RADIAL PULSATIONS OF AN INFINITE CYLINDER IN THE PRESENCE OF MAGNETIC FIELD

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ABSTRACT. The general equation governing the radial pulsations of an infinite cylinder having volume currents has been derived and the integral formulae for the frequency of pulsations are deduced for two models of current density, viz., (i) circular currents and (ii) line currents. It is found that the cylinder remains dynamically stable for the two models of current systems.

The pulsations of an infinitely long cylindrical mass are of great significance for the cosmic bodies, e.g., spiral arm, solar ion streams etc. Chandrasekhar and Fermi (1953) have investigated the radial pulsations of such cylindrical masses in the presence of axial magnetic field. In the present note we investigate a similar problem in the presence of volume currents following the method adopted in our earlier paper (Talwar and Tandon, 1956) for the radial pulsations of spherical mass. Here, two special cases, (i) circular currents and (ii) axial line currents are discussed.

The equation of continuity and motion for the cylindrical fluid subjected to electromagnetic field can be written in the form

$$\frac{r}{r_0} \cdot \frac{\rho}{\rho} \cdot \frac{dr}{dr_0} = 1 \qquad \dots (1)$$

and

$$\rho \frac{\partial^2 r}{\partial t^2} = -\frac{\partial p}{\partial r} - \frac{2Gm(r)}{r} \rho + (\boldsymbol{j} \times \boldsymbol{H})_{radial} \qquad \dots (2)$$

Here all the physical quantities have their usual meaning and the electromagnetic field vectors $\boldsymbol{H}$ and $\boldsymbol{j}$ satisfy the usual Maxwellian relations. viz.,

$$\text{curl } \boldsymbol{H} = 4\pi \boldsymbol{j} \qquad \dots (3)$$

and

$$\operatorname{div} \boldsymbol{H} = 0 \tag{4}$$

Distinguishing the values of various parameters for the equilibrium configuration by a subscript zero, we write

$$r = r_0 + \delta r, \qquad p = p_0 + \delta p, \qquad \rho = \rho_0 + \delta\rho \quad \ldots \; (5)$$

$$\boldsymbol{H} = \boldsymbol{H}_0 + \delta\boldsymbol{H} \quad \text{and} \quad \boldsymbol{j} = \boldsymbol{j}_0 + \delta\boldsymbol{j}$$

The variations in these parameters are assumed to be small so that the powers higher than the first can be neglected.

The equation (1) then gives

$$-\frac{\delta\rho}{\rho_0} = \frac{1}{r_0}\frac{\partial}{\partial r_0}(r_0\delta r) \quad \ldots \; (6)$$

Taking the various terms in equation (2) in turn, we get

$$\rho\frac{\partial^2 r}{\partial t^2} = \rho_0\frac{\partial^2 \delta r}{\partial t^2} \quad \ldots \; (7)$$

$$\frac{2Gm(r)}{r}\rho = \frac{2Gm(r_0)}{r_0}\rho_0\left(1 - \frac{\partial\delta r}{\partial r_0}\right) \quad \ldots (8)$$

and

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial r_0}\frac{\partial r_0}{\partial r}$$

$$= \frac{\partial p_0}{\partial r_0}\left(1 - \frac{\partial\delta r}{\partial r_0}\right) + \frac{\partial\delta p}{\partial r_0} \quad \ldots \; (9)$$

But for the equilibrium configuration equation (2) gives

$$\frac{\partial p_0}{\partial r_0} = -\frac{2Gm(r_0)}{r_0}\rho_0 + (\boldsymbol{j}_0 \times \boldsymbol{H}_0)_{radial} \quad \ldots \; (10)$$

and hence

$$\frac{\partial p}{\partial r} = \left[-\frac{2Gm(r_0)}{r_0}\rho_0 + (\boldsymbol{j}_0 \times \boldsymbol{H}_0)_{radial}\right] \quad \ldots \; (11)$$

$$\times\left(1 - \frac{\partial\delta r}{\partial r_0}\right) + \frac{\partial\delta\rho}{\partial r_0}$$

