ON THE AXIAL STABILITY OF A DEFORMED NUCLEUS I

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ABSTRACT. The energy eigen values of a single particle moving in an anisotropic harmonic oscillator potential have been calculated assuming that the equipotential surface coincides with the nuclear surface. It has been found that in the rare-earth region the shell model energy levels of a single particle depend sensitively on the deviation of the nuclear shape from cylindrical symmetry. Calculations have been done upto Ca^{40} core and it is found that the energetically stable form of the nucleus corresponds to either $\gamma = 0$ or $2\pi/3$. It is also found that the total energy of an oblate nucleus is about 2 Mev greater than that of a prolate one, which is consistent with the observed positive excess of quadrupole moment in the rare-earth region.

1. INTRODUCTION

In the region of the rare-earth nuclei (A — 150 to 190) very large positive quadrupole moments have been observed, which indicates that these nuclei are strongly deformed. Moreover, it has been found (A. Bohr, 1954) that the first excited level of the majority of these nuclei are of collective rotational origin. The fact that the energies of these levels are proportional to $I\ (I + 1)$ (where I is the spin of the level), indicates that these nuclei have cylindrical symmetry. From these two observations one can infer that the nucleon configuration in the rare-earth region prefers energitically a prolate deformation. The object of the present paper is to see whether this is so, if the nucleons are assumed to be moving in an average deformed potential.

From the self-consistency condition it is obvious that the strength of the nuclear potential is proportional to the particle density inside the nucleus. Hence one can conclude that at the nuclear surface the equipotential surface more or less coincides with the nuclear surface. Starting from this the energy eigen values of a single particle have been calculated by various workers, (Moszkowski, 1955 ; Nilsson, 1955 ; Gottfried, 1956) using rectangular well of infinite depth, harmonic oscillator potential and rectangular well of finite depth respectively. But they have tentatively assumed a cylindrical symmetry of the nucleus. Gottfried (1956), however, showed that for a rectangular well of finite depth the nucleus prefers a cylindrical symmetry. But in his calculations the positive and negative quadrupole moments are found to be equally probable.

We have taken here an anisotropic harmonic oscillator potential and applying the usual perturbation treatment have calculated the energy eigen values for different ellipsoidal shapes. This paper contains the results up to $N = 20$, while the exact machine calculation for higher levels are in progress and will be reported in due course.

2. MATHEMATICAL FORMULATION

Let the hamiltonian of a single particle of mass M, moving in an average field, $V(\mathbf{r})$ be

$$H = -\frac{\hbar^2}{2M}\Delta' + V(\mathbf{r}') + C\,\mathbf{l}.\mathbf{S} + D\mathbf{l}^2 \qquad \ldots (1)$$

where the last two terms are added to get the zero-order shell structure levels. (Nilsson 1955).

For anisotropic harmonic oscillator field one can write,

$$V(\mathbf{r}') = \tfrac{1}{2} M(\omega^2_{x'} x'^2 + \omega^2_{y'} y'^2 + \omega^2_{z'} z'^2). \qquad \ldots (2)$$

where $\mathbf{r}'$ gives the position of the particle with respect to a body-fixed system of axes (X', Y', Z') It is customary to choose (X', Y', Z') as the principal axes of the nucleus. (A. Bohr 1953) whose surface is

$$r_S = \lambda^{-1} R_0[1 + a_0 Y_{20}(\theta'\phi') + a_2\{Y_{22}(\theta'\phi') + Y_{2-2}(\theta'\phi')\}] \qquad \ldots (3)$$

The parameter λ in (3) preserves the nuclear volume. By straightforward integration one can show from (3) that

$$\lambda^3 = 1 + \frac{3}{4\pi}(a_0^2 + 2a_2^2) + \frac{16}{35}\left(\frac{5}{16\pi}\right)^{3/2}(a_0^4 - 6a_0a_2^2) \qquad \ldots (4)$$

(without any approximation)

