FINITE SPHERICAL INHOMOGENEITIES IN CONCENTRIC SHELLS

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ABSTRACT Inclusion and inhomogeneity problems in infinitesimal elasticity have been studied by various authors, but such problems in finite elasticity have no tbeen attempted so far. The present paper is concerned with spherical inhomogeneities when the elastic deformation is large. The shells considered are isotropic and incompressible. The problem has been solved for two shells and later generalised for n shells embedded within each other. Further, the outermost and the inner-most boundaries of the system have been subjected to uniform normal pressures. It will be observed that the final equation determining the parameter giving the equilibrium boundary has irrational roots and could be solved numerically or graphically. Because of the complete symmetry with respect to r, the elastic field would be function of r only If W be the elastic potential density at r, U the complete elastic potential r, and T^{ij} the stress components then

$$U = \int_{R_1}^{R_2} 4\pi R^2 W(R)dR.$$

$$\frac{4}{3}\pi(R_1^3-r_1^3)\Pi - \frac{4}{3}\Pi(R_2^3-r_2^3)\Pi_2 = U + \frac{4}{3}\pi r_1^3 W(R_1) - \frac{4}{3}\pi r_2^3 W(R_2) \quad .. \quad (1)$$

$$T^{ij} = \phi g^{ij} + \psi B^{ij} + pG^{ij} \quad .. \quad (2)$$

Equation (1) is a very important relation between boundary pressures, boundary displacements and the predetermined strain-energy of the body Equation (2) gives the stress field everywhere For derivation of these equations etc., please refer to [3].

SPHERICAL INCLUSION

Two shells case

Let a homogeneous, isotropic spherical shell made of incompressible material with its outer and inner radii $a_1(1+\delta)$, δ being finite, and a_2 respectively be embedded into another similar shell of a different incompressible material with outer and inner radii a_0 and a_1 respectively. The former shell will be called 'inclusion' and the latter 'matrix'. Further, let the outer boundary of the matrix be subjected to a uniform normal pressure p_0 and the inner boundary of the inclusion be subjected to a similar pressure p_i in the equilibrium position. It is further assumed that no relative slipping takes place between the two shells. Due to misfit both the shells will be stressed. Let r be the inner radius of the inclusion and R the outer radius of matrix in the strained state, similarly let $a_1(1+\epsilon)$, ϵ finite, be the radius of the common interface. Let p be the pressure at the equilibrium boundary Let W_i, U_i respectively be the elastic potential per unit volume and the total elastic potential of the inclusion in the undeformed state and W_m, U_m, the corresponding, quantities in the case of the matrix.

From (1), for the inclusion we have

$$\frac{4}{3}\pi(r^3-a_2^3)p_i-\frac{4}{3}\pi a_1^3\{(1+\epsilon)^3-(1+\delta)^3\}$$

$$= U_i+\frac{4}{3}\pi a_2^3W_i(r)-\frac{4}{3}\pi a_1^3(1+\delta)^3W_i\{a_1(1+\epsilon)\} \quad \ldots \quad (3)$$

The incompressibility condition gives

$$r^3-a_2^3 = a_1^3\{(1+\epsilon)^3-(1+\delta)^3\} \quad \ldots \quad (4)$$

Hence from equations (3) and (4)

$$a_1^3\{(1+\epsilon)^3-(1+\delta)^3\}(p_i-p)=\frac{3}{4\pi}U_i+a_2W_i(r)-a_1^3(1+\delta)^3W_i\{a_1(1+\epsilon)\}$$

The right hand side of the above equation is a known quantity say I, where

$$I=\frac{3}{4\pi}U_i+a_2W_i(r)-a_1^3(1+\delta)^3W_i\{a_1(1+\epsilon)\}.$$

Hence

$$a_1^3\{(1+\epsilon)^3-(1+\delta)^3\}(p_i-p)=I \quad \ldots \quad (5)$$

For the matrix in the deformed state the inner boundary is $a_1(1+\epsilon)$ and the outer boundary is R As in the case of inclusion we shall have

$$a_1^3\{(1+\epsilon)^3-1\}(p-p_0)=M \quad (6)$$

where

$$M=\frac{3}{4\pi}U_m+a_1^3W_m\{a_1(1+\epsilon)\}-a_0^3W_m(R)$$

Eliminating p from (5) and (6)

$$a_1^3(p_i-p_0)=\frac{I}{(1+\epsilon)^3-(1+\delta)^3}+\frac{M}{(1+\epsilon)^3-1}$$

or $$a_1^3(p_i-p_0)(1+\epsilon)^6-[a_1^3(p_i-p_0)\{(1+\delta)^3+1\}+I+M](1+\epsilon)^3$$
$$+\{a_1^3(p_i-p_0)+M\}(1+\delta)^3+I=0 \quad (7)$$

This will give the value of ϵ in terms of known quantities

$$\delta,\ p_i, p_0,\ I,\ M$$

The equilibrium pressure p can also be determined from (5) and (6)

$$2p=p_0-p_i+\frac{M}{(1+\epsilon)^3-1}-\frac{I}{(1+\epsilon)^3-(1+\delta)^3} \quad (8)$$

where ϵ is given by (7).

