STUDY OF AN OSCILLATOR WITH TWO DEGREES OF FREEDOM BY A DIFFERENTIAL ANALYSER

B. R. NAG

INSTITUTE OF RADIO PHYSICS AND ELECTRONICS, UNIVERSITY OF CALCUTTA

(*Received, June* 17, 1958)

ABSTRACT. Equations giving the stable amplitudes of oscillation and the conditions of stability of all the possible modes of oscillation of an oscillator with two degrees of freedom and stabilised by a non-linearity which can be described by a third degree polynomial are given. The use of a differential analyser for the verification of these equations is illustrated. Also a method of graphically representing the transient oscillations on the analyser is described.

INTRODUCTION

Oscillators described by two simultaneous differential equations of the second order have two possible frequencies of oscillation. If the non-linearity of the circuit is neglected, it follows that oscillations will occur independently at both the frequencies, the amplitude at any one frequency being determined only by losses at that frequency. However, the non-linearity which is essential for limiting the amplitudes of oscillation introduces interdependence. Due to this interdependence there are two distinct modes of oscillations. In the one, the oscillator may oscillate at any one of the two frequencies and the amplitude of oscillation is then determined by the losses at the oscillation frequency; in the other it may oscillate at both the frequencies simultaneously and the amplitudes in that case are determined by the losses at both the frequencies. The detailed characteristics of such an oscillator may be obtained by solving the differential equations taking into account the contributions due to non-linearity. In the present paper, the solutions as obtained by the variation of parameter method are presented (Van der Pol, 1922; Fontana 1951; Schaffner, 1954).

Experimental verification of the theoretical results were obtained earlier by actual oscillator circuits, the non-linear terms being realised by vacuum tubes (Fontana, 1951). A closer and more detailed representation of the non-linear terms is possible with a differential analyser. The author has made use of a differential analyser with a view to verify the theoretical derivations for non-linearities expressible by a polynomial of the third degree. The results obtained are presented in this paper.

In visualising the growth of a particular mode of oscillation from the initial conditions, plots of the transient oscillations are very helpful. Theoretically, the transient plots are obtained by the method of isoclines and involve considerable labour (Schaffner, 1954). A simple method of obtaining graphical representation of the transient oscillations by depicting experimentally the trajectories in the $A_1 - A_2$ plane on the analyser is also described.

TYPICAL OSCILLATORS DESCRIBED BY TWO SIMULTANEOUS SECOND ORDER DIFFERENTIAL EQUATIONS

In general, oscillators consisting of two separately tuned circuits coupled together are described by two simultaneous differential equations of the second order. The tuned grid tuned plate oscillator is an example. The equivalent circuit of a $TG-TP$ oscillator is shown in figure 1.

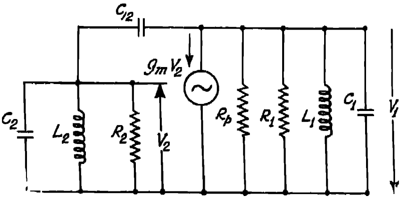


Fig. 1. Equivalent circuit of a tuned grid tuned plate oscillator.

The differential equations describing the oscillator are given by

$$\frac{d^2V_1}{dt^2}+\frac{1}{C_1+C_{12}}\frac{d}{dt}\left(\frac{1}{R_1}+\frac{1}{R_p}\right)V_1+\frac{V_1}{L_1(C_1+C_{12})}+\frac{C_{12}}{(C_1+C_{12})}\frac{d^2V_2}{dt^2}$$

$$=\frac{1}{(C_1+C_{12})}\frac{d(g_mV_2)}{dt}, \quad \ldots \quad (1a)$$

$$\frac{d^2V_2}{dt^2}+\frac{1}{(C_2+C_{12})}\frac{1}{R_2}\frac{dV_2}{dt}+\frac{V_2}{L_2(C_2+C_{12})}+\frac{C_{12}}{C_2+C_{12}}\frac{d^2V_1}{dt^2}=0. \quad \ldots \quad (1b)$$

If R_p is very large compared to R_1, the non-linearity is introduced by g_m only. g_m can be generally expressed as a function of V_2 in the form of a polynomial

Common single tuned circuit oscillators when supplying the output to a tuned load directly coupled to it are also described by two simultaneous differential

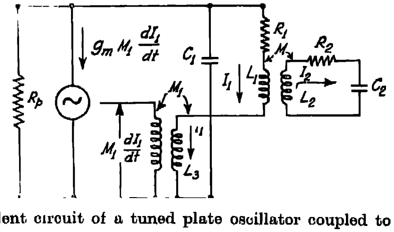


Fig. 2. Equivalent circuit of a tuned plate oscillator coupled to a tuned circuit.

equations of the second order. Such coupling is usual in the R.F. extra high tension supplies. The equivalent circuit for a tuned plate oscillator coupled to a tuned load by mutual inductance is shown in figure 2.

The differential equations for this circuit are

$$\frac{d^2I_1}{dt^2} + \left[\frac{R_1}{(L_3+L_1)} + \frac{1}{C_1R_p} + g_m\frac{M_1}{(L_3+L_1)C_1}\right]\frac{dI_1}{dt} - \frac{M}{(L_3+L_1)C_1R_p}\frac{dI_2}{dt}$$

$$+\frac{I_1}{(L_3+L_1)C_1}\left(1+\frac{R_1}{R_p}\right) - \frac{M}{(L_3+L_1)}\frac{d^2I_2}{dt^2} = 0. \quad \ldots \quad (2a)$$

$$\frac{d^2I_2}{dt^2} + \frac{R_2}{L_2}\frac{dI_2}{dt} + \frac{1}{L_2C_2}I_2 - \frac{M}{L_2}\frac{d^2I_1}{dt^2} = 0. \quad \ldots \quad (2b)$$