Equating the length of the major axes from (2) and (3), and noting that the volume of the nucleus remains constant under deformation, it is easy to see that (2) can be written as

$$V(\mathbf{r}') = \tfrac{1}{2} MKR_0^{-2}\lambda^{-1}[\{1 + f_1(a_0a_2)\}r'^2 + f_2(a_0a_2)r'^2 Y_{20}(\theta'\phi') + f_3(a_0a_2)r'^2 \{Y_{22}(\theta'\phi') + Y_{2-2}(\phi'\theta')\}] \qquad \ldots (5)$$

where $(KR_0^{-2})^{\frac{1}{2}} = \omega_0^{\,o}$ is the zero-order harmonic oscillator frequency, and

$$f_1(a_0a_2) = -\frac{5}{4\pi}\sqrt{\frac{5}{16\pi}}\,a_0^3 + \frac{15}{4\pi}\sqrt{\frac{5}{4\pi}}\,a_0a_2^2 + \frac{75}{256\pi^2}a_0^4 + \frac{75}{64\pi^2}a_0^2\alpha_2^2 + \frac{75}{64\pi^2}\cdot a^4_2 \qquad \ldots (6a)$$

$$f_2(a_0a_2) = -2a_0 + 3\sqrt{\frac{5}{16\pi}}\,a^2{}_0 - \frac{3}{2}\sqrt{\frac{5}{\pi}}\,a^2{}_2 - \frac{25}{256\pi^2}\sqrt{\frac{16\pi}{5}}\,a_0{}^4$$

$$- \frac{75}{64\pi^2}\sqrt{\frac{16\pi}{5}}\,a^2{}_0\,a^2{}_2 + \frac{75}{64\pi^2}\cdot\sqrt{\frac{16\pi}{5}}\,a_2{}^4\,. \qquad \ldots \quad (6b)$$

$$f_3(a_0a_2) = -2a_2 - \frac{15}{8\pi}\sqrt{\frac{16\pi}{5}}\,a_0a_2 - \frac{25}{32\pi^2}\cdot\sqrt{\frac{16\pi}{5}}\,a^3{}_0a_2 \qquad \ldots \quad (6c)$$

neglecting higher order terms.

So (1) becomes

$$H = -\frac{\hbar^2}{2M}\Delta' + \tfrac{1}{2}M\omega_0{}^2(a_0a_2)r'^2[1 + p_2(a_0a_2)Y_{20}(\theta'\phi') + p_3(a_0a_2)$$
$$\{Y_{22}(\theta'\phi') + Y_{2-2}(\theta'\phi')\}C'\mathbf{l}\,.\,\mathbf{s} + D\mathbf{l}^2] \qquad \ldots \quad (7)$$

where
$$\omega_0(a_0a_2) = \omega_0{}^0\lambda^{-2}\{1 + f_1(a_0a_2)\}^{1/2} \qquad \ldots \quad (8a)$$

and
$$p_2(a_0a_2) = \frac{f_2(a_0a_2)}{1 + f_1(a_0a_2)} \qquad \ldots \quad (8b)$$

$$p_3(a_0a_2) = \frac{f_3(a_0a_2)}{1 + f_1(a_0a_2)} \qquad \ldots \quad (8c)$$

Introducing the usual dimensionless co-ordinate

$$x = \sqrt{\frac{M\omega_0(a_0a_2)}{\hbar}}\,x' \text{ etc and two new parameters}$$

$$\chi = -\tfrac{1}{2}\frac{C'}{\hbar\omega^0{}_0} \qquad \ldots \quad (9a)$$

$$\mu = \frac{2D}{C'} \qquad \ldots \quad (9b)$$

equation (7) becomes

$$H = -\frac{\hbar\omega_0(a_0a_2)}{2}(\nabla - r^2) + \hbar\chi\omega_0{}^0R \qquad \ldots \quad (10)$$

where

$$R = \tfrac{1}{2}\xi r^2\Big[Y_{20}(\theta\phi) + \frac{f_3(a_0a_2)}{f_2(a_0a_2)}\{Y_{22}(\theta\phi) + Y_{2-2}(\theta\phi)\}\Big] - 2\mathbf{l}\mathbf{s} - \mu\mathbf{l}^2. \qquad \ldots (11)$$