Now,

$$\frac{\partial \delta p}{\partial r_0} = \frac{\partial}{\partial r_0}\left[\frac{\Gamma p_0}{\rho_0}\delta\rho\right], \text{ Since } \frac{\delta p}{p_0} = \Gamma\frac{\delta\rho}{\rho_0}, \text{ (adiabatic pulsations)}$$

$$= -\Gamma\left[-\frac{2Gm(r_0)}{r_0}\rho_0 + (j_0\times H_0)_{radial}\right] \times \frac{1}{r_0}\frac{\partial}{\partial r_0}(r_0\delta r) - \Gamma p_0\frac{\partial}{\partial r_0}\left\{\frac{1}{r_0}\frac{\partial}{\partial r_0}(r_0\delta r)\right\}, \quad \ldots (12)$$

and hence equation (11) becomes

$$\frac{\partial p}{\partial r} = \left(1-\frac{\partial \delta r}{\partial r_0}\right)\left\{-\frac{2Gm(r_0)}{r_0}\rho_0 + (j_0\times H_0)_{radial}\right\} - \Gamma\left\{-\frac{2Gm(r_0)}{r_0}\rho_0 + (j_0\times H_0)_{radial}\right\} \times \frac{1}{r_0}\frac{\partial}{\partial r_0}(r_0\delta r) - \Gamma p_0\frac{\partial}{\partial r_0}\left\{\frac{1}{r_0}\frac{\partial}{\partial r_0}(r_0\delta r)\right\} \quad (13)$$

Substituting equations (7), (8) and (13) in equation (2), we obtain, after some simplications.

$$\frac{\partial^2\xi}{\partial r_0^2} + \left[\frac{2}{r_0} - \frac{2Gm(r_0)}{r_0}\frac{\rho_0}{p_0} + \frac{\Gamma+1}{\Gamma p_0}(j_0\times H_0)_{radial}\right]\frac{\partial\xi}{\partial r_0} + \left[\frac{4(1-\Gamma)}{\Gamma p_0}\frac{Gm(r_0)}{r_0^2}\rho_0 + \frac{1+2\Gamma}{\Gamma p_0}\frac{(j_0\times H_0)_{radial}}{r_0} + \frac{\sigma^2\rho_0}{\Gamma p_0}\right]\xi + \frac{[(\delta j\times H_0)+(j_0\times\delta H)]_{radial}}{\Gamma p_0 r_0} = 0 \quad \ldots (14)$$

where we have put

$$\frac{\delta r}{r} = \xi = \xi_0 e^{i\sigma t} \quad \ldots (15)$$

The change in the magnetic field, δH, following the notion is given, (in a medium of infinite electrical conductivity), by

$$\delta \boldsymbol{H} = \text{curl } (\delta \boldsymbol{r}\times \boldsymbol{H}) + (\delta\boldsymbol{r}\cdot\text{grad})\,\boldsymbol{H} \quad \ldots (16)$$

In any particular configuration where j_0 and H_0 are known, the change δj in current density, following the motion, can be evaluated by using equation (16) in conjunction with equation (3). Substituting the values of δH and δj thus obtained, in equation (14), we get the required equation for the radial pulsations of a cylindrical fluid, valid for all sorts of current system. We shall now deduce expressions for the frequency of pulsations in two special cases of current density.