HOOP STRESS ON THE EQUILIBRIUM BOUNDARY

For the inclusion, at the equilibrium boundary, let

$$Q = \frac{1+\delta}{1+\epsilon}$$

The Hoop stress $\sigma_i^{22} = R^2 T_i^{22}$ for inclusion is given by

$$\sigma_i^{22} = R^2\left[\left\{\left(\frac{1+\epsilon}{1+\delta}\right)^2 - \left(\frac{1+\delta}{1+\epsilon}\right)^4\right\}\phi_i + \left\{\left(\frac{1+\epsilon}{1+\delta}\right)^4 - \left(\frac{1+\delta}{1+\epsilon}\right)^2\right\}\psi_i + I - p\right]$$

Similarly, for the matrix $Q = \frac{1}{1+\epsilon}$ hence

$$\sigma_m^{22} = R^2\left[\left\{(1+\epsilon)^2 - \frac{1}{(1+\epsilon)^4}\right\}\phi_m + \left\{(1+\epsilon)^4 - \frac{1}{(1+\epsilon)^2}\right\}\psi_m + M - p\right] \quad \dots \quad (10)$$

The jump in hoop stress at the equilibrium boundary is given by

$$\sigma_m^{22} - \sigma_i^{22} = R^2\left[\left\{(1+\epsilon)^2 - \frac{1}{(1+\epsilon)^4}\right\}\phi_m - \left\{\left(\frac{1+\epsilon}{1+\delta}\right)^2 - \left(\frac{1+\delta}{1+\epsilon}\right)^4\right\}\phi_i\right.$$

$$\left. + \left\{(1+\epsilon)^4 - \frac{1}{(1+\epsilon)^2}\right\}\psi_m - \left\{\left(\frac{1+\epsilon}{1+\delta}\right)^4 - \left(\frac{1+\delta}{1+\epsilon}\right)^2\right\}\psi_i + M - I\right]. \quad (11)$$

The ratio of hoop-stress at the equilibrium boundary

$$\frac{\sigma_m^{22}}{\sigma_i^{22}} = \frac{\left\{(1+\epsilon)^2 - \frac{1}{(1+\epsilon)^4}\right\}\phi_m + \left\{(1+\epsilon)^4 - \frac{1}{(1+\epsilon)^2}\right\}\psi_m + M - p}{\left\{\left(\frac{1+\epsilon}{1+\delta}\right)^2 - \left(\frac{1+\delta}{1+\epsilon}\right)^4\right\}\phi_i + \left\{\left(\frac{1+\epsilon}{1+\delta}\right)^4 - \left(\frac{1+\delta}{1+\epsilon}\right)^2\right\}\psi_i + I - p} \quad (12)$$

Special cases of solid incompressible and rigid inclusions can be deduced from (5) and (6). In both these cases it may easily be shown that the equilibrium boundary will coincide with the outer boundary of the inclusion, i.e. $\epsilon = \delta$, which is obvious on physical grounds. When the matrix is rigid the equilibrium boundary will again coincide with the inner boundary of the matrix, i.e. $\epsilon = -\delta$ and the equilibrium pressure is given by (6). Assuming the strain energy form to be that given by linear elasticity, and assuming the infinitesimal displacements, it can be easily verified that ϵ and p as deduced from equations (7) and (8) are the same as obtained in [1], [2], [3].

It may be observed that the solutions given above require the knowledge of nature of the elastic potentials of the materials used for both the inclusion and the matrix. Mooney [1940] has empirically formulated the expression for some materials like rubber. It is

$$W = C_1(I_1-3)+C_2(I_2-3).$$

where C'_1 and C'_2 are known constants of the material and I_r $(r = 1, 2)$ are the strain invariants.

