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Laminar flow of two incompressible immiscible fluids between two parallel plates with suction and injection.

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The paper is devoted to a study of laminar flow of two viscous incompressible immiscible fluids occupying equal heights between two parallel porous plates with suction at the upper plate and an equal injection at the lower plate under the action of constant pressure gradient. The effects of suction and injection on flow field have been investigated and compared with those cases where no suction and injection are present. A critical value of the ratio of suction/injection Reynold's numbers of the two fluids for which there is no shifting of the position of interface has also been determined in terms of fluid viscosities. The velocity distribution and dependence of interface position on suction/injection are also represented graphically and results have been discussed critically.

1. Introduction

Flow through channels with porous walls in presence of suction/injection has been studied by Berman (1958) and others. Bird *et al* (1960) have considered adjacent flows of two immiscible liquids in a horizontal thin slit and Rawat (1968) has discussed the corresponding steady flow problem for power law fluids. We propose to study the laminar flow problem of two incompressible immiscible viscous fluids between two infinite parallel porous plates with suction/injection under the influence of constant pressure gradient.

2. Formulation of the problem

Consider two incompressible immiscible fluids of densities ρ_i ($\rho_1 > \rho_2$) and viscosities μ_i ($i = 1,2$) each occupying a height h, flowing in the $x -$ direction between two flat porous plates, at $y = 0$, $y = 2h$ under the action of a constant pressure gradient, $- \partial p/\partial y$.

A constant uniform suction at the upper plate and an equal injection at the lower plate are applied ; the position of the interface in steady state is $y = h^*$ ($h^* > h$) and we assume that the components u, v, in the direction of x and y are independent of x. Hence, from the equation of continuity, viz.,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad ...(1)$$

we get $\qquad v\,(x,y) = \text{constant} = v_0 \text{ (say)}, \qquad ...(2)$

where v_0 is positive for both injection/suction.

Also, the equations governing the steady flow of two incompressible fluids are

$$v_0 \frac{\partial u_i}{\partial y} = -\frac{1}{\rho_i} \frac{\partial p}{\partial x} + \frac{\mu_i}{\rho_i} \frac{\partial}{\partial y}\left(\frac{\partial u_i}{\partial y} \right), \; (i = 1, 2), \qquad \ldots(3)$$

$$\text{or } \frac{\partial^2 u_i}{\partial y^2} - \frac{R_i}{h} \frac{\partial u_i}{\partial y} = -\frac{P}{\mu_i}, \qquad (4)$$

where $$\frac{\partial p}{\partial x} = -P \text{ and } R_i = \frac{v_0 h \rho_i}{\mu_i} \qquad \ldots(5)$$

are the suction/injection parameters called Reynold's numbers and $u_i(y)$ are the velocities of the two fluids.

The boundary conditions are

$$\left. \begin{aligned} [u_1]_{y=0} &= [u_2]_{y=2h} = 0 \\ [u_1]_{y=h^*} &= [u_2]_{y=h^*} = U_I \quad \text{(say)}, \end{aligned} \right\} \qquad \ldots(6)$$

and assuming the continuity of shear stress at the interface, we have

$$\left[\mu_1 \frac{\partial u_1}{\partial y} \right]_{y=h^*} = \left[\mu_2 \frac{\partial u_2}{\partial y} \right]_{y=h^*}, \qquad \ldots(7)$$

where U_I is the common interface velocity. Suffix 1 refers to the lower fluid and 2 to the upper fluid.

Also, the velocities obtained in (2) are given by

$$\left. \begin{aligned} u_{10} &= \frac{Ph^2}{2\mu_1}\left[\frac{2\mu_1}{\mu_1+\mu_2} + \frac{\mu_1-\mu_2}{\mu_1+\mu_2}\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2 \right], \\ u_{20} &= \frac{Ph^2}{2\mu_2}\left[\frac{2\mu_2}{\mu_1+\mu_2} + \frac{\mu_1-\mu_2}{\mu_1+\mu_2}\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2 \right] \\ \text{and } U_0 &= \frac{Ph^2}{\mu_1+\mu_2}, \end{aligned} \right\} \qquad \ldots(8)$$

where, in this case, y is measured from the interface position taken as x-axis and U_0 is the interface velocity.