In (11) we have introduced a new parameter

$$\xi = \frac{p_2(a_0a_2)}{\chi} \cdot \frac{\omega_0(a_0a_2)}{\omega_0^0}$$

$$= \frac{f_2(a_0a_2)}{\cdots} \cdot \lambda^{-4}. \qquad \ldots (12)$$

3. DETERMINATION OF THE ENERGY-LEVELS

It can be shown that the r-dependent part of the hamiltonian H as given by equation (10) commutes with the operators l, l_z & S_z, where **l** is the orbital angular momentum, l_z its projection on z-axis and S the z-component of the spin **S** of the particle. Hence the base vector $|Nl\,l_zS_z>$ can be obtained from the zero-th order equation of motion

$$-\frac{\hbar\omega_0^0}{2}(\nabla - r^2) \mid Nl\,l_zS_z> = E_0 \mid N\,l\,l_zS_z> \qquad \ldots (13)$$

The radial part of the eigen ket appearing in (13) looks like

$$\mid Nl> \simeq \sqrt{\frac{2\Gamma(n+1)}{[\Gamma(n+l+3/2]^3}}\, r^l e^{-\frac{1}{2}r^2} L_n^{l+\frac{1}{2}}(r^2) \qquad \ldots (14)$$

while the orbital part is $Y_{ll_z}(\theta\phi)$, with usual notation.

In (14) $N=2n+l$ (where n can assume positive integral values, defines the radial quantum number.

Unfortunately none of the above operators commute with the total hamiltonian H as given in equation (10). But it can be readily shown that to a first approximation, j_z commutes with H, where j_z is the z-component of the total angular momentum **j** (= **l**+**s**) of the particle. Thus one can take the actual eigen function of a single particle in an ellipsoidal field as

$$\mid Nj_z> = \sum_l [A_{lj_z-\frac{1}{2}} \mid Nlj_z-\tfrac{1}{2}\,\tfrac{1}{2}> + A_{lj_z+\frac{1}{2}} \mid Nlj_z+\tfrac{1}{2},-\tfrac{1}{2}>] \qquad \ldots (15)$$

Using this wave function the energy eigen values can be easily calculated from equation (10) from a set of secular equations. The expression for total energy of a single particle moving in an ellipsoidal field is

$$E(Nj_za_0a_2) = (N+3/2)\hbar\omega_0(a_0a_2) + \hbar\chi\omega_0^0 < R > \qquad \ldots (16)$$

The evaluation of $< R >$ is shown in the appendix.

The parameters χ and ω_0^0 in (16) are so chosen that the energy levels correspond to the Klinkenberg's (1952) level scheme in the limit of zero deformation, ($a_0 = a_2 = 0$). For $A > 100$, $\hbar\omega_0^0 \sim 8.8$*Mev* and χ is 0.05. The parmeter μ

defined in equation (9b) is of importance only when $N > 2$. In general, its value is ~ 0.4.

In calculating the energy eigen values we have introduced two new deformation parameters β and γ defined by

$$a_0 = \beta \cos \gamma \qquad \ldots \quad (17a)$$

$$a_2 = \frac{1}{\sqrt{2}} \beta \sin \gamma \qquad \ldots \quad (17b)$$

A look at equation (3) will show that if $\gamma = 0$ the surface of the nucleus becomes a prolate spheroid about Z'-axis and if $\gamma = \pi$ it is an oblate spheroid. In between these two values of γ the surface is a general ellipsoid.

