DYNAMICS OF THE VIBRATION OF AN ELASTIC-PLASTIC STRING UNDER TRANSVERSE IMPACT

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ABSTRACT. Dynamics of an elastic-plastic string struck by an inelastic transverse load, has been worked out in this paper following the well known operational method due to Heaviside. The importance of this paper is that unlike the case of an ordinary flexible string the velocities of the transverse wave motion at different points on both sides of the struck-point of the elastic-plastic string, depend mainly on the strains at the corresponding points in the two portions of the string. The study of the displacements and presrure at the struck point due to elastic-plastic wave generation in the string is the main feature of the problem, published in two distinct cases I and II.

In case I, the displacement and pressure have been obtained when the string is struck at the middle point and in case II the general expression for displacement of the string is found when it is struck near one end.

INTRODUCTION

The Dynamics of vibration of string excited by transverse impact have been worked out by a number of workers. The new idea included in the theory of the present topic is that, unlike the case of an ordinary flexible string, the velocities due to transverse wave propagation at different points on both sides of the struck point of the string depend on unknown functions of strains at the corresponding points. A second important assumption, in this paper, is that the tension of the string is known non-linear function of strain but does not depend upon the strain rate. The useful contributions of these assumptions are mainly, the strains, the velocities of transverse waves and their gradients are different from point to point on both sides of the struck point of the string. In fact, the changes in the velocity-gradients at points being assumed to be different unknown functions of strains, being in the idea of elastic-plastic wave generation in the string vibrating under transverse inpact.

The plane motion of the struck-string (Ghosh, 1938) is studied by means of the equation of motion,

$$\rho \frac{d^2y}{dt^2} = \frac{d}{ds}\left(\frac{T}{1+\epsilon}\frac{dy}{ds}\right) \quad \dots \quad (1.0)$$

The complete dynamics of the present problem have been studied using the powerful operational method due to Heaviside, and the results obtained after suitable approximation, agree well with those derived in the case of an ordinary flexible string.

EXPLANATION OF THE SYMBOLS USED

l = Length of the string $= a + b$

a = Shorter segment of the string.

b = Longer segment of the string.

s = Variable measured along length of the string fixed at $s = 0$ and $s = l$

t = Variable time.

y = Displacement of any point at any time t.

y_1 = Displacement of any point, in $0 < s < a$.

y_2 = Displacement of any point, in $a < s < l$.

y_a = Displacement of the struck point, $s = a$.

ρ = Linear density of the string.

m = Mass of the hammer.

T = Tension of the string, a known function of strain.

ϵ = Variable Strain at any point of the string.

$c_1(\epsilon)$ = Velocity of transverse wave motion of the string in the portion $0 < s < a$.

$c_2(\epsilon)$ = Velocity of transverse wave motion of the string in the portion, $a < s < l$.

$c_a(\epsilon)$ = Velocity of transverse wave motion at $s = a$

$\theta_a = 2a/c_a$

v_0 = Velocity of impact.

$t_n = t - n\theta_a$. Where $n = 1, 2, 3, 4$ etc.

$J = mv_0$

P = Pressure exerted by the hammer.

D = Operator d/dt.

SOLUTION OF THE PROBLEM

The equation of motion of the Elastic-Plastic string can be approximately written as,

$$\rho \frac{d^2y}{dt^2} = \frac{d^2}{ds^2}\left(\frac{T}{1+\epsilon}\, y\right) \qquad \cdots \quad (1.1)$$

Equation (1.1) in the operational notation is,

$$\frac{d^2}{ds^2}\,(\rho c^2 y) = \frac{D^2}{c^2}(\rho c^2 y) \qquad \cdots \quad (1.2)$$

where,

$$c^2 = \frac{T}{\rho(1+\epsilon)} \qquad \cdots \quad (1.3)$$

The solution of (1.2) is,

$$\rho c^2 y = A \cosh \frac{D}{c}\, s + B \sinh \frac{D}{c}\, s. \qquad \ldots \quad (1.4)$$

A, B being constants.

