ELASTIC MISFITTING SHELLS

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ABSTRACT. In this paper the problem of n elastic spherical and tubular shells misfitting in each other is considered. Linear simultaneous equations determining the equilibrium boundaries have been formulated, the solution of which gives the values of the parameters determining not only the equilibrium configuration but also the stress-strain field and the related problems in the structure. Results for a particular problem, when the shells are 3 in number, are given for the case of spherical shells.

INTRODUCTION

Consider a spherical shell of outer radius a_0 and inner radius a_1, in which a concentric shell of outer radius $a_1(1+\delta_1)$ and inner radius a_2 is embedded. In this latter shell another one of outer radius $a_2(1+\delta_2)$ and inner radius a_3 is embedded. In this way let a shell of outer radius $a_r(1+\delta_r)$ and inner radius a_{r+1} be embedded into the shell of outer radius $a_{r-1}(1+\delta_{r-1})$ and inner radius a_r. This is schematically shown in the adjoining figure. Each of the δ's are supposed to be within the elastic limits. Further we suppose that no relative slipping takes place and continuity of the material is maintained throughout.

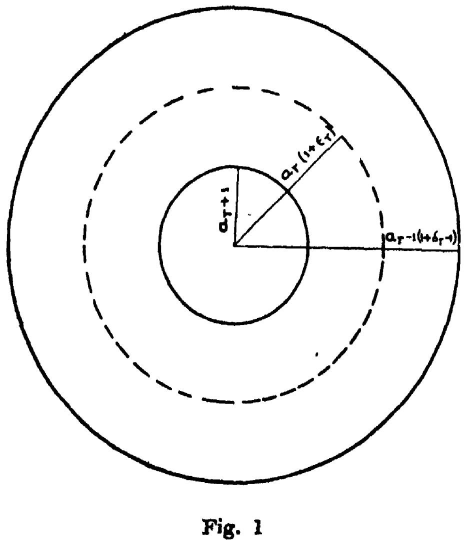

Fig. 1

Due to the misfits in the sizes of the shells stresses develop within the structure. Determination of the elastic field and the equilibrium position form the subject matter of the paper.

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Such problems have been studied by Mott and Nabarro (1940), Frankel (1946), Jaswon and Bhargava (1961), Bhargava and Radhakrishna (communicated) but in each case there was one single solid material inside. In a recent paper Bhargava and Pande (1963), have considered hollow inclusions. This paper generalises the above case in as much as the inner materials are hollow and are more in number. This problem is technically important as it is useful when the boundaries are reinforced. The problem has been solved by Energy Method suggested by one of the authors Bhargava (1963). This consists in taking an arbitrary, physically consistent equilibrium position and finding the energy in the material. That position will give the true equilibrium boundary which minimises the energy.

For ease of exposition we name the shells as follows : The shell whose outer and inner radii are $a_{r-1}(1+\delta_{r-1})$ and a_r respectively be named A_r. It may be noted that for the outermost shell the outer radius is a_r i.e. $\delta_0 = 0$.

On physical grounds the interface both in the case of spherical as well as tubular shells will be concentric spherical or tubular. We thus take the common boundary of A_r and A_{r+1} to be $a_r(1+c_r)$. We find the energy in the medium consisting of all the shells. We first give briefly the case for spherical shells.

Spherical shells : Each shell will be under uniform normal pressure due to the shells above and below it. It is known that for such a case, the normal, hoop and shear stresses p_{rr}, $p_{\theta\theta}$ and $p_{r\theta}$ are respectively of the form

$$p_{rr} = \frac{\zeta}{r^3}+D; \quad p_{\theta\theta} = -\frac{\zeta}{2r^3}+D, \quad p_{r\theta} = 0. \qquad \ldots \quad (1)$$

The radial and transverse displacements are

$$u_r = -\frac{c}{4\mu r^2}+\frac{D}{3K}r\ ; \ u_\theta = 0 \qquad \text{respectively.} \qquad \ldots \quad (2)$$

The radial, hoop and shear strains will respectively be

$$e_{rr} = \frac{c}{2\mu r^3}+\frac{D}{3k}\ ; \ e_{\theta\theta} = -\frac{\zeta}{4\mu r^3}+\frac{D}{3k} \ \text{ and } \ e_{r\theta} - 0 \qquad \ldots \quad (3)$$

μ and K being the shear and bulk moduli of the material. Let μ_{r-1} and K_{r-1} be the shear and bulk moduli for A_r.