3. Solution of the equations

Solving (4), subject to (6), we get.

$$\frac{u_1 \mu_1}{Ph^2} = \frac{1}{R_1}\left[\bar{y} - \lambda \frac{e^{R_1 \bar{y}} - 1}{e^{R_1 \lambda} - 1} \right] + \frac{U_I \mu_1}{Ph^2} \frac{e^{R_1 y} - 1}{e^{R_1 \lambda} - 1}, \qquad \ldots(9)$$

$$\frac{u_2\mu_2}{Ph^2} = \frac{1}{R_2}\left[(\bar{y}-2) + (2-\lambda)\frac{1-e^{-(2-\bar{y})R_2}}{1-e^{-(2-\lambda)R_2}}\right] + \frac{U_I\mu_2}{Ph^2}\frac{1-e^{-(2-\bar{y})R_2}}{1-e^{-(2-\lambda)R_2}}, \quad \ldots(10)$$

and on using (7), (9) and (10), we get

$$\frac{U_I\mu_1}{Ph^2} = \frac{\dfrac{1}{R_2} - \dfrac{1}{R_1} + \dfrac{2-\lambda}{1-e^{(2-\lambda)R_2}} + \dfrac{\lambda}{1-e^{-\lambda R_1}}}{\dfrac{\bar{\mu}R_2}{e^{(2-\lambda)R_2}-1} + \dfrac{R_1}{1-e^{-\lambda R_1}}} \quad \ldots(11)$$

$$= \Phi\ (R_1, R_2, \lambda, \bar{\mu}), \ \text{(say)}, \quad \ldots(11a)$$

where $\bar{\mu} = \frac{\mu_2}{\mu_1}$, $\bar{y} = \frac{y}{h}$ and $\lambda = \frac{h^*}{h}$ $(1 \leqslant \lambda \leqslant 2)$.

Total flux is given by

$$Q = \int_0^{h^*} u_1\, dy + \int_{h^*}^{2h} u_2\, dy = Q_1 + Q_2 \ \text{(say)},$$

where from (9) and (10) respectively, we have

$$\frac{Q_1\mu_1}{Ph^2} = \frac{\lambda}{R_1^2}\left[\frac{\lambda R_1}{2}\coth\frac{\lambda R_1}{2} - 1\right] + \frac{U_I\mu_1}{Ph^2}\frac{1}{R_1}\left[1 + \frac{\lambda R_1}{1-e^{\lambda R_1}}\right], \quad \ldots(13)$$

and

$$\frac{Q_2\mu_2}{Ph^2} = \frac{2-\lambda}{R_2^2}\left[\frac{2-\lambda}{2}R_2\coth\frac{2-\lambda}{2}R_2 - 1\right] + \frac{U_I\mu_2}{Ph^2}\frac{1}{R_2}\left[\frac{(2-\lambda)R_2}{1-e^{-(2-\lambda)R_2}} - 1\right] \quad \ldots(14)$$

The skin frictions at the lower and upper plates are respectively, given by

$$\tau_1 = \left[\mu_1\frac{\partial u_1}{\partial y}\right]_{y=0}$$

$$= Ph\left[\frac{1}{R_1} + \frac{\lambda}{1-e^{\lambda R_1}} + \frac{U_I\mu_1}{Ph^2}\frac{R_1}{e^{R_1\lambda}-1}\right], \quad \ldots(15)$$

$$\text{and } \tau_2 = \left[-\mu_2 \frac{\partial u_2}{\partial y} \right]_{y=2h}$$

$$= -Ph \left[\frac{1}{R_2} - \frac{2-\lambda}{1-e^{-(2-\lambda)R_2}} - \frac{U_I \mu_2}{Ph^2} \frac{R_2}{1-e^{-(2-\lambda)R_2}} \right] \quad \ldots (16).$$

We introduce the following dimensionless variables

$$\bar{u}_i = \frac{u_i}{U_I}, \quad P_m = \frac{Ph^2}{U_I \mu_1} = \frac{1}{\Phi(R_1, R_2, \lambda, \bar{\mu})} \quad .. (17).$$