SOLUTION BY THE VARIATION OF PARAMETER METHOD

The above differential equation may, in general, be written in the form

$$\frac{d^2x_1}{dt_2} + \omega_1^2x_1 + K_2\frac{d^2x_2}{dt^2} = F_1\left(x_1, \frac{dx_1}{dt}, x_2, \frac{dx_2}{dt}\right), \quad \ldots \quad (3a)$$

$$\frac{d^2x_2}{dt^2} + \omega_2^2x_2 + K_1\frac{d^2x_1}{dt^2} = F_2\left(x_2, \frac{dx_2}{dt}, x_1, \frac{dx_1}{dt}\right), \quad \ldots \quad (3b)$$

In what follows it is assumed that

$$F_2\left(x_1, \frac{dx_1}{dt}, x_2, \frac{dx_2}{dt}\right) = -a_2\frac{dx_2}{dt}, \quad \ldots \quad (4a)$$

and

$$F_1\left(x_1, \frac{dx_1}{dt}, x_2, \frac{dx_2}{dt}\right) = a_1\frac{dx_1}{dt} - a_1f(x_1)\frac{dx_1}{dt} \quad \ldots \quad (4b)$$

Eliminating x_2 and neglecting terms involving the product a_1a_2, which is assumed to be small, one gets

$$\frac{d^4x_1}{dt^4} + (\omega_1^2+\omega_2^2)\frac{d^2x_1}{dt^2} + \omega_1^2\omega_2^2x_1 - K_1K_2\frac{d^4x_1}{dt^4}$$

$$= -a_2\left[\frac{d^3x_1}{dt^3} + \omega_1^2\frac{dx_1}{dt}\right] + a_1\left[\frac{d^3x_1}{dt^3} + \omega_2^2\frac{dx_1}{dt}\right]$$

$$-a_1\left[\frac{d^2}{dt^2}f(x_1)\frac{dx_1}{dt} + \omega_2^2f(x_1)\frac{dx_1}{dt}\right]. \quad \ldots \quad (5)$$

When $a_1 = a_2 = 0$, solution of Eqn. 4 is given by

$$x_1 = A_1 \cos(\omega_{10}t + \phi_1) + A_2 \cos(\omega_{20}t + \phi_2), \quad \ldots \quad (6)$$

where

$$\omega_{10}^2 = \frac{\omega_1^2 + \omega_2^2 - [(\omega_1^2 - \omega_2^2)^2 + 4K_1K_2\omega_1^2\omega_2^2]^{\frac{1}{2}}}{2(1 - K_1K_2)} \quad \ldots \quad (7a)$$

$$\omega_{20}^2 = \frac{\omega_1^2 + \omega_2^2 + [(\omega_1^2 - \omega_2^2) + 4K_1K_2\omega_1^2\omega_2^2]^{\frac{1}{2}}}{2(1 - K_1K_2)} \quad \ldots \quad (7b)$$

A_1, A_2, ϕ_1 and ϕ_2 are constants determined by the initial values of x_1 and its derivatives.

When a_1 and a_2 are finite A_1, A_2, ϕ_1 and ϕ_2 are not constant but are functions of time. In that case we may rewrite Eqn. (6) as

$$x_1 = A_1(t) \cos[\omega_{10}t + \phi_1(t)] + A_2(t) \cos[\omega_{20}t + \phi_2(t)], \quad \ldots \quad (7)$$

and substitute this value in Eqn. (5). Since a_1 and a_2 are small, though finite, an approximate solution may be obtained retaining the first order derivatives only. Thus

$$\frac{dA_1}{dt} = -\frac{F \sin(\omega_{10}t + \phi_1)}{\omega_{10}(\omega_{20}^2 - \omega_{10}^2)}, \quad \ldots \quad (8a)$$

$$\frac{dA_2}{dt} = \frac{F \sin(\omega_{20}t + \phi_2)}{\omega_{20}(\omega_{20}^2 - \omega_{10}^2)}, \quad \ldots \quad (8b)$$

$$\frac{d\phi_1}{dt} = -\frac{1}{A_1} \frac{F \cos(\omega_{10}t + \phi_1)}{\omega_{10}(\omega_{20}^2 - \omega_{10}^2)} \quad \ldots \quad (8c)$$

$$\frac{d\phi_2}{dt} = \frac{1}{A_2} \frac{F \cos(\omega_{20}t + \phi_2)}{\omega_{20}(\omega_{20}^2 - \omega_{10}^2)}, \quad \ldots \quad (8d)$$

where

$$(1 - K_1K_2)\, F = -a_2\left[\frac{d^3x_1}{dt^3} + \omega_1^2 \frac{dx_1}{dt}\right] + a_1\left[\frac{d^3x_1}{dt} + \omega_2^2 \frac{dx_1}{dt}\right] -$$

$$-a_1\left[\frac{d^2}{dt^2} f(x_1) \frac{dx_1}{dt} + \omega_2^2 f(x_1) \frac{dx_1}{dt}\right]. \quad \ldots \quad (9)$$

Since a_1 and a_2 are assumed to be small, $\dot{A}_1$, $\dot{A}_2$, $\dot{\phi}_1$, $\dot{\phi}_2$, which vary little over a period of oscillation can be approximately taken to be equal to their average values over the period.

