RESPONSES IN A PIEZO-ELECTRICAL PLATE-TRANSDUCER

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ABSTRACT. The responses of a Piezo Electrical Plate Transducer, vibrating in a thickness mode excited by transient stress input has been worked out following the powerful operational method due to Heaviside The problem is discussed in two different cases, (a) the transducer is open at both ends, (b) to one end of the transducer there is a resistance loading. As a particular case the results corresponding to periodic stress input to the Plate Transducer has also been discussed in the present paper

INTRODUCTION

Rigorous theory for a Piezo-Electric Crystal Vibrator of any form and orientation, vibrating in any desired mode would have to take account of all boundary conditions, size and position of electrodes, losses due to dielectric and mounting non-linear effects, coupling due to different modes of vibration, non-uniformity of electric field, and when the electrodes separated by a gap, including possible resonance effect in the air itself. While no such general theory has been attempted, special problem involving most of these considerations have been attacked by may writers. We consider a few simple cases here. In practice, the commonest type of Piezo-Transducers are the bar, vibrating compressionally lengthwise and the Plate vibrating in the thickness mode. For this purpose one employes an X-cut quartz plate With this cut the vibrations are compressional, the Plate becoming alternately thicker and thinner The experimental evidence of this mode of vibration is that acoustics waves in the air are emitted from the surface. A number of practical problems involve the application of a transient electrical or mechanical signal to Piezo Electrical Transducer and also continuous wave generation for ultrasonic use and detection. Redwood (1961) has studied the characteristics of transient pulses, responses etc. for a Piezo Electric Transducer (both Plate and Bar) in the case when the input exciting function is a step function. In this paper the input is taken as a mathematical function of time. Special cases have been worked out using sinusoidal exciting function both Mechanical and Electrical acting as continuous wave-guide which, as Redwood has remarked elsewhere, is nearer the type of source obtained in practice than those represented by simple mathematical functions. The results worked out here after reasonable approximations agree with the results arrived at by Redwood.

EXPLANATIONS OF THE SYMBOLS USED

X = Length of the Plate

t = Variable time.

x = Variable length, measured along the length of the Plate.

ξ = Mechanical displacement of any particle in the x-direction

ξ_0 = Mechanical displacement of any particle at $x = 0$.

v = Velocity of propagation of wave along the Plate

F = Mechanical stress applied to the surface normal to x-direction at $x = 0$

h = Piezo-electric constant.

Q = Total charge at the surface of the Plate

V = Potential across the Plate from $x = 0$ to $x = X$.

E = Young's Modulus of the material of the Plate

C_0 = Static Capacitance of the Plate-transducer.

R = Resistance due to Plate Transducer.

$D = \frac{d}{dt}$ (Operator).

SOLUTION OF THE PROBLEM

In the present problem we propose to find out electrical response when a Piezo-electric Plate transducer is excited by transient mechanical stress input and vice-versa. The equation of vibration of a Piezo-electric Plate-transducer is,

$$\frac{\partial^2 \xi}{\partial t^2} = v^2 \frac{\partial^2 \xi}{\partial x^2} \quad \ldots \quad (1.0)$$

solution of which is,

$$\xi = A \cosh \frac{D}{v} x + B \sinh \frac{D}{v} x \quad \ldots \quad (1.1)$$

A, B being constants.

We consider that the Plate is vibrating mechanically in a thickness-mode and the x-axis coincides with this particular direction. The potential difference across the Plate transducer is related with the corresponding displacements of the two ends of the Plate by,

$$V = -[(\xi)_{x=X} - (\xi)_{x=0}] + \frac{Q}{C_0} \quad \ldots \quad (2.0)$$

The equation for the plane compressional wave propagation along x-direction at $x = 0$ of the Plate can be written as,

$$F + hQ = E \left(\frac{\partial \xi}{\partial x} \right)_{x=0} \quad \ldots \quad (3.0)$$

The problem has been discussed in two distinct cases. In case I the transducer is open at both ends i.e., the Plate is cemented between two semi-infinite mechanical systems and in case II to one end of the transducer there is a resistance loading analogous to the case of a fixed-free system.