As the transverse displacements are zero throughout we write u_r for the radial displacement. Let the radial displacements for the outer and inner boundaries of A_r respectively be

$$u_0 = -a_{r-1}(\delta_{r-1}-\epsilon_{r-1}) \quad \text{and} \quad u_i = a_r\epsilon_r.$$

On substituting these values in (2) and solving for C and D we get

$$\zeta_r = -\frac{4\mu_{r-1}a^3_{r-1}a_r^3\{\delta_{r-1}-\epsilon_{r-1}+\epsilon_r\}}{a^3_{r-1}-a_r^3}\;;\; D_r = -\frac{3k_{r-1}\{a_r^3\epsilon_r+a^3_{r-1}(\delta_{r-1}-\epsilon_{r-1})\}}{a^3_{r-1}-a^3_r} \tag{3a}$$

for the shell A_r.

The total mechanical energy of the shell A_r is given by

$$W_r = \tfrac{1}{2}\int_{a_r}^{a_{r-1}}\{p_{rr}e_{rr}+2p_{\theta\theta}e_{\theta\theta}\}4\pi r^2dr - \iiint_V F_r dv - \iiint_\Omega f\,d\sigma$$

where the three terms of the right member of this equation give energy due to elastic forces, body forces and the forces on the boundary. But there being no body or surface forces, the last two terms will contribute nothing. Hence on substituting for p_{rr}, $p_{\theta\theta}$, e_{rr} and $e_{\theta\theta}$ and integrating we get the energy for A_r as

$$W_r = 2\pi\{a^3_{r-1}-a^3_r\}\left[\frac{C_r^2}{4\mu_{r-1}a^3_{r-1}a^3_r}+\frac{D_r^2}{3k_{r-1}}\right].$$

It may be noted that for A_1 and A_n the expressions for energy would not be symmetrical to A_r. They would actually be

$$W_1 = \frac{24\pi\mu_0k_0a_1^3(a_0^3-a_1^3)}{4\mu_0a_1^3+3k_0a_0^3}\epsilon_1^2 \quad W_n = \frac{24\pi\mu_{n-1}k_{n-1}a^3_{n-1}(a^3_{n-1}-a^3_n)}{4\mu_{n-1}a^3_{n-1}+3k_{n-1}a^3_n}(\delta_{n-1}-c_{n-1})^2.$$

The energy for the whole system would be $W = \sum_{r=1}^{n} W_r$.

The true values of ϵ_r are those which minimise the value of W. By the known theorem the extreme values of W are obtained by solving $\partial W/\partial\epsilon_r = 0$ ($r = 1, 2, \ldots n-1$). On simplifying we obtain the following set of equations for determining ϵ_r.

$$(B'_0+B_1+a_1^6B_2)\epsilon_1 - (B_1+a_1^3a_2^3B_2)\epsilon_2 = (B_1+a_1^6B_2)\delta_1$$

$$(B_1+a_1^3a_2^3)\epsilon_1-(B_1+a_2^6B_2+B_3+a_2^6B_4)\epsilon_2+(B_3+a_2^3a_3^3B_4)\epsilon_3$$

$$= (B_1+a_2^3a_3^3B_2)\delta_1-(B_3+a_2^6B_4)\delta_2$$

$$(B_3+a_2^3a_3^3B_4)\epsilon_2-(B_3+a_3^6B_4+B_5+a_3^6B_6)\epsilon_3+(B_5+a_3^3a_4^3B_6)\epsilon_4$$

