DISPERSION RELATIONS FOR ELECTRON PLASMA IN AN EXTERNAL MAGNETIC FIELD

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ABSTRACT. The Boltzmann-Vlasov equation is solved by integrating it over the characteristic trajectory of the electrons. The dispersion laws for parallel and perpendicular wave propagation are obtained in terms of a function which belongs to the family of Bessel's function. Explicit expressions for this function has been obtained for low temperature plasma.

INTRODUCTION

The theory of wave propagation in a plasma placed in an external magnetic field has been developed by many authors e.g., by Gross (1951), Sitenks and Stepanov (1957) Bernstein (1958) Allis *et al.* (1962) and others Most of these investigations are concerned with solving the Boltzmann-Vlasov kinetic equation coupled with Maxwell's e.m. field equations. From the condition of the existence of the solution of these equations, one obtains the dispersion relation of the plasma. The dielectric coefficient for a magnetic plasma turns out to be a tensor whose nine elements have extremely complicated form. Only in the case of low temperature plasma, these elements can be simplified to a certain extent. However, it is possible to express all the elements in terms of a single function which makes the mathematical tasks less involved. This will be done in the present paper and in the paper that will be published subsequently.

In this paper we start by solving the Boltzmann-Vlasov kinetic equation, by integrating it over the characteristic trajectory of the electron. The plasma will be assumed "collisionless" and ion motions will be neglected. We shall calculate the dispersion law for waves propagating (a) parallel, and (b) perpendicular, to the magnetic field. In the next paper, we extend the calculation for an arbitrary direction of wave propagation.

CHARACTERISTIC INTEGRATION OF BOLTZMANN-VLASOV EQUATION

The linearised Boltzmann-Vlasov equation for the plasma electrons is

$$\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{r}}-\frac{e}{mc}(\mathbf{v}\times\mathbf{H}_0)\cdot\frac{\partial f_1}{\partial \mathbf{v}}=\frac{e}{m}\mathbf{E}\cdot\frac{\partial f_0}{\partial \mathbf{v}}, \qquad \dots (1)$$

where f_1 is a small perturbation in the distribution function of the electrons over the steady state distribution f_0. $\mathbf{E}$ is the electric field in the plasma created due to f_1, and $\mathbf{H}_0$ is the externally applied steady magnetic field. Other quantities have their usual meaning.

The characteristic trajectory of the electrons is given by

$$\frac{1}{d\tau} = \frac{\mathbf{v}}{d\mathbf{r}} = -\frac{e}{mc}(\mathbf{v}\times\mathbf{H}_0)\cdot\frac{d}{d\mathbf{v}}. \qquad \ldots \quad (2)$$

Let $\mathbf{v}_c(\tau)$, $\mathbf{r}_c(\tau)$ denote the characteristic velocity and position of the electrons as a function of the characteristic time τ, and these obey eqns. (2). Replacing $\mathbf{r}$ by $\mathbf{r}+\mathbf{r}_c(\tau)$, v by $\mathbf{v}_c(\tau)$ and t by $t+\tau$ in eqn. (1), it reduces to the following form :

$$\frac{d}{d\tau} f_1(\mathbf{r}+\mathbf{r}_c(\tau),\ \mathbf{v}_c(\tau),\ t+\tau) - \frac{e}{m}\mathbf{E}(\mathbf{r}+\mathbf{r}_c(\tau),\ t+\tau)\cdot\frac{\partial f_0}{\partial v_c(\tau)}. \qquad \ldots \quad (3)$$

We choose the initial conditions such that at time t (i.e., $\tau = 0$), $\mathbf{r}_c(0) = 0$ and $\mathbf{v}_c(0) = \mathbf{v}$, and at $t = 0$, $f_1 = 0$. With these conditions, we integrate (3) over τ from $\tau = -t$ to $\tau = 0$ and obtain .

$$f_1(\mathbf{r},\mathbf{v},t) = \frac{e}{m}\int_0^t \mathbf{E}(\mathbf{r}+\mathbf{r}_c(-\tau),\ t-\tau)\cdot\frac{\partial f_0}{\partial \mathbf{v}_c(-\tau)}\,d\tau.$$