The two cases are explained following the powerful operational method due to Heaviside which is very straight forward shortcut and gives results of higher accuracy.

Case I

Here the circuit may be taken to be open the condition for which is at

$$x = 0, \quad \xi = \xi_0 \qquad \ldots \quad (4.0)$$

$$x = X, \quad \frac{\partial \xi}{\partial x} = 0 \qquad \ldots \quad (4.1)$$

$$Q = 0$$

Relation (1.1) together with these end-conditions give

$$\xi = \xi_0 \frac{\cosh (D/v)(x - X)}{\cosh (D/v) X} \qquad \ldots \quad (5.0)$$

and the equation for compressional wave propagation is,

$$F = E\left(\frac{\partial \xi}{\partial x}\right)_{x=0} \qquad \ldots \quad (5.1)$$

From (5.0) and (5.1)

$$\xi_0 = -\frac{Fv}{E} \cdot \frac{1}{D} \coth \frac{D}{v} X \qquad \ldots \quad (5.2)$$

Thus the potential difference as obtained by the help of (5.2) and (2.0) is

$$V = -\frac{hF_0 v}{E}\left[\tanh \left(\frac{1}{2}\frac{D}{v} X\right)\right] H(t) \qquad \ldots \quad (6.0)$$

where, $H(t) = \int \phi(t)dt, \quad F(t) = F_0\phi(t)$ ... (6.1)

The equation (6.0) representing the series can also be summed up like a geometrical progression when

$$V = \frac{hvF_0}{E}(-1)^{\left[t / \frac{X}{v} \right]} \quad (t > 0) \qquad \ldots \quad (6.2)$$

The average value of $t/(X/v)$ for t between nX/v and $\frac{(n+1)X}{v}$ is,

$$n + \tfrac{1}{2} = \left[t / \frac{X}{v} \right] + \tfrac{1}{2}$$

where $\left[t\Big/\frac{X}{c}\right]$ is the greatest integer but not greater than $t\Big/\frac{X}{c}$. Then the quantity

$$t\Big/\frac{X}{c} - \left[t\Big/\frac{X}{c}\right] - \frac{1}{2} = -\frac{1}{2} \text{ at } t = 0$$

increases uniformly to $+\frac{1}{2}$ as $t \to X/c$ drops discontinuously at $t = X/c$ to $-\frac{1}{2}$ and then repeats itself periodically. This elementary theory does not take into account the attenuation due to losses within the material (Jefferys 1956).

If the mechanical stress applied be periodic step input i.e. if

$$F(t) = F_0 \cos \omega t \; H(t) \; \frac{\sin \omega t}{\omega} \qquad \dots (7.0)$$

From (6.0) and (7.0) it follows that there corresponds a periodic step voltage response in agreement with that derived by Sinha (1962).

Case II

Here the transducer is connected to a finite electrical impedence or resistance R ohms. The boundary conditions are at

$$\left.\begin{array}{lll} x = 0 & \xi = \xi_0 & \\ x = X & \xi = \xi_x = 0 \end{array}\right\} \qquad \dots (8.0)$$

The relation (1.1) with the above boundary conditions give

$$\xi = \xi_0 \frac{\sinh \frac{D}{c}(X - x)}{\sinh \frac{D}{c} X} \qquad \dots (9.0)$$

and the electro-mechanical stress relation at the input end of the Plate is given by.

$$F = hQ - E\left(\frac{\partial \xi}{\partial x}\right)_0 \qquad \dots (10.0)$$

From (9.0) and (10.0)

$$\xi_0 = \frac{(F - hQ)c}{E} \cdot \frac{1}{D} \tanh \frac{D}{c} X \qquad \dots (10.1)$$

The expression for the electrical response can now be obtained by the help of the equations (10.1) (2.0) and the relation $V = DQR$ to be

$$V = \frac{hFc}{E} \cdot \frac{1}{\psi(D)} \qquad \dots (11.0)$$

where,

$$\psi(D) = \left[1 + \frac{1}{DC'_0R} - \frac{h^2v}{ER} \cdot \frac{1}{D^2} \tanh \frac{D}{a} X \right] \Big/ \frac{1}{D} \tanh \frac{D}{v} X \qquad \ldots \text{(11.1}$$

