THE EFFECTS OF GAUSSIAN NOISE ON THE FREQUENCY RESPONSE CHARACTERISTICS OF A NONLINEAR FEEDBACK CONTROL SYSTEM

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(*Received September* 15, 1964)

ABSTRACT. In this paper, a quasi-linearisation technique is desscribed which gives a parameter for approximately representing a memory-type nonlinearity on the basis of an input containing a sinusoidal signal and a Gaussian noise with mean value zero. The parameter is termed 'complex equivalent gain' and this is used for investigating the effects of a Gaussian noise on the frequency response characteristics of a stable feedback control system incorporating the nonlinearity. A simple second-order position control system with backlash in the output coupling is considered as an example and the results obtained are verified experimentally with the help of an electronic analogue computer.

INTRODUCTION

When the input of a stable feedback control system incorporating a memory-type nonlinearity is subjected to a sinusoidal signal, the freauency response characteristics of the system can be determined approximately by the use of linearisation techniques (Stein and Thaler 1958, Sen, 1964). In the application of these techniques, a quasi-linearised transfer function is used to represent the nonlinear element in the system which is called 'complex describing function'.

But, when the input of the nonlinear system considered becomes contaminated with a Gaussian noise then it is found that all the parameters of the frequency response characteristics, namely, the bandwidth, resonant frequency and the height of the resonant peak previously obtained for a particular value of the impressed sinusoidal amplitude, change considerably. This is due to the fact that the transmission property of the nonlinearity alters due to the presence of the noise. In this paper, an analytical method will be proposed for investigating these effects of the Gaussian noise on the frequency response characteristics of a memory-type nonlinear system, where the nonlinearity considered is assumed to be amplitude-sensitive alone. The importance of this investigation arises from the fact that the inputs of all practical control systems are usually contaminated with such external disturbances.

In order to carry out the above analytical investigation, a quasi-linearisation technique will be adopted in which a 'complex equivalent gain' (Sen, 1955) will be obtained for making an approximate representation of the nonlinearity under the assumption that the input is composed of only a sinusoidal signal and a Gaussian noise with mean value zero. An outline of the proposed quasi-linearisation technique has been presented in the following section.

THE PROPOSED QUASI-LINEARISATION TECHNIQUE

When a linear element is subjected to an input consisting of a sine wave and a Gaussian signal of mean value zero it is found that the response of the element will also contain only the components of the input signal and the original shape of the input wave will be maintained at the output. But, when the element becomes nonlinear, distortion will appear in the shape of the output wave and it will become difficult to make any rigorous analysis of the response of the element in this case. However, it can be seen that, for the assumed input, the response of the nonlinearity can be separated into two parts—one representing the correlated component that exactly reproduces the input spectrum, while the remainder is called the 'distortions' comprising the harmonics and intermodulation components and then a quasi-linearisation technique can be adopted to make an approximate representation of the nonlinearity. The use of this linearisation technique assumes the presence of only the correlated component at the output and defines a quasi-linearised transfer function for the nonlinearity that relates the input to the output correlated component. For the case of nonlinearities that involve memory, phase-shift will be introduced to each of the frequency components at the output and therefore, in such cases, the quasi-linearised transfer function obtained for the nonlinearity becomes a complex quantity and may be called 'complex equivalent gain'. The magnitude of this complex equivalent gain is given by the ratio of the r.m.s. value of the output correlated component to that of the input, while the phase is assumed to be frequency-independent. The above definition of a 'complex equivalent gain' has been made for a memory type nonlinearity under the assumption that the components of the signal assumed at the input of the nonlinearity lie within a narrowband frequency spectrum.