From this equation, by taking Fourier space-transform and Laplace time-transform of the form

$$f_1(\mathbf{k},\mathbf{v},\omega) = \int_0^\infty dt\int d\mathbf{k} f_1(\mathbf{r},\mathbf{v},t)e^{i(\omega t-\mathbf{k}.\mathbf{r})},$$

we obtain

$$f_1(\mathbf{k},\mathbf{v},\omega) = \frac{e}{m}\mathbf{E}(\mathbf{k},\omega)\,.\,\Omega(\mathbf{k},\mathbf{v},\omega), \qquad \ldots \quad (4)$$

where
$$\Omega(\mathbf{k},\mathbf{v},\omega) = \int_0^\infty e^{i\mathbf{k}.\mathbf{r}_c(-\tau)+i\omega t}\frac{\partial f_0}{\partial \mathbf{v}_c(-\tau)}\,d\tau. \qquad \ldots \quad (5)$$

DISPERSION RELATION

Let us assume that a test particle of charge q, mass M, and moving with a velocity $\vec{V}_0(t)$, disturbs the plasma from the steady state. The two Maxwell's equations in that case, are given by :

$$\nabla\times\mathbf{E}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}, \qquad \ldots \quad (6a)$$

$$\nabla\times\mathbf{H}(\mathbf{r},t) = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} - \frac{4\pi e}{c}\int f_1(\mathbf{r},\mathbf{v},t)\mathbf{v}\,d\mathbf{v} + \frac{4\pi q}{c}\delta(\mathbf{r}-\mathbf{r}_1)\vec{V}_0(t). \qquad \ldots \quad (6b)$$

In (6b), the position $\mathbf{r}_1$ of the test particle is exactly given by the Dirac's δ-function.

We take Fourier-Laplace transform of eqns. (6) as before. Then eliminating the two quantities **H** and f_1, from the resulting equations and eqn. (4), we obtain an equation for the electric field in the plasma :

$$\left(k^2-\frac{\omega^2}{c^2}\right)\mathbf{E}(\mathbf{k},\omega)-(\mathbf{k}\cdot\mathbf{E})\,\mathbf{k}+i\,\frac{4\pi e^2\omega}{mc^2}\int d\mathbf{v}\,\mathbf{v}\,\mathbf{E}(\mathbf{k},\omega)\Omega(\mathbf{k},\mathbf{v},\omega)$$

$$=i\,\frac{\omega}{c^2}\,4\pi q\vec{\Gamma}, \tag{7}$$

where
$$\vec{\Gamma}=\int_0^\infty \vec{V}_0(t)e^{i\mathbf{k}\cdot\mathbf{r}_1+i\omega t}\,dt\,. \tag{8}$$

It we define a matrix

$$A_{ij}=\left(k^2-\frac{\omega^2}{c^2}\right)\delta_{ij}-k_ik_j+i\,\frac{4\pi e^2\omega}{mc^2}\int d\mathbf{v}\,v_i\,\Omega_i\,(\mathbf{k}.\,\mathbf{v},\,\omega),$$

where the subscripts $i,j=x,y,z$ and

$$\delta_{ij}=0,\ \text{for}\ i\neq j$$
$$=1,\ \text{for}\ i=j.$$

then eqn. (7) becomes

$$A_{ij}E_j=i\,\frac{\omega}{c^2}\,4\pi q\,\Gamma_i. \tag{10}$$

The condition for vanishing of the determinent of the matrix A_{ij} gives the dispersion relation

$$\det|A_{ij}|=0. \qquad \ldots \tag{11}$$

We take the magnetic field $\mathbf{H}_0$ along the positive z-direction of cylindrical coordinates, and write

$$\mathbf{k}=\mathbf{k}_z+\mathbf{k}_\rho.$$
$$\mathbf{v}=\mathbf{v}_z+\mathbf{v}_\rho, \qquad \ldots \tag{12}$$

where the subscripts z and ρ denote components along and at right angles to the magnetic field. Then solving eqns. (2) we get an expression for the characteristic path of the electrons in plasma;

$$\mathbf{r}_c(\tau)$$
$$=\mathbf{e}_xa[\sin(\theta_0+\omega_c\tau)-\sin\theta_0]-\mathbf{e}_ya[\cos(\theta_0+\omega_c\tau)-\cos\theta_0]+\mathbf{e}_zv_z\tau. \qquad \ldots \tag{13}$$