$$= (B_3+a_3^3a_4^3B_4)\delta_2-(B_5+a_3^6B_6)\delta_3$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$(B_{2r-3}+a_{r-1}^3a_r^3B_{2r-2})\epsilon_{r-1}-(B_{2r-3}+a_r^6B_{2r-2}+B_{2r-1}+a_r^6B_{2r})\epsilon_r$$

$$+(B_{2r-1}+a_r^3a_{r+1}^3B_{2r})\epsilon_{r+1}=(B_{2r-3}+a_{r-1}^3a_r^3B_{2r-2})\delta_{r-1}-(B_{2r-1}+a_{r+1}^6B_{2r})\delta_r$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \tag{3b}$$

$$(B_{2n-5}+a_{n-2}^3a_{n-2}^3B_{2n-4})\epsilon_{n-2}-(B_{2n-5}+a_{n-1}^6B_{2n-4}+B_n')\epsilon_{n-1}$$
$$= (B_{2n-5}+a_{n-2}^3a_{n-1}^3B_{2n-4})\delta_{n-2}-B_n'\delta_{n-1}$$

where $$B'_0 = \frac{12\mu_0k_0a_1^3(a_0^3-a_1^3)}{4\mu_0a_1^3+3k_0a_0^3}\ ;\ B_{2r-1} = \frac{4\mu_ra_r^3a_{r+1}^3}{a_r^3-a_{r+1}^3}\ ;$$

$$B_{2r} = \frac{3k_r}{a_r^3-a_{r+1}^3}\ ;\ B'_n = \frac{12\mu_{n-1}k_{n-1}a_{n-1}^3(a_{n-1}^3-a_n^3)}{4\mu_{n-1}a_{n-1}^3+3k_{n-1}a_n^3}.$$

Note that all B_k are constants.

These equations can be more systematically put in the matrix form

$$L\epsilon = M\delta$$

where ϵ is a column vector $\{\epsilon_1, \epsilon_2, \ldots\ldots \epsilon_{n-1}\}$, L is a symmetric matrix of order $(n-1)\times(n-1)$.

$$\begin{bmatrix} -(B_0'+B_1+a_1^6B_2) & (B_1+a_1^3a_2^3B_2) & & 0 & 0 & \ldots & 0 & 0 \\ (B_1+a_1^3a_2^3B_2) & -(B_1+a_2^6B_2+B_3+a_2^6B_4) & (B_3+a_2^3a_3^3B_4) & 0 & 0 & \ldots & 0 & 0 \\ 0 & (B_3+a_2^3a_3^3B_4) & -(B_3+a_3^6B_4+B_5+a_3^6B_6) & (B_5+a_3^3a_4^3B_6)\ldots & & & 0 & 0 \\ — & — & — & — & \ldots & — & — & — \\ — & — & — & — & \ldots & — & — & — \\ 0 & 0 & 0 & 0 & & -(B_{2n-7}+a_{n-2}^6B_{2n-6}+B_{2n-5}+a_{n-2}^6B_{2n-4}) & & (B_{2n-5}+a_{n-2}^3a_{n-1}^3B_{2n-4}) \\ 0 & 0 & 0 & 0 & \ldots & (B_{2n-5}+a_{n-2}^3a_{n-1}^3B_{2n-4}) & & -(B_{2n-5}+a_{n-1}^6B_{2n-4}+B_n') \end{bmatrix}$$

The determinant of the above matrix is called the continuant matrix. It is comparatively easy to find a recurring inversion formula for such a matrix.

M is the matrix of order $(n-1)(n-1)$

$$\begin{bmatrix} -(B_1+a_1^6B_2) & 0 & 0 & \ldots & \ldots & 0 \\ (B_1+a_1^3a_2^3B_2) & -(B_3+a_2^6B_4) & 0 & \ldots & 0 & 0 \\ 0 & (B_3+a_2^3a_3^3B_4) & -(B_5+a_3^6B_6) & \ldots & 0 & 0 \\ — & — & — & \ldots & — & — \\ — & — & — \ldots & — & — & \\ 0 & 0 & 0 & & (B_{2n-5}+a^3_{n-2}a^3_{n-1}B_{2n-4}) & -B^1_n \end{bmatrix}$$

δ is the column vector $\{\delta_1, \delta_2 \ldots \delta_{n-1}\}$.