Now, if $f(x_1)$ is given by a polynomial, F can be expanded into a Fourier series of the two fundamental periods $\frac{2\pi}{\omega_{10}}$ and $\frac{2\pi}{\omega_{20}}$. Let a_{10}, a_{20}, b_{10}, b_{20}, be the coefficients of $\sin(\omega_{10}t + \phi_1)$, $\sin(\omega_{20}t + \phi_2)$, $\cos(\omega_{10}t + \phi^1)$, $\cos(\omega_{20}t + \phi_2)$ respectively in the expansion, then

$$\frac{dA_1}{dt} = -\frac{1}{2}\frac{a_{10}}{\omega_{10}(\omega_{20}^2 - \omega_{10}^2)}, \qquad \ldots \quad (10a)$$

$$\frac{dA_2}{dt} = \frac{1}{2}\frac{a_{20}}{\omega_{20}(\omega_{20}^2 - \omega_{10}^2)}, \qquad \ldots \quad (10b)$$

$$\frac{d\phi_1}{dt} = -\frac{1}{2}\cdot\frac{b_{10}}{A_1\,\omega_{10}(\omega_{20}^2 - \omega_{10}^2)}, \qquad \ldots \quad (10c)$$

$$\frac{d\phi_2}{dt} = \frac{1}{2}\frac{b_{20}}{A_2\,\omega_{20}(\omega_{20}^2 - \omega_{10}^2)}. \qquad \ldots \quad (10d)$$

If ω_{10} and ω_{20} are not integrally related, the only conditions for equilibrium are $\dot{A}_1 = 0$, $\dot{A}_2 = 0$, and the equilibrium values of A_1 and A_2 may be obtained therefrom Also, the equilibrium is stable if the roots of the equation

$$\begin{vmatrix} p - \dfrac{\partial \dot{A}_1}{\partial A_1} & -\dfrac{\partial \dot{A}_1}{\partial A_2} \\ -\dfrac{\partial \dot{A}_2}{\partial A_1} & p - \dfrac{\partial \dot{A}_2}{\partial A_2} \end{vmatrix} = 0 \qquad \ldots \quad (11)$$

have their real parts negative for the equilibrium values of A_1 and A_2.

If ω_{10} and ω_{20} are integrally related, *i.e.*, $\frac{\omega_{20}}{\omega_{10}} = \frac{p}{q}$, $\frac{p}{q}$ being an integral ratio, then for equilibrium in addition to the conditions $\dot{A}_1 = 0$, $\dot{A}_2 = 0$, it is required that

$$\dot{\phi} = p\dot{\phi}_1 - q\dot{\phi}_2 = 0. \qquad \ldots \quad (12)$$

In this case stable equilibrium will require the roots of the equation

$$\begin{vmatrix} p - \dfrac{\partial \dot{A}_1}{\partial A_1} & -\dfrac{\partial \dot{A}_1}{\partial A_2} & -\dfrac{\partial \dot{A}_1}{\partial \phi} \\ -\dfrac{\partial \dot{A}_2}{\partial A_1} & p - \dfrac{\partial \dot{A}_2}{\partial A_2} & -\dfrac{\partial \dot{A}_2}{\partial \phi} \\ -\dfrac{\partial \dot{\phi}_1}{\partial A_1} & -\dfrac{\partial \dot{\phi}}{\partial A_2} & p - \dfrac{\partial \dot{\phi}}{\partial \phi} \end{vmatrix} = 0 \qquad \ldots \quad (13)$$

to have negative real parts. The derivatives are to be evaluated at the equilibrium values of A_1, A_2 and ϕ.

It may be noted that small deviations in the ratio of $\frac{\omega_{20}}{\omega_{10}}$ from the integral value may be compensated by $\dot{\phi}$. In that case for equilibrium condition (12) will be replaced by

$$\Delta\omega + \dot{\phi} = 0,$$

where
$$\Delta\omega = p\omega_{10} - q\omega_{20}. \quad \ldots \quad (14)$$

An oscillator, stabilised by a non-linearity characterised by $a_1 f(x_1) = cx_1^2\ bx_1$, will now be considered in the light of the above general analysis. For such an oscillator three distinct cases are to be considered; these are discussed below :

Case I : ω_{20}/ω_{10} has a value other than 2/1 or 3/1. In this case,

$$(1-K_1K_2)a_{10} = a_2A_1(\omega_1^2-\omega_{10}^2)\omega_{10} - a_1A_1(\omega_2^2-\omega_{10}^2)\omega_{10} + \frac{c}{4}A_1(\omega_2^2-\omega_{10}^2)\omega_{10}(A_1^2+2A_2^2), \quad \ldots \quad (15a)$$

$$(1-K_1K_2)a_{20} = -a_2A_2(\omega_{20}^2-\omega_1^2)\omega_{20} + a_1A_2(\omega_{20}^2-\omega_2^2) - \frac{c}{4}A_2(\omega_{20}^2-\omega_2^2)\omega_{20}(A_2^2+2A_1^2), \quad \ldots \quad (15b)$$

$$b_{10} = b_{20} = 0 \quad \ldots \quad (15c)$$

Putting

$$\frac{\omega_1^2-\omega_{10}^2}{\omega_2^2-\omega_{10}^2} = \rho, \quad \frac{\omega_{20}^2-\omega_1^2}{\omega_{20}^2-\omega_2^2} = \sigma,$$

$$\frac{4}{c}(\rho a_2 - a_1) = -A_{10}^2.$$

and
$$\frac{4}{c}(\sigma a_2 - a_1) = -A_{20}^2,$$

Eqn. (15a) and (15b) may be written as

$$(1-K_1K_2)a_{10} = \frac{c}{4}(\omega_2^2-\omega_{10}^2)[-A_{10}^2+A_1^2+2A_2^2]A_1, \quad \ldots \quad (16a)$$

$$(1-K_1K_2)a_{20} = -\frac{c}{4}(\omega_{20}^2-\omega_2^2)[-A_{20}^2+A_2^2+2A_1^2]A_2 \quad \ldots \quad (16b)$$