In order to determine the phase function attributed to the quasi-linearised model of the nonlinearity, the simple procedure as outlined below is to be adopted. Consider a nonlinear element (Fig. 1), the input of which is impressed upon by a signal z comprising a sine wave $z_1 = A_z \sin \omega_c t$ and a Gaussian noise z_2 having mean value zero and variance σ_n^2. If the component of the Gaussian noise is expressed in the form $z_2 = \sum_{n=0}^{\infty} a_n \sin(\omega_n t + \phi_n)$ where a_n describes the power spectrum of the noise and ϕ_n is randomly distributed with a uniform probability distribution from 0 to 2π, then, neglecting the harmonics and intermodulation components, the approximate output of the nonlinearity can be written as :

$$y = |H(\sigma)| [A_z \sin\{\omega_c t + \theta(\sigma)\} + \sum_{n=0}^{\infty} a_n \sin\{\omega_n t + \phi_n + \theta(\sigma)\}] \quad \dots \quad (1)$$

where $|H(\sigma)|$ is the magnitude and $\theta(\sigma)$ the phase of the complex equivalent gain defined for the nonlinearity. In order to attribute the proper sign to the phase function $\theta(\sigma)$, it should be remembered that $\theta(\sigma)$ is negative for those nonlinearities that introduce a lagging phase-shift to the output frequency components for a sinusoidal input, while it is positive for those introducing a leading phase-shift.

Now, the difference between the approximate output and the input multiplied by the magnitude of the complex equivalent gain is given by

$$e = y - |H(\sigma)|z$$

$$= 2|H(\sigma)|\sin\frac{\theta(\sigma)}{2}\left[A_z\cos\left\{\omega_e t+\frac{\theta(\sigma)}{2}\right\} + \sum_{n=0}^{\infty} a_n\cos\left\{\omega_n t+\phi_n+\frac{\theta(\sigma)}{2}\right\}\right] \quad \dots (2)$$

or, the rms value of the quantity e is given by

$$\sigma_e = 2|H(\sigma)|\sigma_z\sin\frac{\theta(\sigma)}{2} \quad \dots (3)$$

whence

$$\theta(\sigma) = 2\sin^{-1}\frac{\sigma_e}{2|H(\sigma)|\sigma_z} \quad \dots (4)$$

where σ_z represents the rms value of the total signal at the nonlinearity input.

In practice, however, as the magnitude of the complex equivalent gain defined for the nonlinearity, cannot be easily determined, an approximate measure of the parameter can be obtained by taking the ratio of the rms value of the actual output to that of the input of the nonlinearity. Evidently, this measurement will give a somewhat increased value of the parameter $|H(\sigma)|$ due to the presence of the distortion components at the output of the nonlinearity.

Thus, the procedure for having an approximate measure of the complex equivalent gain of a memory-type nonlinearity for the assumed input can be summarised as follows :

(1) The rms values of the input and the output of the nonlinearity are first measured and then the parameter $|H(\sigma)|$ is computed for different rms values of the input.

(2) The rms values of the quantity e are measured for different values of the input by arranging the set-up as shown in Fig. 1, each time using proper value of the quantity $|H(\sigma)|$ obtained from the procedure in step (1).

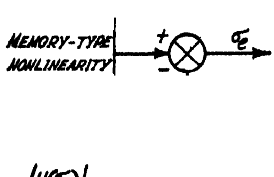

Fig. 1. Set-up for measuring the phase of complex equivalent gain.

(3) Finally, eqn. (4) is used to compute the parameter $\theta(\sigma)$ of the complex equivalent gain.

For different values of the quantities A_z/δ and σ_n/δ the complex equivalent gain of a simple backlash as measured by the above method is presented in Fig. 2, where δ represents the backlash half-width.

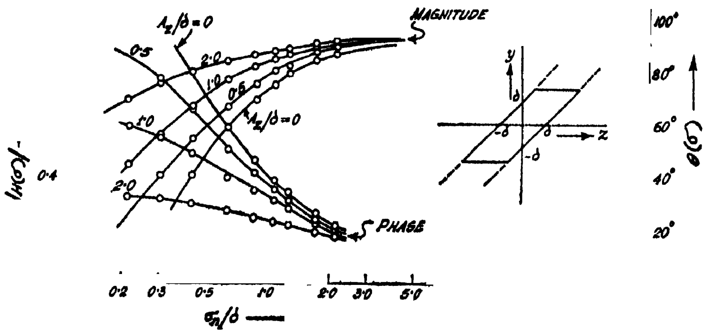

Fig. 2. The complex equivalent gain of a simple backlash.