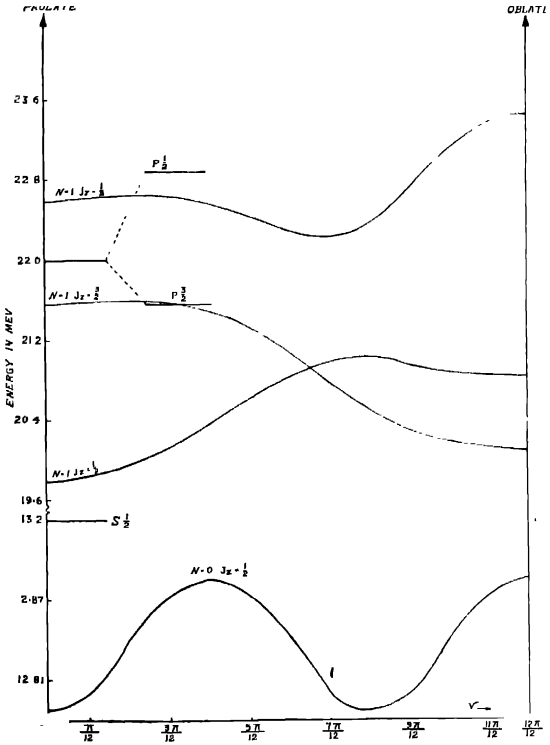

Fig. 1. The variation of the energy eigen values of a single particle moving in an anisotropic harmonic oscillator potential with γ for $N = 0$ and 1.

We have taken β as 0.3 which approximately corresponds to the deformation of Lu^{175} (as calculated from the intrinsic quadrupole moment). The variations of the energy eigen value of a single particle with γ are presented in figures 1 and

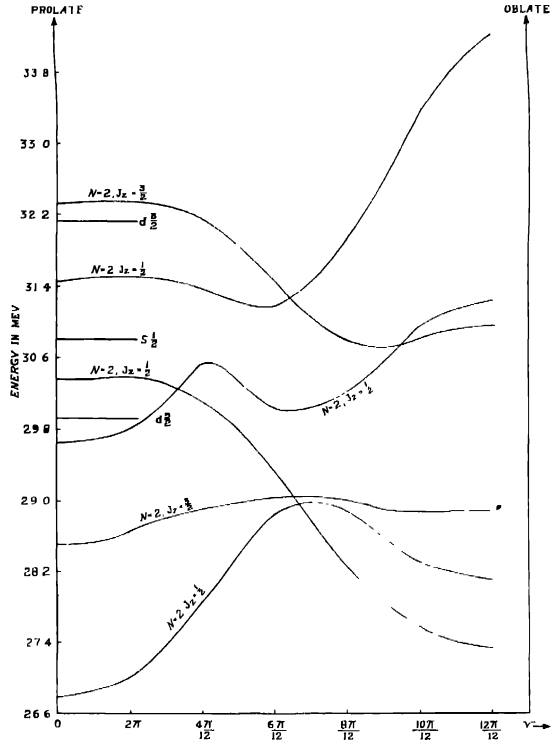

Fig. 2. The variation of the energy eigen values of a single particle with γ for $N = 2$.

2 for $N = 0$, 1 and 2 respectively. From these figures it is obvious that the positions of the energy levels depend sensitively on the choice of γ. The energy levels of a single particle in a central field approximation ($a_0 = a_2 = 0$) have also been shown in figures 1 and 2. Figure 3 illustrates the variation of the parameter ξ with γ.

Since the interactions between the nucleons are neglected in assuming an average field, the total energy of the nucleus is simply given by the sum of the energies of the individual particles. In this way total energy of the Ca^{40}-core part of Lu^{175} is plotted against γ in figure 4, from which one can see that there are

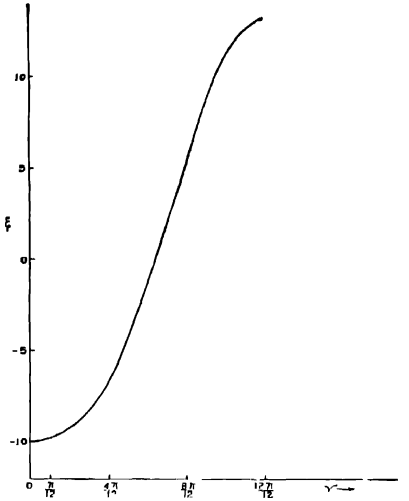

Fig. 3. The deformation parameter ξ against γ.

