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A note on the deformation of a rotating inhomogeneous piezoelectric thick disc

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The equations of elasticity, the Maxwell's electromechanical equations and the constitutive equations of piezoelectricity have been made use of in solving the problem. The results arrived at essentially agree with the results known for aeolotropic media and which are free from piezoelectric excitations.

1. Introduction

In recent years the problems of piezoelectricity have assumed importance in view of their applications in the field of ultrasonics and acoustic. Toupin (1959), Paria (1960), Paul (1961), Sinha (1962, 1963), Giri (1964), Das (1966) and Bakshi (1967) in their papers distinguished the problems of piezoelectrity as analogues of the well-known classical problems of elasticity. The present note is a piezoelectric analogue of the elastic problem concerning the determination of the deformations in a rotating thick disc with inhomogeneous material parameters. The inhomogeneities are supposed to appear due to the impurities of the crystal structure of the material.

2. The problems, fundamental equations and boundary conditions.

The problem, as stated above is electromechanical in character and is therefore to be solved by the equations of elasticity and electro magnetic Maxwell's equations. The problem has been considered as that of a plane strain in cylindrical co-ordinates (r, θ, z) with origin at the lower face and z-axis parallel to its thickness and direction of the electric field along z-axis. The equation of equilibrium is

$$\frac{\partial T_{rr}}{\partial_r} + \frac{T_{rr} - T_{\theta\theta}}{r} + \rho\omega^2 r = 0 \qquad \ldots(1)$$

where T_{rr} and $T_{\theta\theta}$ are the stress components, ρ the density and ω the uniform velocity of the rotating disc. The Maxwell's equations are given by

$$\text{rot}\,\vec{E} = 0, \quad \text{div}\,\vec{D} = 0 \qquad \ldots(2)$$

where $\vec{E}$ and $\vec{D}$ are the electric intensity and electric displacement vectors, respectively.

The constitutive equations of piezoelectricity in Cartesian co-ordinates are given by Sinha (1968)

$$\begin{aligned}
T_{xx} &= C^E_{11}S_{xx} + C^E_{12}S_{yy} + C^E_{13}S_{zz} - e_{31}E_z \\
T_{yy} &= C^E_{12}S_{xx} + C^E_{11}S_{yy} + C^E_{13}S_{zz} - e_{31}E_z \\
T_{zz} &= C^E_{13}(S_{xx} + S_{yy}) + C^E_{33}S_{zz} - e_{33}E_z \\
T_{zx} &= C^E_{44}S_{zx},\ T_{yz} = C^E_{44}S_{yz} \\
D_x &= e_{14}S_{yz} + e_{15}S_{zx} \\
D_y &= e_{15}S_{yz} + e_{14}S_{zx} \\
D_z &= e_{31}(S_{xx} + S_{yy}) + e_{33}S_{zz} + \epsilon_{33}E_z
\end{aligned} \qquad \ldots(3)$$

where T—stress components, S—strain compoments, D—electric displacements, C—elastic compliances, e—piezoelectric constants, E–dielectric permittivities

In cylindrical co-ordinates the above equations give

$$\begin{aligned}
T_{rr} &= \cos\theta\, T_{xx} + 2\sin\theta\cos\theta\, T_{xy} + \sin^2\theta\, T_{yy} \\
T_{\theta\theta} &= \sin^2\theta\, T_{xx} - 2\sin\theta\cos\theta\, T_{xy} + \cos^2\theta\, T_{yy} \\
T_{r\theta} &= \sin\theta\cos\theta\,(T_{yy} - T_{xx}) + (\cos^2\theta - \sin^2\theta)\, T_{xy} \\
T_{rz} &= \cos\theta\, T_{zx} + \sin\theta\, T_{yz} \\
T_{\theta z} &= -\sin\theta\, T_{xz} + \cos\theta\, T_{yz} \\
T_{zz} &= T_{zz} \\
D_r &= D_x\cos\theta + D_y\sin\theta \\
D_\theta &= D_y\cos\theta - D_x\sin\theta \\
D_z &= D_z.
\end{aligned} \qquad \ldots(4)$$

Again the components of displacements (U, V, W) are given by

$$U = u\cos\theta, \quad V = v\sin\theta, \quad W = \omega \qquad \ldots(5)$$

whence $u(rz)$ and $\omega\,(rz)$ are the radial and axial components which are assumed to be independent of θ. The strain components are given by

$$\begin{aligned}
S_{xx} &= \frac{u}{r} + \frac{x^2}{r}\,\frac{\partial}{\partial r}\left(\frac{u}{r}\right) \\
S_{yy} &= \frac{u}{r} + \frac{y^2}{r}\,\frac{\partial}{\partial r}\left(\frac{u}{r}\right) \\
S_{zz} &= \frac{\partial\omega}{\partial z} \\
S_{xy} &= \frac{2xy}{r}\,\frac{\partial}{\partial r}\left(\frac{u}{r}\right) \\
S_{yz} &= \frac{y}{r}\left(\frac{\partial u}{\partial z} + \frac{\partial\omega}{\partial r}\right) \\
S_{zx} &= \frac{x}{r}\left(\frac{\partial u}{\partial z} + \frac{\partial\omega}{\partial r}\right)
\end{aligned} \qquad \ldots(6)$$