APPLICATION OF THE PROPOSED QUASI-LINEARISATION TECHNIQUE

In the preceding section, a quasi-linearisation technique has been developed which yields a 'complex equivalent gain' as a parameter for approximately representing a memory-type nonlinearity with an input function comprising a sinusoidal signal and a Gaussian noise with mean value zero. When the nonlinear element considered occurs as a part of a feedback system which is also subjected to a similar input, the application of the quasi-linearisation technique is facilitated by replacing the nonlinearity with the help of the quasi-linearised gain and then the analysis is carried out by obtaining two separate linearised versions for the over-all nonlinear system—one for the sinusoidal portion and the other for the Gaussian component of the impressed signal (Sawaragi and Sugai, 1959). The justification in using the two separate linearised systems for the analysis can be seen from the fact that, in either case, the effect of the remaining signal simultaneously present in the system is included in the quasi-linearised gain obtained for the nonlinearity. Though the presence of the nonlinearity will destroy the nature of the signals impressed upon the system, but it will be assumed that the signal fedback to the input of the nonlinearity will contain only a sinusoidal and a Gaussian component, and, possibly, this assumption will be justified in practice because of the narrow-band characteristic of the feedback system.

A POSITION CONTROL SYSTEM WITH BACKLASH

Consider a position control system as shown in Fig. 3, incorporating backlash in the output coupling and is subjected to a sinusoidal and a Gaussian signal at the point in the loop as indicated in the figure. Assuming the signal at the input of the nonlinearity to contain only the components of the impressed wave, the

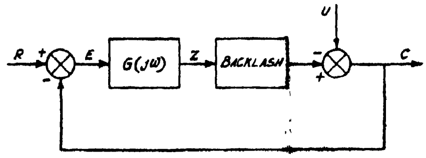

Fig. 3. A position control system with backlash.

application of the quasi-linearisation technique yields the two linearised versions of the nonlinear system as presented in Figs. 4(a) and 4(b).

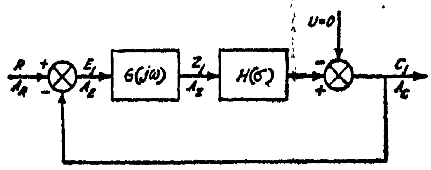

Fig. 4(a). The linearised version of the position control system for sinusoidal portion of the impressed input.

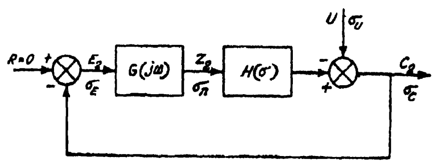

Fig. 4(b). The linearised version of the position control system for Gaussian component of the impressed input.

Confining our attention to the evaluation of the frequency response characteristics of the nonlinear system at the point '*Z*' alone, we get from Fig. 4(*a*),

$$z_1 = \frac{RG(jw)}{1+H(\sigma)G(jw)} \qquad \dots \quad (5)$$

while Fig. 4(b) gives

$$\psi z_2(s) = \psi_U(s) \cdot \frac{G(s)}{1+H(\sigma)\, G(s)} \qquad \dots \quad (6)$$

where $\psi z_2(s)$ represents the complex frequency spectrum of the Gaussian noise assumed at the point z, which is the input of the nonlinearity and $\psi_U(s)$ is the complex frequency spectrum of the impressed noise. Of the above two equations, it can be readily seen that the first equation gives the required frequency response characteristics of the nonlinear system for different assumed values of the amplitudes of the sine wave and also the Gaussian noise at the input of the nonlinearity, while, with the help of the second equation, the rms values of the impressed noise are computed in terms of the rms noise present at the input point '*Z*' of the nonlinearity.