Hence from Eqns. (10a) and (10b)

$$2(1-K_1K_2)\dot{A}_1 = -\frac{c}{4}\frac{(\omega_2^2-\omega_{10}^2)}{(\omega_{20}^2-\omega_{10}^2)}[-A_{10}^2+A_1^2+2A_2^2]A_1, \quad \ldots \quad (17a)$$

$$2(1-K_1K_2)\dot{A}_2 = -\frac{c}{4}\frac{(\omega_{20}^2-\omega_2^2)}{(\omega_{20}^2-\omega_{10}^2}[-A_{20}^2+A_2^2+2A_1^2]A_2 \quad \ldots \quad (17b)$$

Conditions of equilibrium are given by

$$A_1(-A_{10}^2+A_1^2+2A_2^2) = 0,$$
$$A_2(-A_{20}^2+A_2^2+2A_1^2) = 0.$$

These conditions are satisfied by either of the following three possible combinations of A_1 and A_2 ·

(1) $A_2 = 0, \qquad A_1 = A_{10}$. ... (18a)

(2) $A_1 = 0, \qquad A_2 = A_{20}$. ... (18b)

(3) $A_1^2 = \frac{2A_{20}^2 - A_{10}^2}{3}, \; A_2^2 = \frac{2A_{10}^2 - A_{20}^2}{3}$. ... (18c)

The equilibrium corresponding to the combination $A_2 = 0$, $A_1 = A_{10}$ is stable if $\frac{A_{10}^2}{A_{20}^2} > \frac{1}{2}$ and that corresponding to the combination $A_1 = 0$, $A_2 = A_{20}$ is stable if $\frac{A_{10}^2}{A_{20}^2} < 2$. The third combination gives unstable equilibrium (Van Der Pol 1922).

Thus, the oscillator will oscillate at one frequency at a time. Further, if $\frac{1}{2} < \frac{A_{10}^2}{A_{20}^2} < 2$ the oscillator may choose any of the two frequencies, the choice being determined by the initial values of A_1, A_2 and their derivatives.

Case II : $\omega_{20} : \omega_{10} = 3 : 1.$

Here,

$$(1-K_1K_2)a_{10} = -[A_{10}^2-(A_1^2+2A_2^2)-A_1A_2\cos\phi]\frac{c}{4}\omega_{10}A_1(\omega_2^2-\omega_{10}^2), \quad \ldots \quad (19a)$$

$$(1-K_1K_2)a_{20} = \left[A_{20}^2-(A_2^2+2A_1^2)-\frac{A_1^3}{3A_2}\cos\phi\right]\frac{c}{4}\omega_{20}A_2(\omega_{20}^2-\omega_2^2), \quad \ldots \quad (19b)$$

$$(1-K_1K_2)b_{10} = -\frac{c}{4}\omega_{10}(\omega_2^2-\omega_{10}^2)A_1^2A_2\sin\phi. \quad \ldots \quad (19c)$$

$$(1-K_1K_2)b_{20} = -\frac{c}{4}\omega_{20}(\omega_{20}^2-\omega_2^2)\frac{A_1^3}{3}\sin\phi. \quad \ldots \quad (19d)$$

Thus, from Eqns. (10a), (10b), (10c) and (10d) and remembering Eqn. (12),

$$2(1-K_1K_2)\dot{A_1} = \frac{c}{4}\frac{(\omega_2^2-\omega_{10}^2)}{(\omega_{20}^2-\omega_{10}^2)}\left[A_{10}^2-(2A_2^2+A_1^2)-A_1A_2\cos\phi\right]A_1, \quad \ldots \quad (20a)$$

$$2(1-K_1K_2)\dot{A_2} = \frac{c}{4}\frac{(\omega_{20}^2-\omega_2^2)}{(\omega_{20}^2-\omega_{10}^2)}\left[A_{20}^2-(2A_1^2+A_2^2)-\frac{A_1^3}{3A_2}\cos\phi\right]A_2 \quad \ldots \quad (20b)$$

$$2(1-K_1K_2)\dot{\phi} = \frac{c}{4}\left[\frac{3(\omega_2^2-\omega_{10}^2)}{(\omega_{20}^2-\omega_{10}^2)}A_2^2+\frac{(\omega_{20}^2-\omega_2^2)}{(\omega_{20}^2-\omega_{10}^2)}\frac{A_1^2}{3}\right]\frac{A_1}{A_2}\sin\phi. \quad \ldots \quad (20c)$$

Equilibrium conditions are obtained for either of the two possible combinations:

(1) $$A_1 = 0, \qquad A_2 = A_{20}. \quad \ldots \quad (21a)$$

(2) $$A_{10}^2 = 2A_2^2 + A_1^2 + A_1A_2\cos\phi \;(\sin\phi = 0), \quad \ldots \quad (21b)$$

$$A_{20}^2 = 2A_1^2 + A_2^2 + \frac{A_1^3}{3A_2}\cos\phi \;(\sin\phi = 0). \quad \ldots \quad (21c)$$

Of these the equilibrium given by the first combination will be stable if $\frac{A_{10}^2}{A_{20}^2} < 2$. The equilibrium corresponding to the second combination will be stable if $\cos\phi = -1$, *i.e.* $\phi = \pi$ and if A_1 and A_2, given by equations (21b) and (21c) satisfy the condition

$$(2A_1^2 - A_1A_2)\left(2A_2^2 + \frac{A_1^3}{3A_2}\right) - (4A_1A_2 - A_1^2)^2 < 0.$$

Writing $A_2 = nA_1$, n being a positive real number, the above inequality reduces to

$$n^3 + 6n^2 + \frac{2}{3} - 4n - \frac{1}{3n} > 0,$$

Whence, $$n < 0.54 \quad \ldots \quad (22)$$

i.e. the combination 2 is stable if the ratio between the equilibrium values of A_2 and A_1 is less than 0.54.