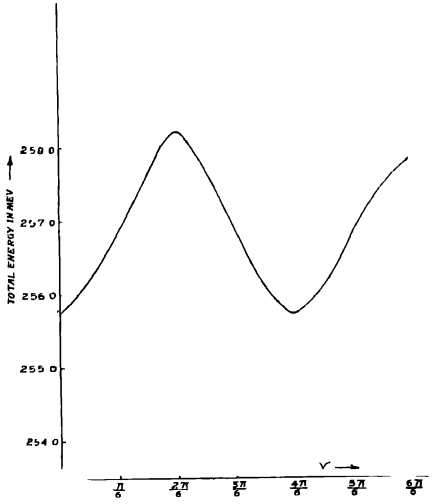

Fig. 4. The calculated total energy for the Ca^{40} core plotted against γ.

two minima, one at $\gamma=0$ and the other at $\gamma = 2\pi/3$ of which the first corresponds to a prolate shape. It is rather surprising that such a minimum should occur at $\gamma = 2\pi/3$, which does not preserve axial symmetry of the nucleus. Up till now we have not been able to explain this, but this may be smoothed out if one goes above the magic number 20. It is found that the difference in energy is approximately 2 Mev between a prolate and an oblate nucleus which is rather encouraging since the nuclei under our consideration prefer a prolate form.

Similar calculations for $N>2$ are in progress and will be reported in due course.

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APPENDIX

The evaluation of the matrix R

For $N \leqslant 2$, it is easy to see

$$< R > = < N'l'l'_z S'_z |\ \tfrac{1}{2}\xi r^2 Y_{20} - 2\vec{l}\,.\,\vec{S}\,|\,Nl\,l_z s_z > \qquad \ldots\ (1a)$$

$$= \tfrac{1}{2}\xi \Big[\ \frac{\Gamma(n+1)\Gamma(n'+1)}{\Gamma(n+t-\nu+1)\,\Gamma(n'+t-\nu'+1)}\ \Big]^{1/2} \nu\,!\,\nu'\,!\,.$$

$$\sum_{\sigma} \frac{\Gamma(t+\sigma+1)}{\sigma!(n-\sigma)!(n'-\sigma)!\,(\sigma+\nu-n)!(\sigma+\nu'-n)!}$$

$$\cdot\sqrt{\frac{5}{4\pi}}\sqrt{\frac{2l+1}{2l'+1}}\ <l2l_z 0\,|\,l2l'l'_z> .\ <l200\,|\,l2l'0>.$$

$$-2< l'l'_z S'_z |\,\vec{l}\,.\,\vec{S}\,|\,l\,l_z S_z> \qquad \ldots\ (1b)$$

where $\nu = \tfrac{1}{2}(l'-l+2)$

$\nu' = \tfrac{1}{2}(l-l'+2)$

and $t = \tfrac{1}{2}(l+l'+3)$

The condition on the summation variables σ is

$$\begin{matrix} n \\ n' \end{matrix} \geqslant \sigma \geqslant \begin{matrix} n-\nu \\ n'-\nu' \end{matrix}$$

The Clebsch-Gordon coefficients appearing in equation (1b) are those defined by Condon and Shortley (1936).

The non-vanishing elements of $< l' l_z' S_z' | \vec{l} \cdot \vec{S} | l\, l_z S_z >$

are
$$< l\, l_z \pm 1, \mp \tfrac{1}{2} | \vec{l} \cdot \vec{S} | l\, l_z, \pm \tfrac{1}{2} > = \tfrac{1}{2} \sqrt{(l \mp l_z)(l \pm l_z + 1)} \qquad \ldots \quad (2a)$$

and

$$< l\, l_z, \pm \tfrac{1}{2} | l \cdot \vec{S} | l\, l_z \pm \tfrac{1}{2} > = \pm \tfrac{1}{2} l_z \qquad (2b)$$

In (1a) the $Y_{22}(\theta\phi) + Y_{2-2}(\theta\phi)$ term has been neglected since in the first order approximation, its contribution is negligible. The non-diagonal terms ($N \neq N'$) in equation (1b) are also of second order importance.

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