THE USE OF NICHOLS' CHART

When the transfer functions for the linear and the nonlinear part of the system considered are given as plots on the conventional magnitude-phase plane (shown in Fig. 5), then the frequency response characteristics of the closed-loop system

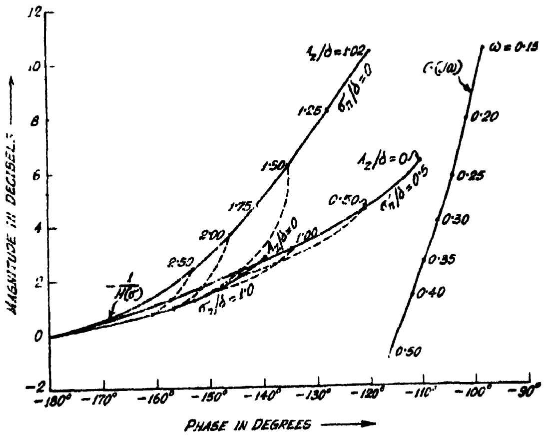

Fig. 5. The magnitude-phase plane plots of the transfer functions for the linear and the nonlinear parts of the position control system.

as given by eqn. (5) can be easily determined by the use of Nichols' chart. Two different approaches can be followed in using the chart as outlined below :

(a) In one approach, the given loci on the magnitude-phase plane are first utilised to obtain different families of curves for the combined transference $H(\sigma)$ $G(j\omega)$ of the linear and the nonlinear components of the system, each family corresponding to a particular value of the rms noise at the input of the nonlinearity. The procedure for obtaining these families of curves for the combined transfer function can be explained as follows :

If the magnitude of the quantity $H(\sigma)$ $G(j\omega)$ be expressed in decibels and its phase in degrees, then denoting the respective quantities as $M_{H\omega}$ and $\theta_{H\omega}$, we have

$$M_{H\omega} = |H(\sigma)| + |G(jw)|$$

$$= \frac{1}{|H(\sigma)|} + |G(jw)| \tag{7}$$

and

$$Q_{H\omega} = /H(\sigma) + /G(jw)|$$

$$= -\left[180^\circ + \Big/ -\frac{1}{H(\sigma)} \right] + /G(jw) \qquad \dots \tag{8}$$

Since the values of the quantities $\frac{1}{|H(\sigma)|}$ and $\Big/ \frac{1}{-H(\sigma)}$ are directly obtainable from the loci of $\Big/ -\frac{1}{H(\sigma)}$ on the magnitude-phase plane and are known for the values of the parameters A_z/δ and σ_n/δ marked on these loci, substitution of these values in the above equations gives the values of both the magnitude and phase of the combined transference for different values of the frequency, and thus, the required families of curves are obtained at different selected values of A_z/δ and σ_n/δ, each curve being graduated with different values of the frequency.

For a particular selected value of the rms noise σ_n/δ and with different values of A_z/δ as a parameter, the family of curves obtained for the combined transference are now superimposed on the contour system of a Nichols' Chart and the points of intersection of these curves with the contours on the Nichols' chart are noted which give the frequency response characteristics for the transfer function

$$\frac{A}{B} = \frac{H(\sigma)\ G(jw)}{1+H(\sigma)\ G(jw)} \qquad \ldots \quad (9)$$

at the selected value of σ_n/δ and for the different chosen values of A_z/δ.

The same procedure as outlined above is then followed for different selected values of σ_n/δ.

Knowing the frequency response characteristics for the transfer function A/B with the help of the Nichols' chart, the frequency response characteristics of the system given by eqn. (5) can now be easily computed and this can be done by determining the values of the parameter $H(\sigma)$ from Fig. 5 at the different selected values of σ_n/δ and A_z/δ and by substituting those values in the relation :

$$\frac{Z_1}{R} = \frac{A}{B} \cdot \frac{1}{H(\sigma)} \qquad \ldots \quad (10)$$

(b) In the other approach, on the other hand, as suggested by Stein and Thaler (1958), the contour system of the Nichol's chart is superimposed on the given plots of the transfer function of the system, locating its origin on the selected values of σ_n/δ and A_z/δ as marked on the $\left[-\frac{1}{H(\sigma)}\right]$ -loci and the same results as represented in eqn. (9) are obtained by observing the points of intersection between the chart-contour and the locus of the given linear transference $G(jw)$ of the system considered. It should be noted, however, that though this latter approach will be useful only when the Nichols' chart is available as contours drawn on a transparent template, but it will be more convenient because of the fact that the laborious computation of the families of curves for the combined transference $H(\sigma)\ G(jw)$ will not be required in this case.