The value of ϵ will be

$$\epsilon = L^{-1}M\delta$$

where L^{-1} is the inverse matrix of L.

This gives the values of ϵ_r in terms of known quantities.

Having known ϵ_r, p_{rr}, $p_{\theta\theta}$, $p_{r\theta}$ can be found from equation (1), after finding the values of C_r, D_r from equations (3*a*). It is difficult, in the general case to prove the continuity of the normal stress p_{rr} at the equilibrium interface. This can, however, be seen indirectly from the following argument. At the interface of A_r and A_{r+1} if p_{rr} is to be continuous, we must have

$$\frac{C_r}{a_r^3(1+\epsilon_r)^3}+D_r = \frac{C_{r+1}}{a_r^3(1+\epsilon_r)^3}+D_{r+1}, \quad \text{i.e.} \quad C_r-C_{r+1} = a_r^3(D_{r+1}-D_r),$$

to the first order of approximation.

This equation is identical with the simultaneous equations obtained above when approximate values of C_r, C_{r+1}, D_r, D_{r+1} are substituted. In fact, the equation is the equation (3b).

Tubular Shells : For the tubular shells we use the same notation as for the spherical shells. In this case also each shell would be under uniform normal pressure due to similar shells above and below it. The normal, hoop and shear stresses in this case will be

$$p_{rr} = \frac{C}{r^2}+D; \quad p_{\theta\theta} = -\frac{C}{r^2}+D; \qquad p_{r\theta} = 0 \qquad \ldots \quad (1)$$

radial and transverse displacements will be

$$u_r = -\frac{C}{2\mu r}+\frac{D}{2(\lambda+\mu)}r; \quad u_\theta = 0 \qquad \ldots \quad (2)$$

and, radial, hoop and shear strains will be

$$e_{rr} = \frac{C}{2\mu r}+\frac{D}{2(\lambda+\mu)}; \quad e_{\theta\theta} = -\frac{C}{2\mu r}+\frac{D}{2(\lambda+\mu)} : \quad e_{r\theta} = 0$$

where λ and μ are the Lame's constants. For A_r let these constants be λ_{r-1}, μ_{r-1}.

As throughout the transverse displacements are zero we write for the radial displacement u_0 for A_r. Let $u_0 = -a_{r-1}(\delta_{r-1}-\epsilon_{r-1})$ be the displacement at the outer boundary and $u_i = a_r\epsilon_r$ the corresponding displacement at the inner boundary.

Substituting these values of u_0, u_i in (2) and solving for C_r and D_r we get

$$C_r = -\frac{2\mu_{r-1}a_{r-1}^2a_r^2(\delta_{r-1}-\epsilon_{r-1}+\epsilon_r)}{a_{r-1}^2-a_r^2} \text{ and } D_r = -\frac{2(\lambda_{r-1}+\mu_{r-1})\{a_r^2\epsilon_r+a_{r-1}^2(\delta_{r-1}\epsilon_{r-1})\}}{a_{r-1}^2-a_r^2}$$

For the outermost and the innermost shells these constants are evaluated from the equations obtained by equating to zero the normal pressure at the outer boundary in the first case and inner boundary in the second case and equating the displacements to $a_1\epsilon_1$, at the inner and $-a_{n-1}(\delta_{n-1}-\epsilon_{n-1})$ at the outer boundary.

Thus the elastic strain energy for A_r will be

$$V_r = \frac{1}{2}\int_{a_r}^{a_{r-1}}(p_{rr}e_{rr}+p_{\theta\theta}e_{\theta\theta})2\pi r\,dr = \pi(a_{r-1}^2-a_r^2)$$

$$\left[\frac{\zeta_r^2}{2\mu_{r-1}a_{r-1}^2a_r^2}+\frac{D_r^2}{2(\lambda_{r-1}++\mu_{r-1})}\right].$$