Thus the simplified form of (4) gives

$$T_{rr} = C_{11}\frac{\partial u}{\partial r} + C_{12}\frac{u}{r} + C_{13}\frac{\partial \omega}{\partial z} - e_{31}E_z$$

$$T_{\theta\theta} = C_{12}\frac{\partial u}{\partial r} + C_{11}\frac{u}{r} + C_{13}\frac{\partial \omega}{\partial z} - e_{31}E_z$$

$$T_{rz} = C_{44}\left(\frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial r}\right)$$

$$T_{zz} = C_{13}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + C_{33}\frac{\partial \omega}{\partial z} - e_{33}E_z$$

$$T_{r\theta} = T_{\theta z} = 0$$

$$D_r = e_{15}\left(\frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial r}\right), \quad D_\theta = -e_{14}\left(\frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial r}\right)$$

$$D_z = e_{31}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + e_3\left(\frac{\partial \omega}{\partial z}\right) + e_{33}E_z. \qquad \ldots(7)$$

The material parameters C_{11}, C_{12}, C_{13}, e_{33} etc. are supposed to vary along the direction of the axis of the disc.

We have to solve (1) and (2) by the help of (7) and also following the boundary conditions.

(i) Electric conditions

$$\left.\begin{aligned} E_z &= b_1 \text{ at } z = l, \quad r = 0 \\ E_z &= b_2 \text{ at } z = l, \quad r = a \\ E_z &= b_3 \text{ at } z = 0, \quad r = 0 \end{aligned}\right\} \qquad \ldots(8)$$

(b_1, b_2, b_3 are constants)

and (ii) mechanical conditions

$$\int_0^1 T_{rr}\,\partial z = 0 \text{ at } r = a \qquad \ldots(9)$$

where l is the thickness of the disc.

2. Solution of the problem

Since the problem is that of plane stress and so $T_{zz} = T_{rz} = 0$

$$C_{13}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+C_{33}\frac{\partial \omega}{\partial z}-e_{33}E_z=0$$

$$\frac{\partial \omega}{\partial r}+\frac{\partial u}{\partial z}=0 \qquad \text{...(10)}$$

From (10) we have

$$E_z=\frac{1}{e_{33}}\left[C_{13}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+C_{33}\frac{\partial \omega}{\partial z}\right] \qquad \text{...(10a)}$$

Then eliminating E_z, we have the first three equations of (7) as

$$T_{rr}=C_1\frac{\partial u}{\partial}+C_2\frac{u}{r}+C_3\frac{\partial \omega}{\partial z}$$

$$T_{\theta\theta}=C_2\frac{\partial u}{\partial r}+C_1\frac{u}{r}+C_3\frac{\partial \omega}{\partial z}$$

$$D_z=e_1\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+e_3\frac{\partial \omega}{\partial z} \qquad \text{...(11)}$$

Because of our assumption of inhomogeneity of the material parameters, we have $C_{11}=C_{11}^{\circ}(1+\kappa z)$, $C_{12}=C_{12}^{\circ}(1+\kappa z)$, $e_{31}=e_{31}^{\circ}(1+\kappa z)$ etc. So that

$$C_1=\left(C_{11}-\frac{C_{13}e_{31}}{e_{33}}\right)=C_1^{\circ}(1+\kappa z)$$

$$C_2=\left(C_{12}-\frac{C_{13}e_{31}}{e_{33}}\right)=C_2^{\circ}(1+\kappa z)$$

$$e_1=\left(e_{31}+\frac{C_{13}e_{33}}{e_{33}}\right)=e_1^{\circ}(1+\kappa z) \qquad \text{...(12)}$$

etc.

whence $C_1^{\circ}, C_2^{\circ}, C_3^{\circ} \ldots e_1^{\circ}, e_2^{\circ}, e_3^{\circ}$ being constants given by

$$C_1^{\circ}=\left(C_{11}^{\circ}-\frac{C_{13}^{\circ}e_{31}^{\circ}}{e_{33}^{\circ}}\right)$$

$$C_2^{\circ}=\left(C_{12}^{\circ}-\frac{C_{13}^{\circ}e_{31}^{\circ}}{e_{33}^{\circ}}\right)$$

$$e_1^{\circ}=\left(e_{31}^{\circ}+\frac{C_{13}^{\circ}e_{33}^{\circ}}{e_{33}^{\circ}}\right)$$

etc.

from (2)

$$\frac{\partial Dz}{\partial z}=0$$

$$Dz = \text{a function of } r = f(r) \text{ say.} \qquad \text{...(13)}$$

Thus the equation (1) gives

$$C_1\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+C_3\frac{\partial^2 \omega}{\partial z\partial r}+\rho\omega^2 r=0 \qquad \text{...(14)}$$

