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ON THE APPLICATION OF AN ELECTRONIC DIFFERENTIAL ANALYSER FOR FINDING THE ROOTS OF A POLYNOMIAL

B. R. NAG

INSTITUTE OF RADIO PHYSICS AND ELECTRONICS, CALCUTTA UNIVERSITY

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ABSTRACT. A method of finding the roots of a polynomial with real coefficients by an electronic differential analyser is described. The principle underlying the method is to set up in the analyser a system having a transfer function, of which the numerator is the polynomial whose roots are to be obtained. The denominator of the transfer function is a suitably chosen polynomial. Roots of the polynomial are then obtained by determining the frequencies at which the system gives zero output.

INTRODUCTION

Different methods of utilising a differential analyser for finding the roots of a polynomial have been investigated. Atkinsen has described a method of obtaining the real roots of a polynomial by electromechanical analysers. The same method can also be applied for obtaining complex roots by breaking the polynomial into real and imaginary parts. However, simpler methods using an electronic analyser have been described, which give both real and complex roots directly.

In the harmonic synthesis method the variable is represented as $re^{j\omega t} = r\cos\omega t + jr\sin\omega t$. The polynomial is constructed by generating the terms $r\cos\omega t$, $r\sin\omega t$, $r^2\cos 2\omega t$, $r^2\sin 2\omega t$......etc., multiplying them by the proper coefficients of the polynomial and adding them. The quantities $\cos\omega t$, $\sin\omega t$, $\cos 2\omega t$, $\sin 2\omega t$, ... etc. are first generated by setting oscillators on the analyser having the frequencies ω, 2ω, ... etc. Multiplication of these quantities by r, r^2 ... etc. is done by a set of ganged potentiometers. Evidently, a root of the polynomial can be obtained by determining the value of r and ωt which make both the real and the imaginary parts of the polynomial vanish simultaneously. The method is simple but has the disadvantage of requiring a large number of operational amplifiers. An n degree polynomial may require as many as $(5n+2)$ operational amplifiers.

In another method suggested by Cahn (1956) the roots of a polynomial are obtained by determining the frequencies of maintained oscillations of a system having a transfer function, the denominator polynomial of which is identical to the given polynomial. This method requires a smaller number of computing elements, an n degree polynomial requiring $(2n+1)$ operational amplifiers. This reduction in the number of operational amplifiers is a great advantage when the polynomial is of high degree. However, in this method, after obtaining one root, the polynomial has to be reduced in order to obtain the remaining roots. This reduction may impair the accuracy of the roots and also involves a new arrangement of the computer for finding each root Determination of the frequency of a maintined oscillation also requires a large time in slow computers

In this paper a method is suggested, which requires a small number of operational amplifiers and no reduction of the polynomial. The method is similar to that employed in potential analogues. The roots are obatined by determining the frequencies of zero output of a system having a transfer function the numerator of which is the given polynomial. The denominator polynomial is, however, chosen suitably. In contrast to the potential analogue method, the residues of the transfer function at the poles are not required to be calculated and the poles can be easily varied without altering the set up of the analyser This facilitates accurate determination of the frequencies of zero output of the system.

PRINCIPLE OF THE METHOD

Let the polynomial be given by

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$$

where $a_n, a_{n-1} \ldots a_0$ are all real.

The variable z may be represented by the complex frequency p Now, theoretically, a system having the transfer function $f(p)$ can be set on an analyser This system requires differentiators, which, when put in cascade, give at their outputs the voltages pE_{in}, $p^2E_{in}, \ldots$ etc when E_{in} is applied to the input of the cascaded system These when multiplied by the respective coefficients and added, gives $f(p)\ E_{in}$. However, in practice, it is difficult to set such a system. Since the system gain increases with frequency, it has a tendency to become unstable due to stray coupling between the output and input In addition, extraneous noise is very much amplified in this system.

On the other hand, a system having the transfer function $\frac{f(p)}{D(p)}$ can be easily constructed where $D(p)$ is a polynomial of the same order as $f(p)$ and has roots with real parts negative and greater than 1.

We first note that $\frac{f(p)}{D(p)}$ can be written as $\dfrac{1}{1+\dfrac{D_1(p)}{f(p)}}$, where $D_1(p) = D(p) - f(p)$. Thus if a system having the transfer function $\frac{D_1(p)}{f(p)}$ is available one can realise a system of transfer function $\frac{f(p)}{D(p)}$ in the manner indicated in figure 1

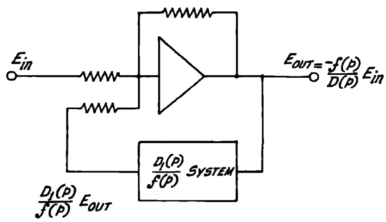

Fig. 1. Arrangement of the analyser for realising a system having the transfer function $\frac{f(p)}{D(p)}$ when a system having the transfer function $\frac{D_1(p)}{f(p)}$ is available.

For obtaining the system with the transfer function $\frac{D_1(p)}{f(p)}$, the analyser is arranged to solve the equation $f(p)E_{out} = E_{in1}$. Then from the outputs of the integrators are obtained the voltages $\frac{p^n}{f(p)} E_{in1}, \frac{p^{n-1}}{f(p)} E_{in1}$ . . etc., which, when multiplied by the coefficients of $D_1(p)$ and added, give $\frac{D_1(p)}{f(p)} E_{in1}$.