Now substituting the exponential values for hyperbolic tangents in equation (11.1) and writing $D+\alpha = D_1$, $D+\beta = D_2$ respectively, we get

$$\psi(D) = \frac{D_1 D_2}{D\left\{1-\exp\left(-\frac{2DX}{v}\right)\right\}} \left[1 + \left\{ 1 - \frac{2\alpha\beta}{D_1 D_2} \right\} \exp\left(-2DX/v\right) \right] \qquad \ldots \text{(11.2)}$$

where, $$D_1 D_2 = (D+\alpha)(D+\beta) = D^2 + \frac{1}{C_0R} D - \frac{h^2v}{ER} \qquad \ldots \text{(12.0)}$$

and $-\alpha$, $-\beta$ are the roots of the equation, $D_1 D_2 = 0$ given by,

$$[\alpha, \beta] = \frac{1}{2} \left[\frac{1}{C'_0R} \pm \left(\frac{1}{C_0^2R^2} + \frac{4h^2v}{ER} \right)^{\frac{1}{2}} \right] \qquad \ldots \text{(12.1)}$$

With the help of (11.0) and (11.2) the difference of potential across the ends of the Piezo-electric Plate transducer can be obtained as,

$$V = -\frac{hvF}{E} \cdot \frac{D}{D_1 D_2} \left\{ 1 - \exp\left(-\frac{2DX}{v}\right) \right\} \left[1 + \left\{ 1 - \frac{2\alpha\beta}{D_1 D_2} \right\} \exp\left(-\frac{2DX}{v}\right) \right]^{-1}$$
$$\ldots \text{(13.0)}$$

Neglecting terms containing higher powers of $\alpha\beta$ other than unity in the multinomial expansion of relation (13.0) we obtain,

$$V = -\frac{hvF}{E} \left[\frac{D}{D_1 D_2} - \left(\frac{2D}{D_1 D_2} - \frac{2\alpha\beta D}{D_1^2 D_2^2} \right) \exp\left(-\frac{2DX}{v}\right) + \left(\frac{2D}{D_1 D_2} - \frac{6\alpha\beta D}{D_1^2 D_2^2} \right) \right.$$

$$\left. \exp\left(-\frac{4DX}{v}\right) - \left(\frac{2D}{D_1 D_2} - \frac{10\alpha\beta D}{D_1^2 D_2^2} \right) \exp\left(-\frac{6DX}{v}\right) + \ldots \right] \qquad \ldots \text{(13.1)}$$

Thus,

$$V = \left[f'_1(t) - \{2f'_1(t) - 2f_2(t)\} \exp\left(-\frac{2DX}{v}\right) + \{2f'_1(t) - 6f_2(t)\} \exp\left\{-\frac{4DX}{v}\right\} \right.$$

$$\left. - \{2f'_1(t) - 10f_2(t)\} \exp\left(-\frac{6DX}{v}\right) + \ldots \right] \qquad \ldots \text{(13.2)}$$

where,
$$f_1(t) = -\frac{hvF}{E}\cdot\frac{1}{D_1D_2}$$

$$= -\frac{hvF}{E}\cdot\frac{1}{(\alpha-\beta)}\left[\frac{1}{\beta}(1-e^{-\beta t})-\frac{1}{\alpha}(1-e^{-\alpha t})\right] \quad \ldots \quad (14.0)$$

and
$$f_2(t) = -\frac{hvF}{E}\cdot\frac{\alpha\beta D}{D_1^2D_2^2}$$

$$= -\frac{hvF}{E}\frac{\alpha\beta}{(\alpha-\beta)^2}\left[\frac{1}{\beta}\left(1-\frac{\alpha+\beta}{\alpha-\beta}+\beta t\right)e^{-\beta t}+\frac{1}{\alpha}\left(1+\frac{\alpha+\beta}{\alpha-\beta}+\alpha t\right)e^{-\alpha t}\right]$$
$$\ldots (14.1)$$

Therefore,

$$V = \left[f'_1(t)-\left\{2f'_1\left(t-\frac{2X}{v}\right)-2f_2\left(t-\frac{2X}{v}\right)\right\}\right.$$
$$+\left\{2f'_1\left(t-\frac{4X}{v}\right)-6f_2\left(t-\frac{4X}{v}\right)\right\}$$
$$\left.-\left\{2f'_1\left(1-\frac{6X}{v}\right)-10f_2\left(t-\frac{6X}{v}\right)\right\}+\ldots\ldots\right] \quad \ldots \quad (15.0)$$

Relation (15.0) is the electrical voltage response corresponding to the given mechanical stress input to the Plate transducer, a result quite in agreement with that derived by Redwood (1961).