AN EXAMPLE OF A SECOND-ORDER SYSTEM WITH BACKLASH

If the position control system considered in the preceding section be of second order with its linear part having the transfer function $G(jw) = K_v/jw(jw+1)$, then the family of curves for the combined transfer function $H(\sigma)\ G(jw)$ obtained at a selected value of σ_n/δ and superimposed on the Nichols' chart will be as shown in Fig. 6. Taking the value of the velocity error constant $K_v = 0.5$, the amplitude

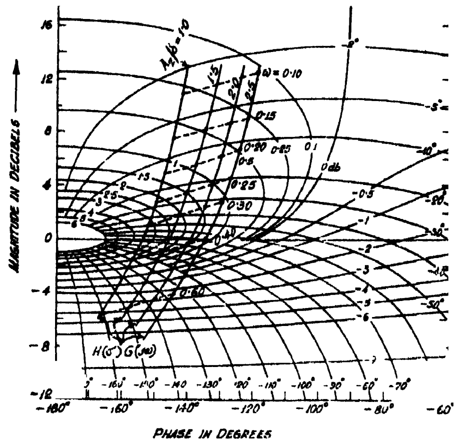

Fig. 6. The loci of the combined transfer function for the linear and nonlinear components of a second-order system superimposed on the contour system of a Nichols' chart.

and the phase response characteristics of the system have been evaluated for different chosen values of σ_n/δ and A_z/δ and the particular characteristics obtained for $\sigma_n/\delta = 0.5$ and for a set of selected values of A_z/δ are presented in Figs. 7 and 8, respectively, where A_z/A_R represents the amplitude response and ϕ_z the phase response of the system at the point Z. With the help of these characteristics evaluated at different constant values of A_z/δ, the frequency response characteristics of the system can be easily determined for different constant values of the amplitude A_R/δ of the sinusoidal signal impressed upon the system and this can be done by first designating at each position of the amplitude response characteristics obtained above with the proper value of A_R/δ and then drawing the locus

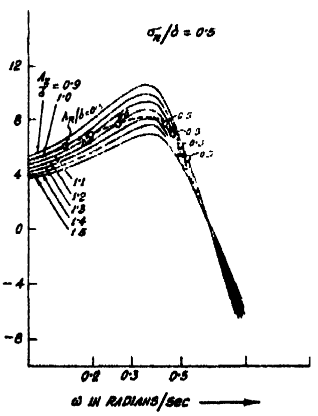

Fig. 7. The amplitude response characteristics of the second-order system for $\sigma_n/\delta = 0.5$ and for $A_z/\delta = 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5$. (The dotted curve shows the corresponding amplitude response characteristics for $A_R/\delta = 0.5$).

of constant-A_R/δ on these characteristics. This is illustrated in Fig. 7. Knowing from the figure the values of w and A_z/δ at different positions on the constant

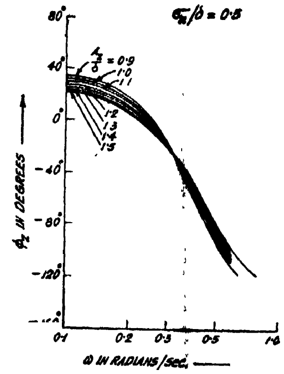

Fig. 8. The phase response characteristics of the second-order system for $\sigma_n/\delta = 0.5$ and for $A_z/\delta = 0.9$, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5.