Further from Eqns. (21b) and (21c), it is observed that n is related to $\frac{A_{10}^2}{A_{20}^2}$ by the relation

$$\frac{A_{10}^2}{A_{20}^2} = \frac{2n^2 + 1 - n}{2 + n^2 - \frac{1}{3n}} \quad \ldots \quad (23)$$

When ω_{20} departs from 3 ω_{10} by a small amount $\Delta\omega$, *i.e.*, $\omega_{20} = 3\omega_{10} \pm \frac{\Delta\omega}{2(1-K_1K_2)}$, equilibrium conditions are given by

$$\Delta\omega = -\frac{c}{4}\left[\frac{3(\omega_2^2 - \omega_{10}^2)}{(\omega_{20}^2 - \omega_{10}^2)} A_2^2 + \frac{(\omega_{20}^2 - \omega_2^2)}{(\omega_{20}^2 - \omega_{10}^2)} \frac{A_1^2}{3}\right] \frac{A_1}{A_2} \sin\phi. \quad \ldots \ (24a)$$

$$A_{10}^2 = 2A_2^2 + A_1^2 + A_1A_2 \cos\phi, \quad \ldots \ (24b)$$

$$A_{20}^2 = 2A_1^2 + A_2^2 + \frac{A_1^3}{3A_2} \cos\phi. \quad \ldots \ (24c)$$

Case III : $\omega_{20} : \omega_{10} = 2 : 1$.

In this case, it may be seen that

$$2(1-K_1K_2)\dot{A}_1 = \frac{c}{4} \frac{(\omega_2^2 - \omega_{10}^2)}{(\omega_{20}^2 - \omega_{10}^2)} \left[A_{10}^2 - (A_1^2 + 2A_2^2) - \frac{2b}{c} A_2 \cos\phi \right] A_1, \ldots \ (25a)$$

$$2(1-K_1K_2)\dot{A}_2 = \frac{c}{4} \frac{(\omega_{20}^2 - \omega_2^2)}{(\omega_{20}^2 - \omega_{10}^2)} \left[A_{20}^2 - (A_2^2 + 2A_1^2) - \frac{2b}{c} \frac{A_1^2}{2A_2} \cos\phi \right] A_2, \ldots \ (25b)$$

$$2(1-K_1K_2)\dot{\phi} = \frac{b}{2}\left[\frac{2(\omega_2^2 - \omega_{10}^2)}{(\omega_{20}^2 - \omega_{10}^2)} A_2^2 + \frac{(\omega_{20}^2 - \omega_2^2)}{(\omega_{20}^2 - \omega_{10}^2)} \frac{A_1^2}{2}\right] \frac{\sin\phi}{A_2}. \quad \ldots \ (25c)$$

Equilibrium conditions are given by either of the two possible combinations :

(1) $$A_1 = 0, \qquad A_2 = A_{20}. \quad \ldots \ (26a)$$

(2) $$A_{10}^2 = A_1^2 + 2A_2^2 + \frac{2b}{c} A_2 \cos\phi \ (b \sin\phi = 0) \quad \ldots \ (26b)$$

$$A_{20}^2 = A_2^2 + 2A_1^2 + \frac{2b}{c} \frac{A_1^2}{2A_2} \cos\phi \ (b \sin\phi = 0) \quad \ldots \ (26c)$$

For the first combination the equilibrium is stable when $\frac{A_{10}^2}{A_{20}^2} < 2$. For the second combination, the equilibrium is stable if $b \cos\phi$ is negative and

$$\frac{b}{c} A_1^2 + \frac{8b}{c} A_2^2 - 6A_2^3 - \frac{2b^2}{c^2} A_2 > 0, \quad \ldots \ (27)$$

where A_1 and A_2 satisfy the equations (26*b*) and (26c).

It may be noted that Eqns. (26b) and (26c) may be solved directly to obtain the equilibrium values of A_1 and A_2. Thus, on eliminating A_1 the following equation is obtained.

$$A_2^3 - \frac{2b}{c} A_2^2 + \frac{1}{3}\left[A_{20}^2 - 2A_{10}^2 + 2\left(\frac{b}{c}\right)^2\right] A_2 + \frac{1}{3}\frac{b}{c} A_{10}^2 = 0, \ (28)$$

For a particular combination of the values of $\frac{b}{c}$, A_{10}^2 and A_{20}^2 there are three values of A_2 which satisfy Eqn. (28). However, only one of them satisfy the inequality (27).

When ω_{20} departs from 2 ω_{10} by a small amount $\Delta\omega$, *i.e.*, $\omega_{20} = 2\omega_{10} \pm \frac{\Delta\omega}{2(1-K_1K_2)}$,

Eqn. (25c) is replaced by

$$\Delta\omega = -\frac{b}{2}\left[2\frac{(\omega_2^3-\omega_{10}^2)}{(\omega_{20}^2-\omega_{10}^2)}A_2^2+\frac{(\omega_{20}^2-\omega_2^2)}{(\omega_{20}^2-\omega_{10}^2)}\frac{A_1^2}{2}\right]\frac{\sin\phi}{A_2} \quad \ldots \quad (29)$$

It should be noted that in both Cases II and III oscillation at the lower frequency alone is not possible. The oscillator may either oscillate at the higher frequency alone or at both simultaneously.

EXPERIMENTAL VERIFICATION

The theoretical results presented in the above section have been verified on an electronic differential analyser. In the following paragraphs the application of the analyser for the verification of Eqns. (18a), (18b), (23), (24a) and (28) has been discussed.