The last equation of (11) reads

$$e_1\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+e_3\frac{\partial \omega}{\partial z}=f(r) \qquad \text{...(15)}$$

Eliminating u between (15) and (16)

$$-\frac{\partial^2\omega}{\partial z\partial r}=\left(\frac{1+\kappa z}{\kappa_1}\right)(C_1{}^\circ f'(r)+e_1{}^\circ\rho\omega^2 r) \qquad \text{...(16)}$$

where $(C_3\,e_1-e_3\,C_1)=\kappa_1$

Eliminating ω from (15) and (16) we get

$$-\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+\left(\frac{1+\kappa z}{\kappa_1}\right)(C_3{}^\circ f'(r)+e_3{}^\circ\rho\omega r)$$

Since $\frac{\partial\omega}{\partial r}+\frac{\partial u}{\partial r}=0,$

$$\frac{\partial^2\omega}{\partial r\partial z}=-\frac{\partial^2 u}{\partial z^2}.$$

Thus, $\frac{\partial^2\omega}{\partial z^2}=\left(\frac{1+\kappa z}{\kappa_1}\right)(C_1{}^\circ f'(r)+e_1{}^\circ\rho\omega^2 r)$

Integrating,

$$u=\left(\frac{z^2}{2\kappa_1}+\frac{\kappa z^3}{6\kappa_1}\right)(C_1{}^\circ f'(r)+e_1{}^\circ\rho\omega r)+f_1(r)z+f_2(r) \qquad \text{...(17)}$$

where $f_1(r)$ and $f_2(r)$ are arbitrary functions of r.

Putting the value of u in (14) we have

$$C_1{}^\circ(1+\kappa z)\frac{\partial}{\partial r}\left\{\left(\frac{z^2}{2\kappa_1}+\frac{\kappa z^3}{6\kappa_1}\right)(C_1{}^\circ f'(r)+e_1{}^\circ\rho\omega^2)+f_1'(r)z+f_2'(r)\right.$$

$$\left.+\left(\frac{z^2}{2\kappa_1}+\frac{\kappa z^3}{6\kappa_1}\right)\left(C_1{}^\circ\frac{f'(r)}{r}+e_1{}^\circ\rho\omega^2\right)+\frac{f_1(r)}{r}z+\frac{f_2(r)}{r}\right\}$$

$$-C_3{}^\circ\frac{(1+\kappa z)^2}{\kappa_1}C_1{}^\circ f'(r)+e_1{}^\circ\rho\omega^2)+\rho\omega^2 r=0 \qquad \text{...(18)}$$

Collecting the co-efficients of different powers of z and solving for $f(r)$, $f_1(r)$ and $f_2(r)$ we have

$$f(r)=A_1+A_2r^2$$

$$f_1(r)=A_3r+\frac{B_1}{r}$$

$$f_2(r)=A_4r+\frac{B_2}{r}+\frac{A_5}{8}\kappa_2'(A_2+\kappa_1'\rho\omega^2)r^3 \qquad \text{...(19)}$$

where $A_1, A_2, A_3 \ldots\ldots B_1, B_2 \ldots\ldots$ are constants

$$\text{and}\ \kappa_2' = \frac{4C_3°C_1°}{\kappa_1},\quad \kappa_1' = \frac{e_1°}{2c_1°} \qquad \ldots(20)$$

Imposing appropriate conditions for the displacement component u we get

$$u = \frac{A_3 A_2' \kappa_2'}{8} r^3 + \frac{C_1°}{3\kappa_1}\left(3\kappa^2 + \kappa z^3\right) A_2' r + A_3 rz + A_4 r \qquad \ldots(21)$$

$$\text{where}\ A'_2 = (A_2 + \kappa'_1 \rho\omega^2) \qquad \ldots(22)$$

$$\therefore\ \frac{\partial u}{\partial r} + \frac{u}{r} = \frac{1}{2} A_3 A_2' \kappa_2' r^2 + \frac{2}{3}\frac{C_1°}{\kappa_1}\left(3z^2 + \kappa z^3\right) A_2' + 2A_3 z + 2A_4.$$

so that from (15)

$$\frac{\partial \omega}{\partial z} = \frac{A_1 + A_2 r^2}{e_3°(1 + \kappa z)} - \frac{e_1°}{e_3°}\left[\frac{2}{3}\frac{C_1°}{\kappa_1}\left(3z^2 + \kappa z^3\right) A_2' + 2A_3 z + 2A_4 + \frac{1}{2} A_3 A_2' \kappa_2' r^2\right] \qquad \ldots(23)$$

Thus from (10a) we can get the value of E_z and then applying the boundary conditions as stated in equations (8) and (9) we can calculate the value of all the constants and finally the values of u and ω. The constants can be expressed in terms of material parameters, radius, and prescribed electric intensity.

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