Equations (9) and (10), on using (17), transform to the dimensionless form

$$\bar{u}_1 = \frac{e^{R_1 \bar{y}} - 1}{e^{R_1 \lambda} - 1} + \frac{Pm}{R_1} \left[\bar{y} - \lambda \frac{e^{R_1 \bar{y}} - 1}{e^{R_1 \lambda} - 1} \right], \quad \ldots (18)$$

$$\bar{u}_2 = \frac{1 - e^{-(2-\bar{y})R_2}}{1 - e^{-(2-\lambda)R_2}} + \frac{Pm}{R_2} \left[(\bar{y} - 2) + (2-\lambda) \frac{1 - e^{-(2-\bar{y})R_2}}{1 - e^{-(2-\lambda)R_2}} \right]. \quad .. (19)$$

If λ is fixed, P_m, U_I, $\bar{u}_1$, $\bar{u}_2$ can be determined for given R_1, R_2 and $\bar{\mu}$. Further we assume that the interface velocity remains the same as in ordinary flow (Bird *et al* 1960) i. e.,

$$U_I = U_0 = Ph^2/(\mu_1 + \mu_2),$$

then from (11) we get

$$\frac{1}{R_1} - \left(\lambda - \frac{R_1}{1+\bar{\mu}} \right) \frac{1}{1 - e^{-\lambda R_1}} = \frac{1}{R_2}$$

$$+ \left[(2-\lambda) + \frac{\bar{\mu}}{1+\bar{\mu}} R_2 \right] \frac{1}{1 - e^{(2-\lambda)R_2}} \quad \ldots (20)$$

which determines λ, the position of interface, and the corresponding velocity distribution can be easily obtained from (9) and (11).

4. Approximate solution for small suction-injection parameters

Considering only first power of R_1 and R_2, neglecting the terms of the order of R_1^2, R_2^2 and also assuming $\lambda R_1 < 1$ and $(2-\lambda) R_2 < 1$ from (13), (19) and (11), respectively, we get

$$\bar{u}_1 = \frac{\bar{y}}{\lambda} + \frac{\bar{y}(\lambda-\bar{y})}{2} P_m \left[1 + \frac{R_1}{6} \left\{ \frac{6}{\lambda P_m} - (2\bar{y} - \lambda) \right\} \right], \quad (21)$$

$$\bar{u}_2 = \frac{2-y}{2-\lambda} + \frac{(2-\bar{y})(\lambda-\bar{y})}{2} P_m \left[1 + \frac{R_2}{6} \left\{ \frac{6}{(2-\lambda) P_m} + 2\bar{y} - \lambda - 2 \right\} \right] \quad ...(22)$$

and
$$\frac{U_I \mu_1}{Ph^2} = \frac{1 + \frac{1}{12}\left[\lambda^2 R_1 - (2-\lambda)^2 R_2 \right]}{\frac{\bar{\mu}}{2-\lambda}\left[1 - \frac{1}{2}(2-\lambda) R_2 \right] + \frac{1}{\lambda}\left[1 + \frac{\lambda R_1}{2} \right]} \quad ...(23)$$

TABLE 1. VALUES OF $\frac{U_I \mu_1}{Ph^2}$

			R_1 = ·1 fixed				
$\bar{\mu}$↓	R_2 = .2	.3	.4	.5	.6	.7	.8
.5	.6528	.6637	.6706	.6799	.6896	.6997	.7100
1	.4334	.4416	.4501	.4594	4685	.4783	.4886
1.5	.3244	.3314	.3389	.3468	.3547	.3634	.3724
			R_2 = .1 fixed				
$\bar{\mu}$↓	R_1 = .2	.3	.4	.5	.6	.7	.8
.5	6345	.6236	.6170	.6085	.6024	5935	.5865
1	.4235	.4204	.4187	.4179	.4160	.4141	.4130
1.5	3178	.3180	.3182	.3183	.3183	.3183	.3183
			$R_1 = R_2$				
$\bar{\mu}$↓	.1	.2	.3	.5			
.5	.6443	.6426	.6415	.6392			
1	.4268	.4312	.4376	.4423			
1.5	.3177	.3178	.3316	.3382			

Table 1 exhibits clearly the behaviour of the interface velocity at λ = 1.4 for different values of R_1, R_2 and $\bar{\mu}$. It indicates that U_I increases with

R_2 for all $\bar{\mu}$ but as R_1 increases it decreases when $\bar{\mu} \leqslant 1$ and increases very slowly to a constant value when $\bar{\mu} > 1$.