The differential equations to be set on the computer are

$$\frac{d^2x_1}{dt^2}+\omega_1^2x_1+(-a_1+bx_1+cx_1^2)\frac{dx_1}{dt}+K_2\frac{d^2x_2}{dt^2}=0, \quad \ldots \quad (30a)$$

$$\frac{d^2x_2}{dt^2}+\omega_2^2x_2+a_2\frac{dx_2}{dt}+K_1\frac{d^2x_1}{dt^2}=0. \quad \ldots \quad (30b)$$

Eqn. (30a) can also be written as

$$\frac{dx_1}{dt}+\omega_1^2\int x_1dt+\left(-a_1x_1+\frac{b}{2}x_1^2+\frac{c}{3}x_1^3\right)+K_2\frac{dx_2}{dt}=0 \quad \ldots \quad (30c)$$

A set-up of the differential analyser for solving Eqns. (30c) and (30b) is shown in figure 3. The non-linear function generator for generating the function $f_1(x_1) = \frac{b}{2}x_1^2+\frac{c}{3}x_1^3$ is of the biased diode type (Burt and Lange, 1956, Meissinger, 1955).

The solution x_1, x_2 and their derivatives appear at the points marked on the figure.

Eqns. (18a) and (18b) relate the losses at the two frequencies with the amplitudes of oscillations when the oscillations occur singly. For verification of these

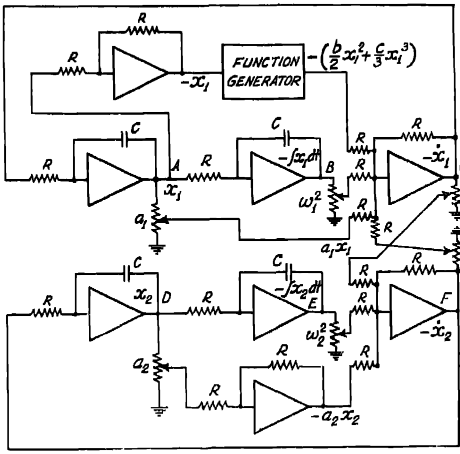


Fig. 3. Set-up of the differential analyser for solving the equations :

$$\ddot{x}_1 - a_1\dot{x}_1 + (bx_1 + cx_1^2)\dot{x}_1 + \omega_1^2 x_1 + k_2\ddot{x}_2 = 0.$$

$$\ddot{x}_2 + a_2\dot{x}_2 + \omega_2^2 x_2 + k_1\ddot{x}_1 = 0.$$

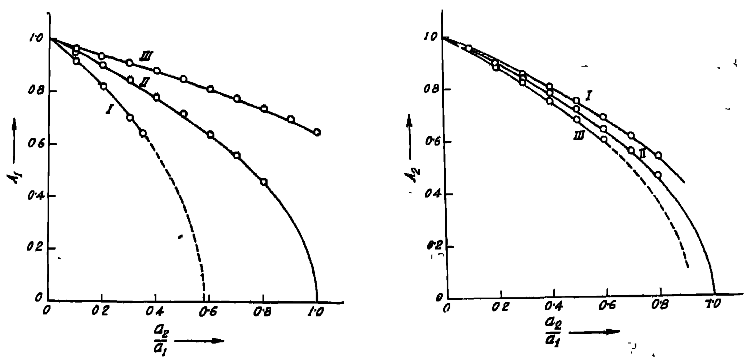


Fig. 4. (a) Plots of A_1 against a_2/a_1. (b) Plots of A_2 against a_2/a_1

———— Theoretical plot for stable oscillations.

– – – – Theoretical plot for unstable oscillation.

o Experimental points.

I – σ = 0.912. ρ = 1.714 II – σ = 1. ρ = 1 III – σ = 1.096 ρ = 0.58

equations oscillations at the two frequencies are excited individually and their amplitudes measured for selected values of the parameters ρ, σ and $\frac{4a_1}{c}$. In figure 4 experimental plots relating A_1 and A_2 with $\frac{a_2}{a_1}$ are shown along with the theoretically calculated curves for three sets of values of ρ and σ for $\frac{4a_1}{c} = 1$.

Eqn. (23) gives the ratio of the amplitudes of simultaneous oscillations at the two frequencies related by the ratio 3 : 1 for different values of $\frac{A_{10}^2}{A_{20}^2}$. Verification of this equation would require determination of the ratio $\frac{A_2}{A_1}$ for different settings of $\frac{A_{10}^2}{A_{20}^2}$. Different values of $\frac{A_{10}^2}{A_{20}^2}$ were set by varying $\frac{a_2}{a_1}$. Values of $\frac{A_{10}^2}{A_{20}^2}$ within the range 0 to ∞ were obtained by making $\omega_1^2 = 1$ and $\omega_2^2 = 0.8$ and values within the range $-\infty$ to 0, by making $\omega_1^2 = 0.8$ and $\omega_2^2 = 1$. For determining the ratio of the amplitudes, oscillations at the two frequencies were first separated by combining the outputs at B and C. The output at A is given by

$$x_1 = A_1 \cos(\omega_{10} t + \phi_1) + A_2 \cos(\omega_{20} t + \phi_2)$$

The outputs at B and C are therefore given by respectively

$$E_B = -\int x_1 dt = -\frac{A_1}{\omega_{10}} \sin(\omega_{10} t + \phi_1) - \frac{A_2}{\omega_{20}} \sin(\omega_{20} t + \phi_2)$$

and $$E_C = -\dot{x}_1 = \omega_{10} A_1 \sin(\omega_{10} t + \phi_1) + \omega_{20} A_2 \sin(\omega_{20} t + \phi_2)$$

Hence,

$$\frac{\omega_{10}^2 E_B + E_C}{\omega_{20}\left(1 - \frac{\omega_{10}^2}{\omega_{20}^2}\right)} = A_1 \sin(\omega_{20} t + \phi_2) \qquad \ldots \text{(31a)}$$

$$\frac{E_B + \frac{E_C}{\omega^2_{20}}}{\frac{1}{\omega_{10}}\left(\frac{\omega_{10}^2}{\omega_{20}^2} - 1\right)} = A_1 \sin(\omega_{10} t + \phi_1) \qquad \ldots \text{(31b)}$$

Thus, the amplitudes of oscillation at the two frequencies can be determined using the Eqns. (31a) and (31b). Experimental values of $n = \frac{A_2}{A_1}$ for different values of $\frac{A_{10}^2}{A_{20}^2}$ along with the theoretical values are given in figure 5.