Here, the **e**'s are three unit vectors along the coordinate axes, θ_0 is the azimuthal angle of $\mathbf{v}_\rho$ and

$$a=\frac{\mathbf{v}_\rho}{\omega_c}$$

$$\omega_c=\frac{eH_0}{mc} \qquad \ldots \tag{14}$$

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are respectively the Larmor radius and cyclotron frequency of electrons.

DISPERSION LAW FOR PARALLEL PROPAGATION

For propagation parallel to $\mathbf{H}_0$, we set $\mathbf{k}_\rho = 0$. Using (13), we get

$$\mathbf{k} \cdot \mathbf{r}_c(-\tau) = -k_z v_z \tau. \qquad \ldots (15)$$

The steady state distribution function f_0 is taken to be Maxwellian :

$$f_0 = n_0(m/2\pi T)^{3/2} \exp\left(-\frac{m}{2T}[v_z^2 + v_\rho^2] \right), \qquad \ldots (16)$$

where n_0 is the number of electrons per c.c., and the plasma temperature T is in ergs.

Writing
$$I_{ij} = i\frac{4\pi e^2 \omega}{mc^2} \int d\mathbf{v} v_i \Omega_j \qquad \ldots (17)$$

and noticing that the characteristic velocity is given by

$$\mathbf{v}_c(\tau) = \mathbf{e}_x v_\rho \cos(\theta_0 + \omega_c \tau) + \mathbf{e}_y v_\rho \sin(\theta_0 + \omega_c \tau) + \mathbf{e}_z v_z, \qquad \ldots (18a)$$

while the random thermal velocity is

$$\mathbf{v} = \mathbf{e}_x v_\rho \cos\theta_0 + \mathbf{e}_y v_\rho \sin\theta_0 + e_z v_z, \qquad \ldots (18b)$$

we can evaluate the various integrals I_{ij} given by (17) in a quite straight forward manner, using eqns. (5), (15), (16) and (18). The result is

$$I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0. \qquad \ldots (19)$$

$$I_{xx} = I_{yy} = I_1 = -\frac{\omega_p^2 \omega}{2c^2} \int_{-\infty}^{+\infty} dv_z F_0(v_z) \left[\frac{1}{k_z v_z - (\omega - \omega_c)} + \frac{1}{k_z v_z - (\omega + \omega_c)} \right], \qquad \ldots (20)$$

$$I_{xy} = -I_{yx} = I_2 = i\frac{\omega_p^2 \omega}{2c^2} \int_{-\infty}^{+\infty} dv_z F_0(v_z) \left[\frac{1}{k_z v_z - (\omega - \omega_c)} - \frac{1}{k_z v_z - (\omega + \omega_c)} \right] \qquad \ldots (21)$$

and

$$I_{zz} = I_3 = -\frac{\omega_p^2 \omega}{c^2} \frac{m}{T} \int_{-\infty}^{+\infty} \frac{dv_z v_z^2 F_0(v_z)}{k_z v_z - \omega}. \qquad \ldots (22)$$

In these equations the quantity $F_0(v_z)$ is given by

$$F_0(v_z) = (m/2\pi T)^{\frac{1}{2}} \exp(-mv^2/2T), \qquad \ldots (23)$$

and ω_p is the plasma frequency

$$\omega_p = (4\pi e^2 n_0/m)^{\frac{1}{2}}. \qquad \ldots \quad (24)$$

Using eqn. (9) and eqns.. (19) to (22) the dispersion law (11) gives rise to the following relations for the three uncoupled waves .

$$\frac{\omega^2}{c^2} - I_3 = 0 \qquad \ldots \quad (25)$$

for longitudinal plasma waves, and

$$k_z^2 - \frac{\omega^2}{c^2} + I_1 + iI_2 = 0 \qquad \ldots \quad (26a)$$

$$k^2 - \frac{\omega^2}{c^2} + I_1 - iI_2 = 0, \qquad \ldots \quad (26b)$$

for transverse circularly polarised e.m waves.