Here the string is clamped and the terminal conditions are, at,

$$\left.\begin{array}{lll} s = 0 & y = 0 & \\ s = a_1 & y = y_a, & c = c_a \end{array}\right\} \qquad \ldots \quad (2.0)$$

The hammer strikes the string at $s = a$ if y be the displacement of the struck point we get from (1.4) and (2.0),

$$y_1 = y_a\left(\frac{c_a}{c_1}\right)^2 \frac{\sinh \frac{D}{c_1} s}{\sinh \frac{D}{c_1} a} (0 < s < a) \qquad \ldots \quad (3.0)$$

$$y_2 = y_a\left(\frac{c_a}{c_2}\right)^2 \frac{\sinh \frac{D}{c_2} (l-s)}{\sinh \frac{D}{c_2} b} (a < s < l) \qquad \ldots \quad (3.1)$$

The string which is straight initially, is supposed to behave like a loaded string attached to $s = a$, excited by an impulse J. The subsequent equation for the load is given by,

$$m \frac{d^2 y_a}{dt^2} = \left(\rho c_2^2 \frac{dy_2}{ds}\right)_{s=a} - \left(\rho c_1^2 \frac{dy_1}{ds}\right)_{s=a} \qquad \ldots \quad (4.0)$$

where the right hand side of (4.0) is the change in the value of $(\rho c^2 dy/ds)$ in crossing the point ($s = a$) in the positive direction.

The corresponding pressure exerted by the hammer is given by the equation,

$$P = m \frac{d^2 y_a}{dt^2} \qquad \ldots \quad (4.1)$$

Now equations (4.0), (3.0) and (3.1), together with boundary conditions give,

$$mD^2 y_a = -2y_a\, \rho c_a \left[\left(\frac{dc_2}{ds}\right)_{s=a} - \left(\frac{dc_1}{ds}\right)_{s=a}\right]$$

$$-y_a \rho c_a D \left[\coth \frac{Da}{c_a} + \coth \frac{Db}{c_a}\right] + DJ \qquad \ldots \quad (5.0)$$

Since the string is elastic-plastic in nature, $(dc_2/ds)_{s=a} - (dc_1/ds)_{s=a}$ can be taken as any function of strain say, $\psi(\epsilon)$.

Equation (5.0) then becomes,

$$mD^2 y_a = -2y_a \rho c_a \psi(\epsilon) - y_a \rho c_a D\left[\coth \frac{Da}{c_a} + \coth \frac{Db}{c_a}\right] + DJ \qquad (5.1)$$

whence we have,

$$y_a = \frac{D}{F(D)} v_0 \qquad \ldots \quad (6.0)$$

where,

$$F(D) = D^2 + \frac{\rho c_a}{m} D\left\{\coth \frac{Da}{c_a} + \coth \frac{Db}{c_a} + \frac{2\rho c_a}{m}\psi(\epsilon)\right\} \qquad \ldots \quad (6.1)$$

STRING SEMI-INFINITE : HAMMER STRIKES AT MIDDLE POINT

In this case we put $b = a$, and equation (6.1) becomes,

$$F(D) = D^2 + qD \coth \frac{Da}{c} + r \qquad \ldots \quad (7.0)$$

where

$$q = \frac{2\rho c_a}{m} \quad \text{and} \quad r = q\psi(\epsilon) \qquad \ldots \quad (7.1)$$

On substituting the exponential values of hyperbolic cotangent in equation (7.0) and writing $D_1 = D+\alpha$ amd $D_2 = D+\beta$ the final form of $F(D)$ is,

$$F(D) = \frac{D_1 D_2}{\left[1 - \exp\left(-\frac{2Da}{c_a}\right)\right]}\left\{1 - \frac{(D-\alpha)(D-\beta)}{D_1 D_2}\exp\left(-\frac{2Da}{c_a}\right)\right\} \qquad (8.0)$$

where,

$$D_1 D_2 = (D+\alpha)(D+\beta) = D^2 + qD + r \qquad \ldots \quad (8.1)$$

and

$$-\alpha \quad -\beta \text{ are the roots of } D^2 + qD + r = 0 \qquad \ldots \quad (8.2)$$

given by,

$$|\alpha, \beta| = \tfrac{1}{2}\{q \pm (q^2 - 4r)^{\frac{1}{2}}\} \qquad \ldots \quad (8.3)$$