Eqn. (24a) relates the phase difference between two approximately synchronous oscillations pulled into synchronism by the non-linearity, with $\Delta\omega$. Verification

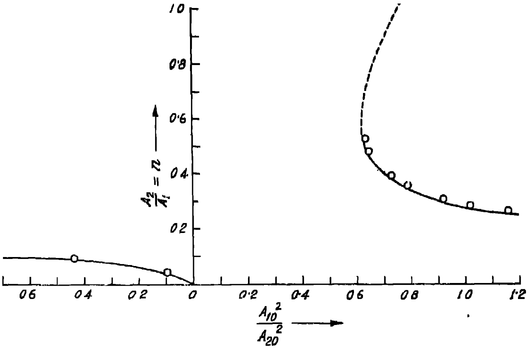


Fig. 5. Plot of $n = \frac{A_2}{A_1}$ against $\frac{A_{10}^2}{A_{20}^2}$ for $\omega_{20} = 3\omega_{10}$.

——— Theoretical plot for stable oscillations

– – – – Theoretical plot for unstable oscillations

o Experimental points.

of this equation requires determination of ϕ for different settings of $\Delta\omega$. This was set to different values by varying the value of K_1K_2. The phase ϕ was measured by an oscilloscope. The arrangement used in shown in figure 6.

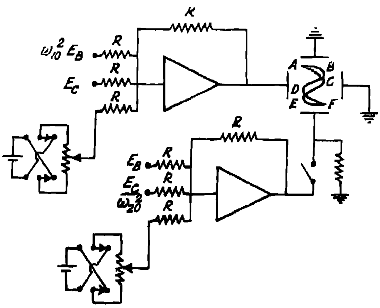


Fig. 6. Experimental arrangement for measuring ϕ when $\omega_{20} = 3\omega_{10} \pm \Delta\omega$.

Oscillations at the two frequencies were separated as described above. The sinusoidal voltage varying at the frequency ω_{20} was fed to the X-plate and that varying at the frequency ω_{10} was fed to the Y-plate. In general, the plot on the oscilloscope screen touches the vertical line at six points, marked A, B, C, D, E and F in figure 6. Let d_{AF}, d_{BE} and d_{CD} denote the distances between A and

F, B and E, C and D respectively. Then $\frac{d_{BE}}{d_{AF}} = \cos\left(\frac{\pi}{3} - \frac{\phi}{3}\right)$ and also $\frac{d_{CD}}{d_{AF}} = \cos\left(\frac{\pi}{3} + \frac{\phi}{3}\right)$. The distances d_{AF}, d_{BE} and d_{OD} were measured by shifting the

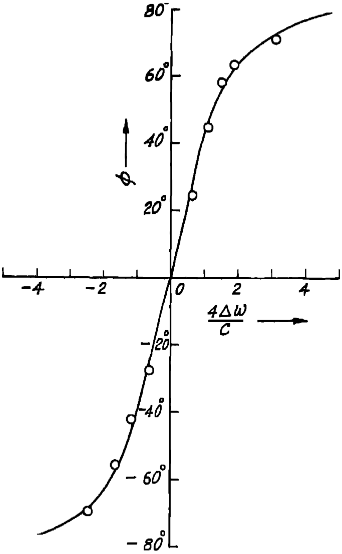


Fig. 7. Plot of ϕ against $\Delta\omega$ for $\omega_{20} = 3\omega_{10} + \Delta\omega$.
——— Theoretical plot.
o Experimental points.

pattern vertically and noting the voltages required for bringing the different points on the zero line, which was put on the oscilloscope by opening a switch at intervals. The shifting voltage was obtained from a calibrated helipot which gives the distance directly. The experimental plot of ϕ against $\Delta\omega$, for $\omega_1^2 = \omega_2^2 = 1$ are shown in figure 7 together with the theoretical plot.

Data for verification of Eqns. (26b) and (26c) were obtained in the same manner as indicated in connection with Eqn. (23). The experimental plots of $\frac{A_2}{A_1}$ against $\frac{2b}{c}$ for three values of A_{20}^2 corresponding to $A_{10}^2 = .2$ is shown in figure 8 along with the theoretically calculated curves.

EXPERIMENTAL TRAJECTORIES IN THE A_1-A_2 PLANE

In all the cases discussed above it is found that the oscillator has two possible modes of stable oscillation. The particular mode chosen by it depends on the initial values of A_1, A_2 and ϕ. The growth of a particular mode from the initial conditions is usually illustrated by drawing trajectories in the A_1-A_2 plane

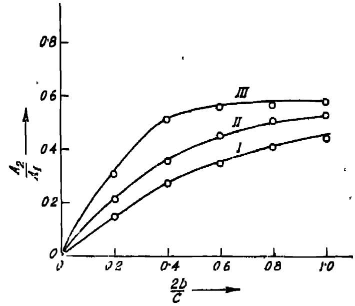


Fig. 8. Plots of A_2/A_1 against $\frac{2b}{c}$ for $\omega_{20} = 2\omega_{10}$.

