Q^{+})}{\partial p} \quad \text{...(15)}$$

f, in equation (11) can be simply written as $\frac{P}{4\pi}f = F$, if the sources of radiation are situated inside the medium and if these sources radiate isotropically, then $F\beta dv$ gives the number of photons emitted by the element of volume dv per unit solid angle. So we can write

$$Q^{+} = \frac{\int QF\beta dv}{\int F\beta dv} \quad \text{...(16)}$$

Here we have this equation in place of equation (11). Q^{+} is the fraction of photons which emerge from the medium for the stipulated radiation sources.

Application to one dimensional semi-infinite medium

During an elementary event, let us assume that the radiation, are emitted with equal probability in either direction with constant frequency.

*The function Q was used before to express number of scatterings of radiation absorbed at a particular point in the medium.

Let $Q(h)$ be the probability of emergence of a radiation absorbed in the medium at the depth h.

Then $Q(h) = \frac{p}{2}\int_0^\infty - e^{(h-t)} Q(t)dt + \frac{p}{2} e^{-h}$ .. (17)

Solution to equation (17) takes the form : $Q(h) = (1 - k)e^{-kh}$ .. (18)

where $k = \sqrt{1-p}$

Substituting (18) into (7), (4), and (8) we have

$$ {}^{\sigma}p\,(h) = 1 - (1-k)e^{-kh} \quad ...(19)$$

$$ {}^{\sigma}p_1(h) = \frac{1}{2k}(1 + k + ph) \quad ...(20)$$

and

$$ {}^{\sigma}p_2(h) = \frac{1}{1-p} - \frac{p}{2k}[1 + (1-k)h]\frac{e^{-hk}}{1-(1-k)e^{-kh}} \quad ...(21)$$

When radiation absorption takes place at the boundary of the medium we have for $h = 0$,

$$ \sigma p(0) = \frac{1}{(1-p)1/2} \quad ...(22)$$

$$ {}^{\sigma}p_1(0) = \frac{1}{2}\left(\frac{1}{(1-p)^{1/2}} + 1\right) \quad ...(23)$$

$$ {}^{\sigma}p_2(0) = \frac{1}{1-p}\left(1 - \frac{p}{2}\right) \quad ...(24)$$

We have for $p < 1$ and h quite large from equations (19) and (21),

$$ {}^{\sigma}p(h) \simeq {}^{\sigma}p_2(h) \simeq \frac{1}{1-p} \quad ...(25)$$

and equation (20) gives ${}^{\sigma}p_1\,(h) \simeq \frac{ph}{2\sqrt{1-p}}$ ...(26)

Substitution of numerical values for p in equations (22), (23), and (24) will produce considerably different values for ${}^{\sigma}p_1$ (o), the mean number of scatterings for radiation going out of the medium, for ${}^{\sigma}p_2$(o) the mean number of annihilated photons, and for ${}^{\sigma}p$(o) the total number of photons in question. Again the last equation shows that the mean number of scatterings of radiation which undergoes absorption at large h and which emerges from the medium is proportional to the depth h. Again the mean number of scatterings for radiation which emerges from the medium will be much greater than the annihilated radiation for extremely large values of h.

For any source of radiation : Let the sources of radiation be located outside the medium and let radiation of intensity I_0 be incident on the boundary of this medium. We can write $f(h) = I_0 e^{-h}$ and

$$Q^+ = \int_0^\infty Q(h)\, e^{-h}\, dh = \frac{1-k}{1+k} \qquad \ldots(27)$$

Substitution of (27) into (13), (14), and (15) gives

$$^{\sigma}p^+ = \frac{2}{p}\left(\frac{1}{(1-p)^{1/2}} - 1\right) \qquad \ldots(28)$$

$$^{\sigma}p^{\,+} = \frac{1}{(1-p)^{1/2}} \qquad \ldots(29)$$

and

$$^{\sigma}p_2^+ = \frac{1+(1-p)^{1/2}}{2(1-p)} \qquad \ldots(30)$$

These equations show that for pure scattering, i.e. for $p = 1$, the mean number of scatterings for radiation or for photons in general in a semi-infinite medium is very great. This is true for any radiation sources.

Reference

Sobolev V. V. 1956 *Radiative Transfer in Stellar and Planetary Atmospheres, Moscow.*