The path of integration in the v_z -plane of the integrals in (20) to (22) can be prescribed by giving a small imaginary part to ω and then deforming the contour arround the pole (See for ex., Stix 1962). It is then possible to wirte the following expression for $I_1 + iI_2$ valid for all ω :

$$I_1 + iI_2 = \frac{1}{k_z} P \int_{-\infty}^{+\infty} \frac{F_0(v_z)dv_z}{v_z - \dfrac{\omega - \omega_c}{k_z}} - i\frac{\pi}{k_z} F_0\left(\frac{\omega - \omega_c}{k_z}\right). \qquad \ldots \quad (27)$$

The principal value part, denoted by P can be easily evaluated and one obtained from eqn. (26a),

$$k_z^2 c^2 - \omega^2 + \frac{c^2(\omega - \omega_c)}{\lambda_D^2 \eta^2 \omega} \phi\left(1, 3/2; -\frac{c^2(\omega - \omega_c)^2}{2\omega^2\eta^2\omega_p\lambda^2_D}\right)$$

$$+ \frac{i\pi}{k_z} \omega\omega_p^2 F_0\left(\frac{\omega - \omega_c}{k_z}\right) = 0. \qquad \ldots \quad (28)$$

In (28), $\eta = ck_z/\omega$ is the plasma refractive index, $\lambda_D = (T/m)^{\frac{1}{2}}\omega_p$ is the Debye length, and ϕ is confluent hypergeometric function, Erdelyi (1953). Writing $\omega = \omega_r + i\omega_i$ with $\omega_i << \omega_r$ in eqn. (28), and separating the real and imaginary parts we obtain

$$^2 - 1 = -\frac{c^2(\omega - \omega_c)}{\lambda_D^2 \eta^2 \omega^3} \phi\left(1, 3/2; -\frac{c^2(\omega - \omega_c)^2}{2\omega^2\eta^2\omega_p^2\lambda_D^2}\right), \qquad \ldots \quad (29)$$

and

$$\omega_i \approx \frac{\pi}{2k_z} \omega_p^2 F_0\left(\frac{\omega-\omega_c}{k_z}\right). \quad \ldots (30)$$

Eqns. (29) and (30) give the dispersion law and damping decrement for the circularly polarised mode (26a). These relations were earlier obtained by Pradhan (1957) by a different method For low temperature plasma, λ_D is small, so that one can use the asymptotic expansion of ϕ in (29). If we keep terms only up to first order in λ_D^2 (i.e. first order in T), we obtain the following expression for the refractive index of the mode (26a).

$$\eta^2 = \left[1-\frac{\omega_p^2}{\omega(\omega-\omega_c)}\right] \Big/ \left[1+\frac{\omega_p^4 \lambda_D^2 \omega}{c^2(\omega-\omega_c)^2}\right]. \quad \ldots (31)$$

The two other modes, (25) and (26b) can be similarly treated.

DISPERSION LAW FOR PERPENDICULAR PROPAGATION

In this case we set $k_z = 0$. and without any loss of generality we can assume that $k_y = 0$, so that $k = k_\rho = k_x$. Using eqn. (13) the quantity $\mathbf{k}\cdot\mathbf{r}_c(-\tau)$ is then given by,

$$\mathbf{k}\cdot\mathbf{r}_c(-\tau) = -2ak_\rho \sin\frac{\omega_c\tau}{2} \cos\left(\theta_0-\frac{\omega_c\tau}{2}\right). \quad \ldots (32)$$