Case (i) : Circular currents;

Let us consider that the circular currents are of the form

$$j = \left(0, \frac{-kr}{4\pi}, 0 \right) \qquad \ldots (17)$$

so that

$$H = \left\{ 0, 0, \frac{K}{2}(r^2 - R^2) \right\} \qquad \ldots (18)$$

and thus the only non-vanishing component for the change in the magnetic field following motion will be given by,

$$\delta H_z = -\left[2\xi + r_0 \frac{\partial \xi}{\partial r_0} \right] \left[\frac{K}{2}(r^2 - R^2) \right] \qquad \ldots (19)$$

here K is a constant and R is the radius of the cylinder. Further, if ξ is assumed to be constant in space then

$$\delta H = -2\xi H_0 \qquad \ldots (20)$$

and the change δj in current density (following motion) is given by

$$\delta j = -3\xi j_0 \qquad \ldots (21)$$

Thus the pulsation equation (14) yields

$$\sigma^2 r_0 = 4(\Gamma - 1) \, . \, G\pi\rho_0 r_0 + \frac{2(2-\Gamma)}{} \times (j_0 \times H_0)_{radial} \qquad \ldots (22)$$

Multiplying equation (22) by r_0 and integrating over the entire mass, we obtain (omitting the subscript zero)

$$\sigma^2 \, . \int^M r^2 dm = 2(\Gamma - 1) G M^2 + 2(2-\Gamma) \int r \, . \, (j \times H) d\tau \qquad \ldots (23)$$

where $d\tau$ is the volume element. Now

$$\int_V r.(j\times H)d\tau = \frac{k^2R^6}{48}$$

$$= 2\int_V \frac{H^2}{8\pi}\, d\tau = 2\mathfrak{M}\ (\text{say}) \qquad \dots \quad (24)$$

Thus we have

$$\sigma^2 \int^M r^2\, dm = 2(\Gamma-1)GM^2+4(2-\Gamma)\mathfrak{M} \qquad \dots \quad (25)$$

It can readily be shown for the equilibrium configuration

$$\int_V r.(j\times H)d\tau = GM^2-2(\Gamma-1)U \qquad \dots \quad (26)$$

where U is the internal energy of the system. Substituting equation (26) in equation (23) we obtain the expression for frequency of pulsation in terms of the internal energy, viz.,

$$\sigma^2 \int^M r^2dm = -4(2-\Gamma)(\Gamma-1)U+2GM^2 \qquad \dots \quad (27)$$

or

$$\sigma^2 \int^M r^2dm = 4(\Gamma-1)^2U+2\int_V \boldsymbol{r}.(\boldsymbol{j}\times\boldsymbol{H})d\tau \qquad \dots \quad (28)$$

An expression similar to that of equation (28) was obtained earlier by Chandrasekhar and Fermi (1953)..

Case (ii). Line currents.

Let us consider the case in which the current density j is constant in the cylinder and is of the form

$$\boldsymbol{j} = \left((0, 0, \frac{K}{2\pi} \right) \qquad \dots \quad (29)$$

where K is constant. The magnetic field will then be given by

$$\boldsymbol{H} = (0, Kr, 0) \qquad \dots \quad (30)$$

For such a configuration

$$\delta\boldsymbol{H} = -\xi\boldsymbol{H}$$

and

$$\delta \boldsymbol{j} = -2\xi \boldsymbol{j} \qquad \cdots \quad (31)$$

where ξ is a constant in space. Substituting these values in equation (14) we find

$$\sigma^2 \int^{M} r^2 dm = 2(\Gamma-1)GM^2 - 2(\Gamma-1)\int_{V} \boldsymbol{r}\,.\,(\boldsymbol{j}\times\boldsymbol{H})\,d\tau \qquad \cdots \quad (32)$$

Using equation (26), we get

$$\sigma^2 \int^{M} r^2 dm = 4(\Gamma-1)^2 U \qquad \cdots \quad (33)$$

as an expression for the frequency of pulsations in the presence of line currents in terms of internal energy, U. It can readily be shown that $\boldsymbol{r}.(\boldsymbol{j}\times\boldsymbol{H})$ is a negative quantity and hence the cylinder will be dynamically stable.

Equations (25) and (32) clearly show that the cylinder is stable for radial pulsations in the presence of circular as well as line currents. The lateral instability of the incompressible infinitely long cylindrical fluid mass has recently been discussed by Auluck and Kothari (1957). They have also shown that the magnetic field in general has a stabilizing effect.

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