Again, putting $U_I = U_0$ and $\lambda = 1 + I_r$, $\left(I_r = \frac{h^* - h}{h} \right)$, in (21) and (24), we get

$$(R_2 - R_1) I_r^4 - 2(R_1 + R_2) I_r^3 - 6\left(2 + \frac{\bar{\mu} R_2 - R_1}{1 + \bar{\mu}} \right) I_r^2$$

$$- 2 I_r \left[6 \frac{\bar{\mu} - 1}{\bar{\mu} + 1} - R_1 - R_2 \right] - \frac{R_1(5 - \bar{\mu}) - R_2(5\bar{\mu} - 1)}{1 + \bar{\mu}} = 0 \quad \text{...(24)}$$

from which I_r (or λ) can be determined.

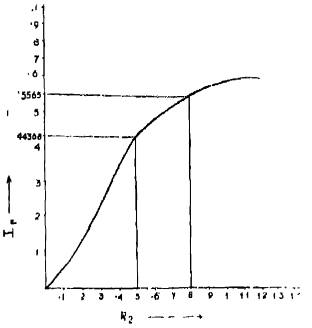

Figure 1. Shifting of common interface with increasing R_2 (R_1=·1)

Equation (24) shows that in the presence of small suction/injection the interface will not shift from the ordinary state position $y = h$, if

$$\frac{R_2}{R_1} = \frac{5 - \bar{\mu}}{5\bar{\mu} - 1}, \; (.2 < \bar{\mu} < 5), \quad \text{...(25)}$$

The plot of I_r against R_2 (figure 1) at fixed R_1 shows that as R_2 increases the common interface shifts towards the less viscous second fluid but never approaches the plate.

5. Results and Discussion

(*i*) When we take the limit of (9), (10), (11), (12), (13), (14), (15) and (16) as R_1 and R_2 tend to zero (i.e., v_0 is zero) we get the results obtained by Bird *et al* (1960).

(*ii*) If μ_1, $\bar{y}$, λ, R_1 are interchanged with μ_2, $\bar{y} - 2$, $\lambda - 2$, R_2 respectively, then u_1 is interchanged with u_2, Q_1 with Q_2 and τ_1 with τ_2. This is the result expected from symmetry also.

(*iii*) Putting $R_i = \frac{v_0 h \rho_i}{\mu_i}$ and keeping ρ_i fixed, (9), (10) and (11) show that u_1, u_2, U_I decrease as μ_1, μ_2 increase.

(*iv*) For fixed $\bar{\mu}$, λ equations (9), (10) and (11) show that the velocities u_1, u_2, U_I increase with increasing suction parameter R_2 and decrease with increasing injection parameter R_1.

(*v*) (a) Comparing the fluid velocities in (21) and (22) with those in ordinary state flow (8) (Bird *et al* 1960) a little consideration shows that $\frac{u_i}{U_0} >, =$ or $< \frac{u_{i0}}{U_0}$ when $\bar{y} <, =$ or $> \frac{\lambda}{2}$, which is also evident from graph (figure 2), which represents the velocity profiles of the two types of flows (with or without suction/injection) at $\lambda = 1.4368$ and $\lambda = 1.2329$ for $\bar{\mu} \lesseqgtr 1$.