Using (32) and (18), the three components of Ω of eqn. (5) can be written down in the following form :

$$\Omega_x = 2v_\rho f'_0(v^2)\int_0^\infty e^{i\omega t - iA\cos\left(\theta_0-\frac{\omega_c t}{2}\right)} \cos(\theta_0-\omega_c t)dt, \quad \ldots (33a)$$

$$\Omega_y = 2v_\rho f'_0(v^2)\int_0^\infty e^{i\omega t - iA\cos\left(\theta_0-\frac{\omega_c t}{2}\right)} \sin(\theta_0-\omega_c t)dt, \quad \ldots (33b)$$

$$\Omega_z = 2v_z f'_0(v^2)\int_0^\infty e^{i\omega t - iA\cos\left(\theta_0-\frac{\omega_c t}{2}\right)} dt, \quad \ldots (33c)$$

where prime denotes differentiation w.r.t. argument, and

$$A = 2\frac{k_\rho v_\rho}{\omega_c}\sin\frac{\omega_c t}{2},$$

Eqns. (33) together with eqn. (17) will give us the various integrals I_{ij} needed to evaluate the elements A_{ij}. We shall briefly indicate how this can be done for the case of I_{xx}. The remaining integrals can be evaluated in the same manner.

From eqns. (17) and (33a), the integral I_{xx} can be written down as follows :

$$I_{xx} = -n_0 \left(\frac{m}{2\pi T}\right) \frac{m}{T} \int_{-\infty}^{+\infty} dv_z\, e^{-\frac{m}{2T} v_z^2} \int_0^{\infty} dv_\rho v_\rho^3\, e^{-\frac{m}{2T} v_\rho^2}$$

$$\int_0^{2\pi} d\theta_0 \cos\theta_0 \int_0^{\infty} dt\, e^{i\omega t - iA\cos\left(\theta_0 - \frac{\omega_c t}{2}\right)} \cos(\theta_0 - \omega_c t).$$

The integral over v_z is straightforward. We integrate over θ_0 first and then over v_ρ. These are standard integrals and by quite straight forward manner one obtains :

$$I_{xx} = -i \frac{\omega_p^2 \nu}{c^2} e^{-z} \left[-z + \frac{\partial}{\partial z} + z \frac{\partial^2}{\partial z^2} \right] S_\nu(z), \qquad \dots \quad (34)$$

where

$$S_\nu(z) = \int_0^{\infty} e^{i\nu t + z\cos t}\, dt, \qquad \dots \quad (35)$$

$$\nu = \frac{\omega}{\omega_c}, \quad z = \frac{k_\rho^2 T}{m\omega_c^2}. \qquad \dots \quad (36)$$

The expression (34) for I_{xx} can be further simplified in the following way : Writing

$$\int_0^{\infty} = \int_0^{2\pi} + \int_{2\pi}^{4\pi} + \dots + \int^{\infty},$$

the function $S\nu(z)$ transforms into the form

$$S\nu(z) = (1 + e^{2\pi\nu i} + e^{4\pi\nu i} + \dots) \int^{2\pi} e^{i\nu t + z\cos t}\, dt.$$

At all points in the complex ν-plane, except where ν = integers the infinite series on the r.h.s. can be replaced by $1/(1 - e^{2\pi\nu i})$, so that $S_\nu(z)$ becomes

$$S_\nu(z) = \frac{i\pi}{\sin\nu\pi} M_\nu(z), \qquad \dots \quad (37)$$

where

$$M_\nu(z) = \frac{1}{\pi} \int_0^{\pi} e^{-z\cos t} \cos\nu t\, dt \qquad \dots \quad (38)$$

The function $M_\nu(z)$ reduces to $I_\nu(-z)$, i.e. modified Bessel's function of the first kind, whenever ν = integers. Otherwise, it is a distinct function and bears the same relation to $I_\nu(-z)$, as Anger functions do to ordinary Bessel functions (Watson 1952). The function $M_\nu(z)$ has the following recurrence relations :

$$M_{-\nu}(z) = M_\nu(z),$$

$$M_{\nu+1}(z)+M_{\nu-1}(z) = -2M'_\nu(z),$$

$$M_{\nu-1}(z)-M_{\nu+1}(z) = \frac{2}{\pi z}\, e^{z} \sin \pi\nu - \frac{2\nu}{z} M_\nu(z). \qquad \ldots \quad (39)$$