———— Theoretical plot

o Experimental points

I — $A_{10}^2 = 0.2$ $A_{20}^2 = 0.1$

II — $A_{10}^2 = 0.2$ $A_{20}^2 = 0.2$

III — $A_{10}^2 = 0.2$ $A_{20}^2 = 0.26$.

applying the method of isoclines. On the differential analyzer these trajectories may also be easily obtained by applying voltages proportional to A_1 and A_2 or some function of A_1 and A_2 to the X and Y plates. It has been described before how voltages proportional to $A_1 \sin(\omega_{10}t+\phi_1)$ and $A_2 \sin(\omega_{20}t+\phi_2)$ may be obtained by combining the outputs at B and C (figure 3) in the steady state. During the transient state also the combined voltages will have amplitudes very nearly proportional to A_1 and A_2 when a_1 and a_2 are small. Similarly by combining the outputs at A and D voltages proportional to $A_1 \cos(\omega_{10}t+\phi_1)$ and $A_2 \cos(\omega_{20}t+\phi_2)$ may be obtained. By squaring these Sine and Cosine voltages and adding them voltages representing A_1^2 and A_2^2 may be obtained.

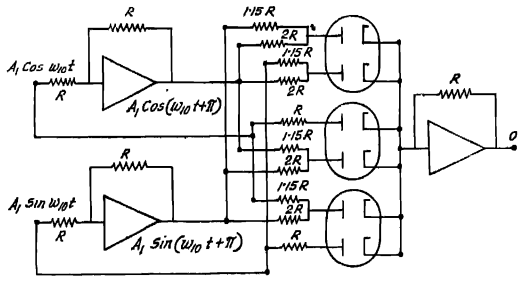
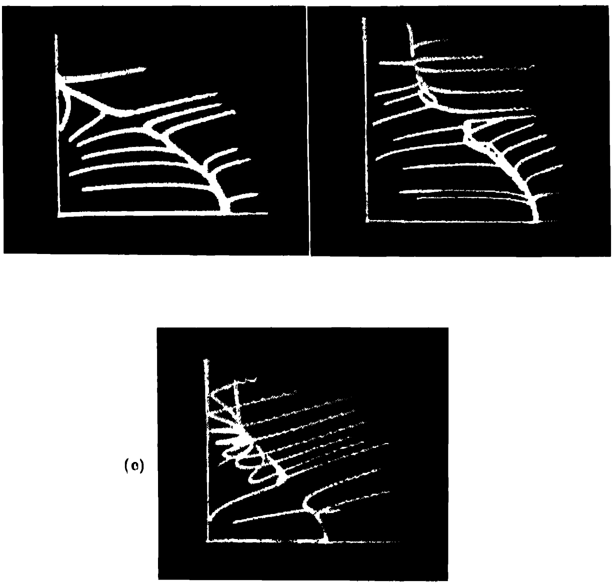


Fig. 9. Experimental arrangement for double three phase rectification.

For getting voltages proportional to A_1 and A_2, one may subject the sine and cosine voltages to multiphase rectification. One arrangement for doing this is shown in figure 9.

It will be noted that the voltage output at O is proportional to A_1 but is mixed up with a certain amount of ripple, which is 3% in this case. The ripple content may be further reduced by quadruple three phase rectification in which case the ripple is only .6%. For obtaining the trajectories it was found advantageous to employ the voltages corresponding to A_1 and A_2 as obtained through the multiphase rectification circuit for it requires less components and is simpler than the squaring circuit.

The trajectories as obtained on the analyser are shown in figure 10. Figure 10(a) gives the trajectories for a particular combination of ω_{20} and ω_{10}, not integrally related. It is seen that in this case the two stable equilibrium points lie on the two axes. The third equilibrium point which is unstable is also clearly indicated.



(a) (b) (c)

Fig. 10. Experimental trajectories in the A_1-A_2 plane.
(a) Trajectories for ω_{20} and ω_{10} non-integrally related.
(b) Trajectories for $\omega_{20} = 3\omega_{10}$
(c) Trajectories for $\omega_{20} = 2\omega_{10}$

Figure 10(b) gives the trajectories for $\omega_{20} = 3\omega_{10}$. In this case one of the stable equilibrium points lies on the A_2 axis, whereas the other one lies at a point for which both A_1 and A_2 have finite values. There is a third equilibrium point which is, however, unstable. In contrast to the previous case, trajectories in the A_1, A_2 plane are not unique in this case since they also depend on ϕ. This explains the crossing of some of the trajectories. Figure 10(c) gives the trajectories for $\omega_{20} = 2\omega_{10}$, the general characteristics of which are similar to those in figure 10(b).

CONCLUSIONS

An oscillator with two degrees of freedom and stabilised by a cubic non-linearity has two possible modes of oscillations. It may oscillate at one of the two possible frequencies at a time or at both simultaneously. The latter mode is stable only if the two frequencies are related approximately by the integral ratio 3 . 1 or 2 . 1. The order of approximation in the integral ratio permitting the simultaneous oscillation is determined by the magnitude of the non-linearity. The stable amplitudes of oscillation as also their conditions of stability as obtained theoretically by the variation of parameter method agree quite closely with those obtained experimentally with the help of a differential analyser. A differential analyser can also be used very usefully for obtaining trajectories in the $A_1 - A_2$ plane which show clearly the growth of oscillations of a particular mode from the various initial conditions, as well as the different possible equilibrium points.

ACKNOWLEDGMENT

The author is deeply indebted to Professor J. N. Bhar for his constant guidance and valuable suggestions.

REFERENCES

Burt, E. G. C. and Lange, O. H., 1956, *Proc. I.E.E., Part C*, **103**, 51.

Fontana, R. E., 1951, *Proc. I.R.E.*, **39**, 945.

Meissinger, H. F., 1955, I.R.E. Convention Records, 150.

N. Kryloff and N. Bogoliuboff, 1943, Introduction to Non-linear Mechanics, 73.

N. Minorsky, 1947, Non-linear Mechanics, 148; 251.

Schaffner, J. S., 1954, *Trans. I.R.E.*, **2**.

Van der Pol., B., 1922, *Phil. Mag.*, **43**, 700.