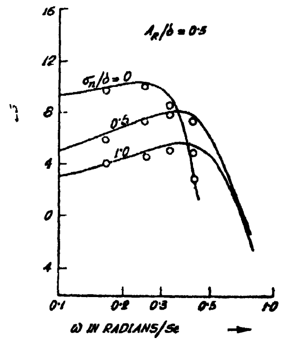

Fig. 9. The amplitude response characteristics of the second-order system for $A_R/\delta = 0.5$ and for $\sigma_n/\delta = 0$, 0.5 and 1.0.
– analytical values
⊙ experimental values.

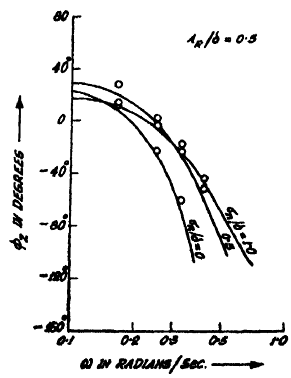

Fig. 10. The phase response characteristics of the second-order system for $A_R/\delta = 0.5$ and for $\sigma_n/\delta = 0$, 0.5 and 1.0.
– analytical values.
⊙ experimental values.

A_R/δ locus, the corresponding phase response characteristics of the system is then determined with the help of Fig. 8. Thus the amplitude and the phase response characteristics of the system are determined for different constant values of A_R/δ and the characteristics evaluated at $A_R/\delta = 0.5$ and for a set of selected values of σ_n/δ are presented in Figs. 9 and 10, respectively.

Since, in the present system considered, the noise impressed upon the system occurs at the output point C, the rms values of the impressed noise corresponding to the different selected values of the rms noise at the point Z are to be determined and this will be done by the use of eqn. (6). Substituting the expression for $G(jw)$, eqn. (6) can be written as

$$\psi_{Z_2}(s) = \psi_U(s) \cdot \frac{K_v}{s(s+1)+K_v H(\sigma)} \qquad \dots \quad (11)$$

Therefore, for a particular input spectrum given by

$$\psi_U(w) = \frac{\omega_0 n}{j\omega+\omega_0} \cdot \frac{\omega_0 n}{-j\omega+\omega_0} \qquad \dots \quad (12)$$

where $s = jw$ and ω_0 is the half-power frequency and n the low frequency amplitude of the noise spectrum, the normalised values of the impressed rms noise in terms of the rms noise at the point z are obtained from the relation :

$$\frac{\sigma_U}{\delta} = \frac{\sigma_n}{\delta} \sqrt{\left| \frac{H(\sigma)}{K_v} \left\{ \omega_0 + \frac{K_v}{\omega_0+1} H(\sigma) \right\} \right|} \qquad \dots \quad (13)$$

Since the parameter $H(\sigma)$ in the above equation is determined by the values of both σ_n/δ and A_z/δ, a set of curves are drawn for the particular system considered by plotting the different values of σ_n/δ as abscissa and the corresponding values of σ_u/δ as ordinate and taking the values of A_z/δ as a parameter. This is shown in Fig. 11. With the help of these curves, it will be possible to obtain the values of σ_U/δ corresponding to the selected values of σ_n/δ and A_z/δ in the above

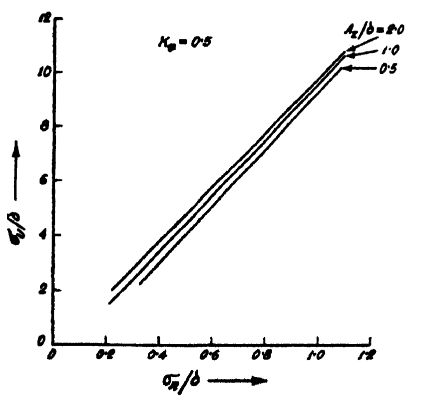

Fig. 11. The plots of σ_U/δ vs. σ_n/δ with different values A_z/δ as a parameter.

analysis or, conversely, for a given value of σ_u/δ, the values of σ_n/δ and $A_z/\delta c$ an also be selected with the help of these curves.