Hence for Mooney materials it may be verified that

$$\Pi_1 - \Pi_1 = C'_1(Q_2^4 + 4Q_2) + 2C'_2\left(Q_2^2 - \frac{2}{Q_2}\right) - C'_1\left(\frac{1}{\lambda_1^4} + \frac{4}{\lambda_1^2}\right)$$
$$-2C'_2\left(\frac{1}{\lambda_1^2} - 2\lambda_1\right).$$

where $\qquad Q_1 = \frac{1}{\lambda_1} = \frac{r_1}{R_1} \qquad$ and $\qquad Q_2 = \frac{r_2}{R_2}$. $\qquad \dots$ (13)

In the present problem for such inclusion materials we have

$$\left.\begin{aligned} K_i(R_2) = {} & C'_1\frac{1+\delta}{1+\epsilon}\left\{\left(\frac{1+\delta}{1+\epsilon}\right)^3+4\right\} + 2C'_2\frac{1+\epsilon}{1+\delta}\left\{\left(\frac{1+\delta}{1+\epsilon}\right)^3-2\right\} \\ & - \frac{a_2C'_1}{\{a_1^3(1+\epsilon)^3 - a_1^3(1+\delta)^3 + a_2^3\}^{1/3}}\left\{\frac{a_2^3}{a_1^3(1+\epsilon)^3 - a_1^3(1+\delta)^3 + a_2^3}+4\right\} \\ & - \frac{2C'_2\{a_1^3(1+\epsilon)^3 - a_1^3(1+\delta)^3 + a_2^3\}^{1/3}}{a_2}\left\{\frac{a_2^3}{a_1^3(1+\epsilon)^3 - a_1^3(1+\delta)^3 + a_2^3}-2\right\} \end{aligned}\right\} \quad (14)$$

where

$$K(R) = \int_{Q_1}^{Q_2}\left[(Q^2+1)\phi + \left(Q - \frac{1}{Q^2}\right)\psi\right] dQ\,;$$

$$K(R_1) = 0, \quad Q = \frac{r}{R}$$

since $\quad Q_2 = \frac{1+\delta}{1+\epsilon}, \quad Q_1 = \frac{1}{\lambda_1}$

$$= \frac{a_2}{\{a_1^3(1+\epsilon)^3 - a_1^3(1+\delta)^3 + a_2^3\}^{1/3}}$$

For the matrix we have

$$\left.\begin{aligned} k_m(R_2) = {} & \frac{a_0C''_1}{\{a_1^3(1+\epsilon)^3 - a_1^3 + a_0^3\}^{1/3}}\left\{\frac{a_0^3}{a_1^3(1+\epsilon)^3 - a_1^3 + a_0^3}+4\right\} \\ & + \frac{2C''_2\{a_1^3(1+\epsilon)^3 - a_1^3 + a_0^3\}^{1/3}}{a_0}\left\{\frac{a_0^3}{a_1^3(1+\epsilon)^3 - a_1^3 + a_0^2}-2\right\} \\ & - \frac{C''_1}{1+\epsilon}\left\{\frac{1}{(1+\epsilon)^3}+4\right\} - 2C''_2(1+\epsilon)\left\{\frac{1}{(1+\epsilon)^3}-2\right\}. \end{aligned}\right\} \quad ..(15)$$

since

$$Q_2 = \frac{a_0}{\{a_1^3(1+\epsilon)^3 - a_1^3 + a_0^3\}^{1/3}}, Q_1 = \frac{1}{\lambda_1} = \frac{1}{1+\epsilon}.$$

Finally we have

$$p_i - p_0 = k_i(R_2) + k_m(R_2) \qquad \ldots \quad (16)$$

Substituting for $k_i(R_2)$ $K_m(R_2)$ and simplifying we get

$$\left.\begin{aligned} p - p_0 = {} & C'_1 \frac{1+\delta}{1+\epsilon}\left\{\left(\frac{1+\delta}{1+\epsilon}\right)^3 + 4\right\} + 2C'_2 \frac{1+\epsilon}{1+\delta}\left\{\left(\frac{1+\delta}{1+\epsilon}\right)^3 - 2\right\} - \\ & - \frac{a_2 C'_1}{\{a_1^3(1+\epsilon)^3 - a_1^3(1+\delta)^3 + a_2^3\}^{1/3}}\left\{\frac{a_2^3}{a_1^3(1+\epsilon)^3 - a_1^3(1+\delta)^3 + a_1^3} + 4\right\} \\ & - \frac{2C'_2\{a_1^3(1+\epsilon)^3 - a_1^3(1+\delta)^3 + a_2^3\}^{1/3}}{a_2}\left\{\frac{a_2^3}{a_1^0(1+\epsilon)^3 - a_1^3(1+\delta)^3 + a_2^3} - 2\right\} + \\ & + \frac{a_0 C''_1}{\{a_1^3(1+\epsilon)^3 - a_1^3 + a_0^3\}^{1/3}}\left\{\frac{a_0^3}{a_1^3(1+\epsilon)^3 - a_1^3 + a_0^3} + 4\right\} + \\ & + \frac{2C''_2\{a_1^3(1+\epsilon)^3 - a_1^3 + a_0^3\}^{1/3}}{a_0}\left\{\frac{a_0^3}{a_1^3(1+\epsilon)^3 - a_1^3 + a_0^3} - 2\right\} - \\ & - \frac{C''_1}{1+\epsilon}\left\{\frac{1}{(1+\epsilon)^3} + 4\right\} - 2C''_2(1+\epsilon)\left\{\frac{1}{(1+\epsilon)^3} - 2\right\}. \end{aligned}\right\} \qquad (17)$$