Also elastic strain energy for A_1 and A_n will be

$$V_1 = \frac{2\pi\mu_0(\lambda_0+\mu_0)a_1^2(a_0^2-a_1^2)}{\mu_0a_1^2+(\lambda_0+\mu_0)a_0^2}\,\epsilon_1^2$$

and $$V_n = \frac{2\pi\mu_{n-1}(\lambda_{n-1}+\mu_{n-1})a_{n-1}^2(a^2{}_{n-1}-a_n^2)}{\mu_{n-1}a_{n-1}^2+(\lambda_{n-1}+\mu_{n-1})a_n^2}\,(\delta_{n-1}-\epsilon_{n-1})^2.$$

The total elastic strain energy of the system, therefore, will be

$$V = \sum_{r=1}^{n} V_r$$

We know that the total mechanical energy for the system

$$W = V-\iiint_V F_r dv-\iint_\Omega f_r d\sigma$$

where the second and third integrals signify the energy due to body forces F_r and the boundary forces f_r of the system. In this case since both F_r and f_r are zero we have, therefore.

$$W = V$$

The true values of ϵ_r as in the spherical shell case will minimise W.

Thus equating $\partial W/\partial\epsilon_r = 0$ and simplifying we get the following set of equations.

$$(B''_0+B'_1+a_1^4B'_2)\epsilon_1-(B'_1+a_1^2a_2^2B'_2)\epsilon_2 \qquad = (B'_1+a_1^4B'_2)\delta_1$$

$$(B'_1+a_1^2a_2^2B_1^2)\epsilon_1-(B'_1+a_2^4B'_2+B'_2+a_2^4B'_4)\epsilon_2+(B'_3+a_2^2a_3^2B'_4)\epsilon_3$$
$$=(B'_1+a_1^2a_2^2B'_2)\delta_1-(B'_2+a_2^4B'_3)\delta_2$$

$$(B'_3+a_2^2a_3^2B'_4)\epsilon_2-(B'_3+a_3^4B'_4+B'_5+a_3^4B'_6)\epsilon_3+(B'_5+a_3^3a_3^4B'_6)\epsilon_4$$
$$= (B'_3+a_2^2a_3^2B'_4)\delta_2-(B'_4+a_3^4B'_5)\delta_3)$$

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$$(B'_{2r-3}+a^2_{r-1}a_r^2B'_{r-2})\epsilon_{r-1}-(B'_{2r-3}a_r^4B'_{2r-2}+B'_{2r-1}+a_r^4B'_{2r})\epsilon_r$$
$$+(B'_{2r-1}+a_r^2a^2_{r+1}B'_{2r})\epsilon_{r-1} = (B'_{2r-3}+a^2_{r-1}a_r^2B'_{2r-2})\delta_{r-1}$$
$$-(B'_{2r-2}+a_r^6B'_{2s-1})\delta_r$$

— — — — —

$$(B'_{2n-5}+a^2_{n-2}a^2_{n-1}B'_{2n-4})\epsilon_{n-2}-(B'_{2n-5}+a^4_{n-1}B'_{2n-4}+B''_n)\epsilon_{n-1}$$
$$= (B'_{2n-5}+a^2_{n-2}a^2_{n-1}B'_{2n-4})\delta_{n-2}-B''_n\delta_{n-1}$$

$$B''_o = \frac{\mu_0a_1^2(a_0^2-a_1^2)}{\mu_0a_1^2+(\lambda_0+\mu_0)a_0^2}\,; \quad B'_{2r-1} = \frac{2\mu_ra_r^2a^2_{r-1}}{a_r^2-a^2_{r-1}}$$

$$B'_{2r} = \frac{2(\lambda_{r-1}+\mu_{r-1})}{a^2_{r-1}-a^2_r}\,; \qquad B''_n = \frac{\mu_{n-1}a^2_{n-1}(a^2_{n-1}-a^2_n).}{\mu_{n-1}a^2_{n-1}+(\lambda_{n-1}+\mu_{n-1})a^2_n}$$

We give below the results for the particular case when there are only 3 shells in the spherical case.