Expressing the functions $S_\nu(z)$ in terms of $M_\nu(z)$, and using eqn. (39), one can reduce I_{xx} to the following simple form .

$$I_{xx} = \frac{\omega_p^2}{c^2}\, \frac{\nu^2}{z}\, [1+i\nu e^{-z}\, S_\nu(z)]. \qquad \ldots \quad (40)$$

In a similar manner we deduce

$$I_{yy} = I_{xx}+i\, \frac{\omega_p^2}{c^2}\, (2\nu z)\, \frac{\partial}{\partial z}\, e^{-z}S_\nu(z), \qquad \ldots \quad (41)$$

$$I_{zz} = -i\, \frac{\omega_p^2}{c^2}\, \nu(e^{-z}S_\nu(z)), \qquad \ldots \quad (42)$$

$$I_{xy} = -I_{yx} = -\frac{\omega_p^2}{c^2}\, \nu^2\, \frac{\partial}{\partial z}\, (e^{-z}S_\nu(z)), \qquad \ldots \quad (43)$$

and

$$I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0. \qquad \ldots \quad (44)$$

We thus notice that if expressed in terms of the function $e^{-z}\, S_\nu(z)$, the formidable looking integrals I_{ij} reduce to particularly simple expression. The dispersion relation (11) in this case has the following form :

$$\begin{pmatrix} -\frac{\omega^2}{c^2}+I_{xx} & I_{xy} & 0 \\ I_{yx} & k_\rho^2-\frac{\omega^2}{c^2}+I_{yy} & 0 \\ 0 & 0 & k_\rho^2-\frac{\omega^2}{c^2}+I_{zz} \end{pmatrix} = 0.$$

We shall conclude the calculation in this paper by giving the explicit expression of I_{xx} etc. for low temperature plasma. When T is small, the quantity z

given by eqn. (36) can be taken as less than unity. Hence, to a first approximation we can expand the quantity $e^{-z\cos t}$ which occurs in the expression for $M_\nu(z)$ in powers of z and integrate term by term to obtain an expression for $M_\nu(z)$ and hence that of $S_\nu(z)$. We also expand, $e^{-z} = 1-z$. Using these result is eqn. (40) [or in (34) along with eqn. (39)], and keeping only terms upto the first power of z (i.e. first order in T), we obtain

$$I_{xx} = \frac{\omega_p^2}{c^2}\left[\frac{1}{1-\beta^2} + \frac{3z\beta^2}{(1-4\beta^2)(1-\beta^2)}\right]$$

where

$$\beta = \frac{1}{\nu} = \frac{\omega_c}{\omega}.$$

In a similar manner we deduce

$$I_{yy} = \frac{\omega_p^2}{c^2}\left[\frac{1}{1-\beta^2} + z\beta^2\frac{1+8\beta^2}{(1-4\beta^2)(1-\beta^2)}\right]$$

$$I_{zz} = \frac{\omega_p^2}{c^2}\left[1 + \frac{z\beta^2}{1-\beta^2}\right]$$

$$I_{xy} = -I_{yx} = -i\,\frac{\omega_p^2}{c^2}\,\beta\left[\frac{1}{1-\beta^2} + \frac{6z\beta^2}{(1-\beta^2)(1-4\beta^2)}\right].$$

It may be noted that these relations for I_{ij}'s reduce to the usual expression for zero magnetic field for $\beta = 0$. If higher temperature correction is desired one must keep terms upto second order in z in the expansion of $M_\nu(z)$.

CONCLUSIONS

From the calculation given in the last section we may notice one interesting point. In case of perpendicular propagation, the problem of finding the dispersive behaviour of magnetic plasma, reduces to discussions of the properties of the function $M_\nu(z)$, or $S_\nu(z)$. This function, as given in (35) or (37) includes the plasma temperature (z) and the ratio of wave frequency to cyclotron frequency (ν) as parameters. In comparison to ordinary Bessel functions, we may refer to ν and z as the order and argument respectively of the function $S_\nu(z)$. If we can find suitable expression for $S_\nu(z)$ for different combinations of small and large ν and z, we shall be able to solve the problem completely.

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