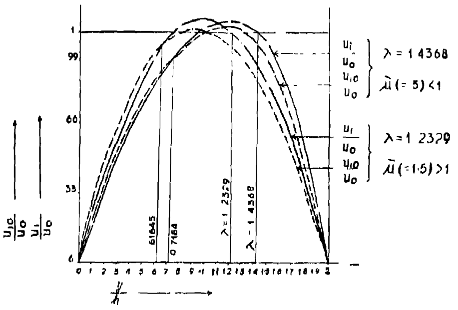

Figure 2. Velocity profiles with or without suction and injection

(b) Comparing interface velocities of the two kinds of flows, we have $U_I >, =$ or $< U_0$ according as

$$\Phi(R_1, R_2, \lambda, \bar{\mu}) >, = \text{ or } < \frac{1}{1+\bar{\mu}} \qquad \ldots(26)$$

and for small R_1, R_2

$$\frac{\bar{\mu}R_2 - R_1}{1+\bar{\mu}} + \frac{R_1\lambda^2 - R_2(2-\lambda)^2}{6}$$

$$>, = \text{ or } < \frac{2(\lambda-1)}{\lambda(2-\lambda)} \cdot \frac{(1+\bar{\mu})\lambda - 2}{1+\bar{\mu}} \qquad ...(27)$$

$$\text{and } \frac{R_2}{R_1} >, = \text{ or } < \frac{5-\bar{\mu}}{5\bar{\mu}-1} \text{ (when } \lambda = 1, .2 < \bar{\mu} < 5) \qquad ..(28)$$

(*vi*) τ_1 decreases with increasing R_1 and τ_2 numerically increases with increasing R_2. Q_1 and Q_2 also behave in a similar manner.

6. Uniqueness of Velocity Maximum

To find velocity maximum, we have

$$\frac{\partial u_i}{\partial y} = 0, (i = 1, 2), \qquad ...(29)$$

which on using (9) and (10), give

$$\frac{y_1}{h} = \lambda + \frac{1}{R_1} \log \frac{1 - e^{\lambda R_1}}{\lambda R_1 - \Phi(R_1, R_2, \lambda, \bar{\mu}) R_1^2} \qquad ...(30)$$

$$= \lambda + \frac{1}{R_1} \log \frac{f_1}{f_2} \text{ (say)}, \qquad ...(30a)$$

and

$$\frac{y_2}{h} = \lambda + \frac{1}{R_2} \log \frac{e^{(2-\lambda)R_2} - 1}{(2-\lambda)R_2 + \bar{\mu} R_2^2 \phi(R_1, R_2, \lambda, \bar{\mu})} \qquad ...(31)$$

$$= \lambda + \frac{1}{R_2} \log \frac{F_1}{F_2} \text{ (say)}, \qquad ...(31a)$$

where $\Phi(R_1, R_2, \lambda, \bar{\mu})$ is defined in (11a).

The velocity maximum will occur in the first or second fluid according as $0 < \frac{y_1}{h} < \lambda$ or $\lambda < \frac{y_2}{h} < 2$. The velocity maximum will be at the common interface (*i. e.*, at $y = h^*$) if $f_1 = f_2$, $F_1 = F_2$ or $\bar{\mu} = \mu^*$,

$$\text{where, } \mu^* = \frac{e^{(2-\lambda)R_2} - (2-\lambda)R_2 - 1}{R_2^2} \Big/ \frac{e^{-\lambda R_1} + R_1\lambda - 1}{R_1^2}, \qquad ...(32)$$

which is the critical value of the ratio of the viscosity coefficients of the two fluids.

Also, we observe the following facts :

(*i*) If $\bar{\mu} > \mu^*$, we have $F_1/F_2 < 1$ so that $y_2/h < \lambda$ and $f_1/f_2 < 1$ so that $y_1/h < \lambda$, which shows that the maximum velocity will occur in the first fluid and not in the second fluid.

(*ii*) If $\bar{\mu} < \mu^*$, we have $f_1/f_2 > 1$ so that $y_1/h > \lambda$ and also $F_1/F_2 > 1$ so that $y_2/h > \lambda$ showing that the maximum velocity will occur in the second fluid and not in the first fluid.

Besides this the velocity as well as its derivatives are continuous at all points except at the interface where the derivative can change only in magnitude but not in sign and velocity vanishes at the plates. Hence, from these considerations, we conclude that there is a unique velocity maximum and it occurs in the lower fluid, interface or upper fluid according as $\bar{\mu} >, =$ or $< \mu^*$ and its position is given by (30) or (31).

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