DISCUSSIONS

Discussion I : If further approximation be made in the right hand side expression for the value V in (15.0) by neglecting terms containing higher powers of h other than one, the expression for V is,

$$V = \left[f'_1(t)-2f_1'\left(t-\frac{2X}{v}\right)+2f'_1\left(t-\frac{4X}{v}\right)-\ldots\right] \quad \ldots \quad (16.0)$$

The first term of which is,

$$f'_1(t) = -\frac{hvF}{E}\cdot\frac{D}{D_1D_2} \quad \ldots \quad (17.0)$$

$$= -\frac{hvF}{E}\cdot\frac{e^{-\beta t}-e^{-\alpha t}}{\alpha-\beta} \quad \ldots \quad (17.1)$$

$$= \frac{hvc_0RF}{E}\left(1 - e^{-t/c_0R}\right) \qquad (17.2)$$

If now R is very large

$$V'_1(t) = \frac{hvF}{E}\cdot t \qquad (18.0)$$

Thus to the first order of approximation

$$V = \frac{hvF}{E}\cdot t \qquad (19.0)$$

Equation (19.0) shows V is linearly dependent on time t.

Discussion II. If periodic stress is applied i.e. if

$$F = F_i e^{i\omega t} = F_i \frac{D}{D - i\omega} \qquad (20.0)$$

The first term of the expression of V in equation (16.0) is

$$V'_1(t) = \frac{hvF_i}{E}\frac{D}{(D - i\omega)D_1 D_2} \qquad (21.0)$$

Since $-\alpha$, $-\beta$, $i\omega$ are the simple zeros of the denominator, by Heaviside's expansion theorem we get

$$V'_1(t) = \frac{hvF_i}{E}\left[\frac{\alpha e^{-\alpha t}}{(\alpha + i\omega)(\alpha - \beta)} + \frac{\beta e^{-\beta t}}{(\beta + i\omega)(\beta - \alpha)} + \frac{i\omega e^{i\omega t}}{(\alpha + i\omega)(\beta + i\omega)}\right] \qquad (21.1)$$

The real part of the right hand side of (21.1) is given by,

$$\frac{hvF_i}{E}\left[\frac{\alpha^2 e^{-\alpha t}}{(\alpha^2 + \omega^2)(\alpha - \beta)} + \frac{\beta^2 e^{-\beta t}}{(\beta^2 + \omega^2)(\beta - \alpha)} + \frac{\omega^2(\alpha + \beta)}{(\alpha^2 + \omega^2)(\beta^2 + \omega^2)}\cos\omega t\right.$$

$$\left. + \frac{\omega(\alpha\beta - \omega^2)}{(\alpha^2 + \omega^2)(\beta^2 + \omega^2)}\sin\omega t\right] \quad \ldots \quad (22.0)$$

Since the higher powers of h is neglected we have, $\beta = 0$, $\alpha = 1/c_0R$. If now R is very large $\alpha \to 0$ and the first termof V becomes

$$V = \frac{hvF_i}{E}\cdot\frac{1}{\omega}\sin\omega t \quad \ldots \quad (23.0)$$

quite similar to that obtained by Sinha in the case of an open circuit.

If further ω is very small (23.0) becomes

$$\frac{hvF_1}{E} \qquad \ldots \quad (24.0)$$

Thus the electrical voltage response under different cases and approximations can be obtained by results in relations (6.3) (7.0) (15.0) (23.0) and (24.0).

The converse problem i.e the Mechanical stress response corresponding to voltage input is under consideration of the authors and will be published soon.

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