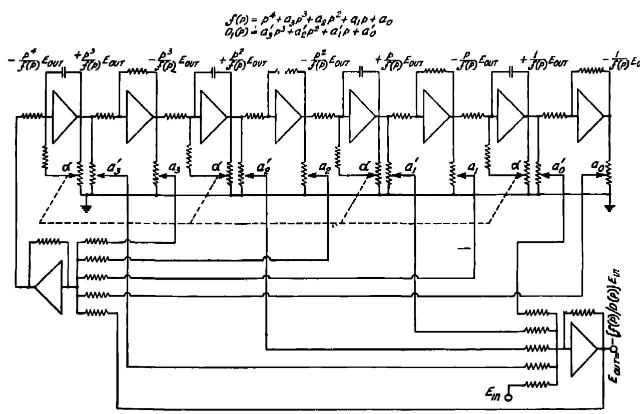

Fig. 2. Arrangement of the analyser for finding the roots of a fourth degree polynomial

The above principle is illustrated in figure 2 which shows the set up of the analyser for finding the roots of a polynomial of fourth degree. $a_3', a_2' \ldots a_0'$ are the coefficients of $D_1(p)$.

Evidently, the values of p for which this system gives zero output are the roots of the polynomial.

METHOD OF SEARCHING THE ROOTS

Roots of the polynomial may be real, imaginary or complex.

For searching real and complex roots, there should be provison for transforming the system with transfer function $\frac{f(p)}{D(p)}$ into one having the transfer function $\frac{f(p+\alpha)}{D(p+\alpha)}$, where α is a real quantity and can be varied between $+1$ to -1. This transformation is done by applying feedback from the output of the integrators to the inputs through potentiometers set to α. For making α negative, the feedback is applied from the output of the inverter following the integrator. On ganging all the α-potentiometers, as shown in figure 2, α can be easily varied between the requred limits

For searching real roots a d.c. voltage is applied to the input and the α-potentiometers adjusted for zero output. The setting of α which gives a zero E_{out} is a real root of the polynomial.

For searching complex roots a sinusoidal voltage is applied to the input. The frequency of this voltage and α are varied alternately till a zero output is obtained. It may be noted that on transforming the system $\frac{f(p)}{D(p)}$ to the system $\frac{f(p+\alpha)}{D(p+\alpha)}$, effectively the imaginary axis in the p plane is shifted to the right by α. Hence if α is so adjusted that the complex root lies on the shifted imaginary axis the output becomes zero when the input has a frequency equal to the imaginary part of the root. Thus the frequency of the input and the setting of α for zero output give respectively the imaginary and real part of the complex root. To make the searching of the complex root systematic, the input voltage may be supplied by a sweep frequency oscillator and then one has only to vary α to find the root. Once a root is approximately located it may, however, be located accurately later by close searching

For searching an imaginary root it is obvious that one needs only set α zero and find the frequency of the input voltage for zero output.

The method enables location of roots having real parts respectively within the range $+1$ to -1 and imaginary within the range $\omega_h\, CR$ to $\omega_l\, CR$, where ω_h and ω_l are the highest and lowest frequencies of the oscillator and CR is the time constant of the integrators. Roots lying outside these limits can be found by properly transforming the polynomial.

Choice of $D(p)$

The roots of $D(p)$ are the poles of the transfer function of the system and since, while searching the roots of the polynomial, α may be set to -1, the real part of the roots of $D(p)$ should be negative and greater than 1. The roots of $D(p)$ may be made real and placed far away from the region where the roots of $f(p)$ are searched. In that case the output of the system will vary exactly in accordance with the nature of the polynomial. However, placing of the roots of $D(p)$ far away results in reduction of the gain of the system on one hand, and on the other, increases the values of the coefficients of $D(p)$. These effects obviously necessitate the use of a high gain null detector and large gain operational amplifiers. To avoid these difficulties the roots of $D(p)$ are placed only slightly away to the left of -1, and to ensure that the d.c. loop gain is not inconveniently increased when α is set -1, the roots are chosen to be complex.

Once the roots of $f(p)$ are approximately known, $D(p)$ may be modified by altering the coefficients of $D_1(p)$ so that its roots lie very near to the left of the root of $f(p)$ being searched. This would facilitate location of the roots of $f(p)$, specially when they are close together

ACCURACY OF ROOT LOCATION

The straightforward method of determining the accuracy of the roots obtained by an experimental method is to check them against the actual roots of the polynomial. However, a good measure of the accuracy of a root is also provided by the value of the polynomial obtained by substituting this root in it. Obviously if this value is zero, it indicates that the root is cent per cent accurate. In polynomials, in which the value depends very critically on the values of the coefficients this method of estimating the accuracy may indicate large error. In such cases one may alternatively judge the accuracy by forming a polynomial with the roots obtained and checking how far the coefficients of the polynomial so found agree with those of the original polynomial. If the variation is found to lie within the tolerance limits expected from instrumental limitations, the roots are accepted

The limits of tolerance expected of the analyser as a whole may be estimated by ascertaining the errors introduced due to imperfections of the computing elements and to inaccuracies in the values of the computing networks. The amplifiers used in the adders and integrators have finite gain and band-width, hence they perform the operations required of them over a limited range of frequency with a certain prescribed accuracy. However, when the time scale of the computer is properly selected, the adders introduce errors which may be included in the values of the network elements. The integrators, however, always operate by $\frac{1}{p+\alpha_0}$ instead of by $\frac{1}{p}$, α_0 being determined by the d.c gain of the amplifier and loss resistance of the integrating capacitor. This results in an error in the value of α However, α_0 can be determined and corrected for in the value of the root obtained. The inaccuracy of the computing networks introduces inaccuracy in the coefficients of the polynomial and can be estimated on knowing the tolerance of the network elements for any particular polynomial.

The method discussed here has been applied to determine the roots of a few polynomials employing the electronic differential analyser, installed at the Institute of Radio Physics and Electronics, Calcutta The roots were found to be accurate up to the second decimal place.

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