The displacement y_a of the struck point can now be obtained by the equations (6.0) and (8.0),

$$y_a = \frac{D}{D_1 D_2}\left\{1 - \exp\left(-\frac{2Da}{c_a}\right)\right\}\left[1 - \frac{(D-\alpha)(D-\beta)}{D_1 D_2}\exp\left(-\frac{2Da}{c_a}\right)\right] v_0 \ldots \quad (9.0)$$

Expanding multinomially, the right hand side, eqn. (9.0) becomes,

$$y_a = \Bigg[\frac{D}{D_1 D_2} - \frac{2(\alpha+\beta)D^2}{D_1^2 D_2^2} \exp(-D\theta_a)$$

$$+ \left\{ \frac{4(\alpha+\beta)^2 D^3}{D_1^3 D_2^3} - \frac{2(\alpha+\beta)D^2}{D_1^2 D_2^2} \right\} \exp(-2D\theta_a)$$

$$- \frac{8(\alpha+\beta)^3 D^4}{D_1^4 D_2^4} - \frac{8(\alpha+\beta)^2 D^3}{D_1^3 D_2^3} + \frac{2(\alpha+\beta)D^2}{D_1^2 D_2^2} \Big\}$$

$$\exp(-3D\theta) + \ldots \Bigg] v_0 \quad \ldots \quad (9.1)$$

now writing,

$$f_1(t) = \frac{1}{D_1 D_2} v_0 \qquad \ldots \quad (10.1)$$

$$f_2(t) = \frac{(\alpha+\beta)D}{D_1^2 D_2^2} v_0 \qquad \ldots \quad (10.2)$$

etc.,

$$f_n(t) = \frac{(\alpha+\beta)^{n-1} D^{n-1}}{D_1^n D_2^n} v_0 \qquad \ldots \quad (10.3)$$

and remembering that, $t_n = t - n\theta_a$ we get,

$$y_a = f_1'(t), -2f'_2(t_1), +4f'_3(t_2) - 2f'_2(t_2), -8f'_4(t_3) + 8f'_3(t_3) - 2f'_2(t_3) + \ldots \qquad \ldots \quad (11.0)$$

Now $f'_1(t)$, $f'_2(t)$ etc., can be obtained as follows :

$$f'_1(t) = \frac{v_0 A}{(\alpha+\beta)} [e^{-\alpha t} - e^{-\beta t}] \qquad \ldots \quad (12.0)$$

$$f'_2(t) = \frac{v_0 A^2}{(\alpha+\beta)} [(A - \alpha t)e^{-\alpha t} - (A + \beta t)e^{-\beta t}] \qquad \ldots \quad (12.1)$$

$$f'_3(t) = \frac{v_0 A^3}{(\alpha+\beta)} \Bigg[\left\{ -\frac{1}{2} + \frac{3A^2}{2} - \frac{1}{2}(3A+1)\alpha t + \frac{\alpha^2 t y}{2!} \right\} e^{-\alpha t}$$

$$+ \left\{ \frac{1}{2} - \frac{3A^2}{2} - \frac{1}{2}(3A-1)\beta t - \frac{\beta^2 t y}{2!} \right\} e^{-\beta t} \Bigg] \qquad \ldots \quad (12.2)$$

and so on, where $A = \dfrac{\beta+\alpha}{\beta-\alpha}$, α and β being given by (8.3).