which gives ϵ and the equilibrium boundary is determined. Further, the equilibrium pressure is given by

$$2p = p_0 - p + K_m(R_2) - K_i(R_2) \qquad \ldots \quad (18)$$

n Shells' Case

The problem can be extended to the case of n elastic shells embedded within each other. Let an isotropic, incompressible, spherical shell A_1 with outer and inner radii a_0, a_1 respectively have another similar shell A_2 of different material and outer and inner radii $a_1(1+\delta_1)$, a_2 respectively embedded within it. Further let a third shell A_3 with outer and inner radii $a_2(1+\delta_2)$, a_3 respectively be embedded in A_2. Like this, let the shell A_{r+1} with outer and inner radii $a_r(1+\delta_r)$, a_{r+1}, be embedded in A_r whose outer and inner radii are $a_{r-1}(1+\delta_{r-1})$, a_r respectively Finally, let the outer and inner radii of A_n be $a_{n-1}(1+\delta_{n-1})$, a_n respectively it is assumed that δ_r are finite. Let the outer boundary of A_1 be subjected to uniform compressive pressure p_0 and the inner boundary of A_n be subjected to a similar pressure p_i. It is assumed that all the shells are homogeneous isotropic and made of different incompressible materials. Due to misfits strains will develop in the system. Let the outer radius of A_1 in the deformed state be R and the inner one of A_n in the same state be r, also let the equilibirum boundaries of A_r and A_{r+1} be $a_r(1+\epsilon_r)$, ϵ_r being finite and $r = 1, 2, \ldots, n-1$. Let the equilibrium pressures on the interfaces be $p_r (r = 1, 2, \ldots, n-1)$. The problem gives rise to $2n$ unknown quantities, viz., ϵ_r, p_r $(r = 1, 2, \ldots, n-1)$ r and R. Which are determined by the equations connecting the boundary pressures and the strain energy of the n shells A_r alongwith another set of n equations giving the incompressibility conditions as shown below

For A_1 we have from (11)

$$a_1^3\{(1+\epsilon_1)^3-1\}p_1-(R^3-a_0^3)p_0 = \frac{3}{4\pi}U_1+a_1^3W_1\{a_1(1+\epsilon_1)\}-a_0^3W_1(R)$$

$$= \Omega_1 \quad \text{(say)} \tag{1}$$

The incompressibility condition gives $a_0^3 - a_1^3 = R^3 - a_1^3(1+\delta_1)^3$ (1′)

Similarly for A_2 we have

$$a_2^3\{(1+\epsilon_2)^3-1\}p_2 - a_0^3\{(1+\epsilon_1)^3-(1+\delta_1)^3\}p_1 = \Omega_2 \tag{2}$$

and

$$a_1(1+\delta_1)^3 - a_2^3 = a_1^3(1+\epsilon_1)^3 - a_2^3(1+\epsilon_2)^3 \tag{2'}$$

For A_r

$$a_r^3\{(1+\epsilon_r)^3-1\}p_r - a^3_{r-1}\{(1+\epsilon_{r-1})^3-(1+\delta_{r-1})^3\}p_{r-1} = \Omega_r \quad \ldots \text{(r)}$$

and from incompressibility condition

$$a^3_{r-1}(1+\delta_{r-1})^3 - a_r^3 = a^3_{r-1}(1+\epsilon_{r-1})^3 - a^3_r(1+\epsilon_r)^3 \quad \ldots \text{(r′)}$$

where $\Omega_r = \frac{3}{4\pi}U_r + a_r^3W_r\{a_r(1+\epsilon_r)\} - a^3_{r-1}W_r\{a_{r-1}(1+\epsilon_{r-1})\}$

Finally for A_n we have

$$(r^3-a_n^2)p_i - a^3_{n-1}\{(1+\epsilon_{n-1})^3-(1+\delta_{n-1})^3\}p_{n-1} = \Omega'_n \quad \ldots \text{(n)}$$

and

$$a^3_{n-1}(1+\delta_{n-1})^3 - a_n^3 = a^3_{n-1}(1+\epsilon_{n-1})^3 - r^3 \quad \ldots \text{(n′)}$$

where $\Omega'_n = \frac{3}{4\pi}U_n + a^3_nW_n(r) - a^3_{n-1}W_n\{a_{n-1}(1+\epsilon_{n-1})\}$.

This set of $2n$ equations completely determines the $2n$ unknowns ϵ_r, p_r $(r = 1, 2, \ldots n-1)$, r and R.

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