These equations giving the values of ϵ_1 and ϵ_2 are the following :

$$(B_0'+B_1+a_1^6B_2)\epsilon_1-(B_1+a_1^3a_2^3B_2)\epsilon_2 = (B_1+a_1^6B_2)\delta_1$$

$$(B_1+a_1^3a_2^3B_2)\epsilon_2-(B_1+a_2^6B_2+B_3')\epsilon_2 = (B_1+a_1^3a_2^3B_2)\delta_1-B_3'\delta_2$$

where

$$B_0' = \frac{12\mu_0k_0a_1^3(a_0^3-a_1^3)}{4\mu_0a_1^3+3k_0a_0^3}\,; \quad B_1 = \frac{4\mu_1a_1^3a_2^3}{a_1^3-a_2^3}\,;$$

$$B_2 = \frac{3k_1}{a_1^3-a_2^3} \qquad B_3' = \frac{12\mu_2k_2a_2^3(a_2^3-a_3^3)}{4\mu_2a_2^3+3k_2a_3^3}\,.$$

Solving these equations and substituting the values of B's we have

$$\epsilon_1 = \frac{\left[\left\{\frac{a_1^3(4\mu_1a_2^3+3k_1a_1^3)}{(a_1^3-a_2^3)^2}\left(\frac{a_1^3(4\mu_1a_2^3+3k_1a_1^3)}{a_1^3-a_2^3}+\frac{4\mu_2k_2a_2^3(a_2^3-a_3^3)}{4\mu_2a_2^3+3k_2a_3^3}\right)-\frac{a_1^6a_2^6(4\mu_1+3k_1)^2}{(a_1^3-a_2^3)^2}\right\}\delta_1-\frac{4\mu_2k_2a_2^3(a_2^3-a_3^3)}{4\mu_2a_2^3+3k_2a_3^3}\,\delta_2\right]}{\left[\left\{\frac{4\mu_0k_0a_1^3(a_0^3-a_1^3)}{4\mu_0a_1^3+3k_0a_0^3}+\frac{a_1^3(4\mu_1a_2^3+3k_1a_1^3)}{a_1^3-a_0^3}\right\}\left\{\frac{a_1^3(4\mu_1a_2^3+3k_1a_1^3)}{a_1^3-a_2^3}+\frac{4\mu_2k_2a_2^3(a_2^3-a_3^3)}{4\mu_2a_2^3+3k_2a_3^3}\right\}-\frac{a_1^6a_2^6(4\mu_1+3k_1)^2}{(a_1^3-a_2^3)^2}\right]},$$

$$\left[\frac{4\mu_2 k_2 a_2^3(a_2^3-a_3^3)}{4\mu_2 a_2^3+3k_2 a_3^3} \left\{ \frac{4\mu_0 k_0 a_1^3(a_0^3-a_1^3)}{4\mu_0 a_1^3+3k_0 a_0^3} + \frac{a_1^3(4\mu_1 a_2^3+3k_1 a_1^3)}{a_1^3-a_2^3} \right\} \delta_2 - \right.$$

$$\left. - \frac{4\mu_0 k_0 a_1^3(a_0^3-a_1^3)}{4\mu_0 a_1^3+3k_0 a_0^3} \left\{ \frac{a_1^3 a_2^3(4\mu_1+3k_1)}{a_1^3-a_2^3} \right\} \delta_1 \right]$$

$$\epsilon_2 = \left[\left\{ \frac{4\mu_0 k_0 a_1^3(a_0^3-a_1^3)}{4\mu_0 a_1^3+3k_0 a} + \frac{a_1^3(4\mu_1 a_2^3+3k_1 a_1^3)}{a_1^3-a_2^3} \right\} \left\{ \frac{a_1^3(4\mu_1 a_2^3+3k_1 a_1^3)}{a_1^3-a_2^3} + \right. \right.$$

$$\left. \left. + \frac{4\mu_2 k_2 a^3(a_2^3-a_3^3)}{4\mu_2 a_2^3+3k_2 a_3^3} \right\} - \frac{a_1^6 a_2^6(4\mu_1+3k_1)^2}{(a_1^3 a_2^3)^2} \right].$$

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