Thus the displacements of the struck point at different intervals of time are, as follows :

during, $0 < t < \theta_a$,

$$y_a = f'_1(t) = \frac{v_0 A}{\beta+\alpha} [e^{-\alpha t} - e^{-\beta t}] \quad \dots \quad (13.0)$$

during, $\theta_a < t < 2\theta_a$,

$$y_a = y_a(0 < t < \theta_a) - \frac{2v_0 A^2}{(\alpha+\beta)} [(A-\alpha t_1)e^{-\alpha t_1} - (A+\beta t_1)e^{-\beta t_1}] \quad \dots \quad (13.1)$$

similarly during, $2\theta_a < t < 3\theta_a$,

$$y_a = y_a(\theta_a < t < 2\theta_a)$$

$$+ \frac{4v_0 A^3}{(\alpha+\beta)} \left[\left\{ -\frac{1}{2} + \frac{3A^2}{2} - \frac{1}{2}(3A+1)\alpha t_2 + \frac{\alpha^2 t_2^2}{2!} \right\} e^{-\alpha t_2} \right.$$

$$\left. + \left\{ \frac{1}{2} - \frac{3A^2}{2} - \frac{1}{2}(3A-1)\beta t_2 - \frac{\beta^2 t_2^2}{2!} \right\} e^{-\beta t_2} \right]$$

$$- \frac{2v_0 A^2}{(a+\beta)} [(A-\alpha t_2)e^{-\alpha t_2} - (A+\beta t_2)e^{-\beta t_2}] \quad \dots \quad (13.2)$$

and so on.

Equations (13.0), (13.1), (13.2) etc. are the expressions of the displacements of the struck point at different intervals of time. These equations together with equations (3.0) and (3.1) will enable us to determine the general displacement of any point of the elastic-plastic string.

It is interesting to note in this connection that if $\psi(\epsilon) = 0$ we have from (8.3), $\beta = 0$ and the expressions for the displacements of the struck-point at different intervals of time takes up the following forms :

During, $0 < t < \theta_a$,

$$y_a = \frac{v_0}{q}(1 - e^{-qt}) \quad \dots \quad (14.0)$$

During, $\theta_a < t < 2\theta_a$,

$$y_a = y_a(0 < t < \theta_a) - \frac{2v_0}{q}[1 - (1+qt_1)e^{-qt_1}] \quad \dots \quad (14.1)$$

During, $2\theta_a < t < 3\theta_a$,

$$y_a = y_a(\theta_a < t < 2\theta_a) + \frac{2v_0}{q}[1 - (1+qt_2+q^2t_2^2)e^{-qt_2}] \quad \dots \quad (14.2)$$

Equations (14.0), (14.1) and (14.2) are the expressions for the displacements of the struck point at different epoch with the above approximations. These are exactly the same as those derived by Ghose (1938) in the case of an ordinary flexible string.

The pressure at different epochs exerted by the hammer on the string can be obtained by means of equations (4.1) and (11.0) as follows :

During, $0 < t < \theta_a$,

$$P_1 = \frac{mv_0}{(q^2-4r)^{\frac{1}{2}}}\left[\alpha^2 e^{-\alpha t} - \beta^2 e^{-\beta t}\right] \quad \ldots \quad (15.0)$$

During, $\theta_a < t < 2\theta_a$,

$$P_2 = P_1 + \frac{2mv_0 q}{q^2-4r}\left[\alpha^2(2+A-\alpha t_1)e^{-\alpha t_1} + \beta^2(2-A-\beta t_1)e^{-\beta t_1}\right] \quad \ldots \quad (15.1)$$