COMPUTER STUDY

In order to have an experimental check on the results obtained analytically, the nonlinear system considered in the example, is simulated on an electronic analogue computer and the arrangements as shown in Fig. 12, is made for measur-

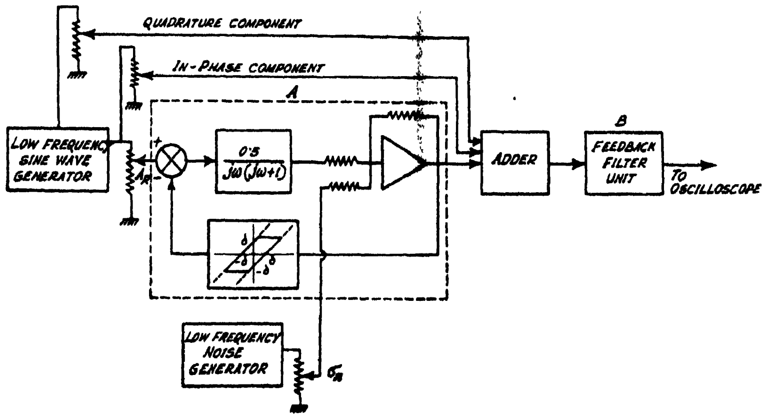

Fig. 12. The experimental arrangement for measuring the amplitude and phase response characteristics of the second-order system with noise injected at the nonlinearity input.

ing both the amplitude and the phase response characteristics of the system for different values of the rms noise present at the point Z in the system loop.

In the arrangement, the block A is the simulated system, under investigation, where the output terminal represents the point Z in the system loop at which the frequency response characteristics are proposed to be evaluated. The block B represents a feedback filter unit having a very narrow pass-band around a centre frequency equal to ω_c and the centre frequency is ganged to the frequency of the oscillator supplying the sinusoidal signal impressed upon the input of the simulated system. The filter unit will be used in conjunction with an oscilloscope for detection of the condition of balance of the fundamental component of the impressed sine wave present at the output of the simulated system.

The procedure adopted for the measurement can be outlined as follows : First of all, the amplitude of the sine wave of a particular frequency and also the rms value of noise impressed upon the system is set at the selected values and then the fundamental component appearing at the output of the simulated system due to the impressed sine wave is balanced out by the addition of suitable fractions of the in-phase and quadrature -components of the same sinusoidal signal and the balance is detected with the help of the oscilloscope. As the output point of the simulated system will be contaminated with noise, the point of exact balance will not correspond to zero output of the filter unit, but, instead, a low frequency noise component will appear on the oscilloscope due to the finite bandwidth of the filter. However, the adjustments could be made such that the departure from the point of balance could readily be detected for a few millivolts change in the fundamental balancing signal from the value at which the balance is obtained. Finally, the amplitude values of the in-phase and the quadrature components of the fundamental balancing signal corresponding to the point of balance are noted and these are used for computing the required amplitude and phase response of the system at the particular value of the frequency of the impressed sine wave.

The above procedures are then repeated for different frequencies and at different selected rms values of the impressed noise and the results obtained are presented and are indicated as circles on Figs. 9 and 10.

CONCLUSION

The quasi-linearisation technique described in the first part of this paper has been found to be useful for investigating the effect of a Gaussian noise on the frequency response characteristics of a feedback control system incorporating a memory-type nonlinearity. As a graphical aid to the evaluation of the closed-loop equation for obtaining the frequency response characteristics of the system, Nichols' chart has been used and the two possible ways of using the chart have been outlined. It has been observed that, by assuming the Gaussian noise to be impressed at the input of the nonlinearity, the effect of the rms noise is to decrease both the amplitude and the phase response characteristics of the system at the lower frequencies, while increasing them towards the high frequency end of the characteristics. The experimental results obtained from a computer study of the system are found to corroborate the above observations.

ACKNOWLEDGMENT

The author is grateful to Prof. J. N. Bhar, D.Sc., F.N.I., for his keen interest in the work and to Dr. A. K. Choudhury, M.Sc., D.Phil. for his guidance and many helpful suggestions and discussions. The author also thankfully acknowledges the award of a research fellowship by the Council of Scientific and Industrial Research, New Delhi.

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