During, $2\theta_a < t < 3\theta_a$,

$$P_3 = P_2 + \frac{4mv_0 q^2}{(q^2-4r)^{3/2}}\Big[\alpha^2\left\{-\frac{1}{2}+\frac{3A^2}{2}-\frac{1}{2}(3A+1)\alpha t_2+\frac{\alpha^2 t_2^2}{2}\right\}e^{-\alpha t_2}$$

$$+2\alpha\left\{\frac{1}{2}(3A+1)\alpha-\alpha^2 t_2\right\}e^{-\alpha t_2}+\alpha^2 e^{-\alpha t_2}$$

$$+\beta^2\left\{\frac{1}{2}-\frac{3A^2}{2}-\frac{1}{2}(3A-1)\beta t_2-\frac{\beta^2 t_2^2}{2!}\right\}e^{-\beta t_0}$$

$$+2\beta\{\tfrac{1}{2}(3A-1)\beta+\beta^2 t_2\}e^{-\beta t_2}-\beta^2 e^{-\beta t_2}\Big]$$

$$+\frac{2mv_0 q}{(q^2-4r)^{\frac{1}{2}}}\left[\alpha^2(2+A-\alpha t_2)e^{-\alpha t_2}+\beta^2(2-A-\beta t_2)e^{-\beta t_2}\right] \quad \ldots \quad (15.2)$$

Here also if we take $\psi(\epsilon) = 0$ the plastic behaviour disappears from the string and the different expressions for pressure as stated in (15.0), (15.1) and 15.2) become quite similar to those obtained by Ghose (1938) for a flexble string.

STRING SEMI-INFINITE : HAMMER STRIKES NEAR ONE END

If hammer strikes the elastic-plastic string at $s = \alpha$ which is too small compared to b, and as $\lim_{b\to\infty} \text{Coth}\,(Db/c_a) = 1$ (6.0) becomes,

$$F(D) = D^2 + \frac{1}{2}\frac{\rho c_a}{m} D\left(\coth\frac{D\alpha}{c_a}+1\right)+r \quad \ldots \quad (16.0)$$

Expanding $\coth(Da/c_a)$ in a power series of Da/c_a and retaining terms containing a/c_a only, it is found,

$$F(D) = D^2\left(1+\frac{\rho a}{3m}\right)+\frac{\rho c_a}{m}D+\left(\frac{\rho c_a^2}{am}+r\right) \quad \ldots \quad (16.1)$$

From (6.0) and (16.1) we have,

$$y_a = \frac{m}{m_0}\frac{D}{(D-q)(D-p)} \cdot v_0 = \frac{m}{m_0}\frac{v_0}{q-p}(e^{qt}-e^{pt})) \quad \ldots \quad (17.0)$$

where $\frac{m_0}{m} = 1 + \frac{\rho a}{3m}$ and q, p are the roots of,

$$D^2 + \frac{\rho c_a}{m_0} D + \frac{m}{m_0}\left(\frac{\rho c_a^2}{am} + r\right) = 0 \quad \ldots \quad (17.1)$$

given by,

$$p = -\mu - i\nu \quad \ldots \quad (18.0)$$

$$q = -\mu + i\nu \quad \ldots \quad (18.1)$$

whence,

$$\mu = \frac{\rho c_a}{2m_0} \quad \ldots \quad (19)$$

and,

$$\nu = \left[\left(\frac{\rho c_a^2}{m_0} + \frac{m}{m_0} r\right) - \frac{\rho^2 c_a^2}{4m_0^2}\right]^{\frac{1}{2}} \quad \ldots \quad (19.1)$$

Thus from (17.0) and (18.0), (18.1) we have,

$$y_a = \frac{m}{m_0} \cdot \frac{v_0}{\nu} e^{-\mu t} \sin \nu t \quad \ldots \quad (20.0)$$

Equation (20.0) shows that the displacement curve is of the damped oscillatory nature. Clearly the damping is introduced due to the plastic nature of the string. Also the frepuency is affected by the strain at the corresponding struck-point. Thus unlike the case of an ordinary flexible string, the frequency of vibration of the elastic-plastic string is found to increase by an amount $\frac{m}{m_0} r = \frac{2\rho c}{m_0} \psi(\epsilon)$ as in the frequency equation (19.1) when the string is struck near one end.

In the case of flexible string $\psi(\epsilon) = 0$ and the corresponding frequency equa-equation reduces to that derived by Ghosh (loc. cit).

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