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DETERMINATION OF THE BETA RAY ENERGY SPECTRUM FROM THE ABSORPTION CURVES OF BETA RAYS

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ABSTRACT. An attempt is made to calculate the energy-distribution of the β-ray spectrum of RaE starting from the observed absorption curve of these β-rays. The absorption law of homogeneous β-rays, required as the basis of this calculation, is obtained here by a rigorous curve fitting of the existing data. Further, Katz and Penfold's (K.P.) remarkable range-energy relation for β-rays is utilised, which seems to avoid most of the limitations of the earlier relations Somewhat improved results compared to previous workers' are obtained in the present work, but still some divergence with the results of magnetic spectrograph remains. Possible causes of the divergence are discussed.

1. INTRODUCTION

Side by side with the elaborate standard methods of β-ray spectrometry, (e.g., the methods of magnetic focussing, scintillation and proportional counters etc.), the method of measuring the total intensity of transmitted beta -rays through various thicknesses of aluminium absorbers (the absorption method) may be considered as a simpler alternative. This method has been widely used in early days for quick and fairly accurate determination of the end-points of β-ray spectra In spite of its inherent defects, the simplicity of the absorption method and the much weaker beta-ray source that it needs compared to the standard methods, make it worthwhile to investigate whether the absorption method can be usefully extended to the determination of the energy spectrum of the continuous beta-rays. The problem, as we shall see presently, resolves essentially to the analysis of the mechanism of absorption of homogeneous beta-rays in a material and the effect of simultaneous absorption of a large number of such homogeneous groups present in the whole spectrum. Attempts have been made by Dass Gupta and Chaudhury (1948), Evans (1950), and Schopper (1951) to solve this problem with varying degrees of success. A review of these works, however, makes it clear that the problem is not yet satisfactorily solved. The energy spectrum determined by this method does not agree well with the results of the magnetic spectrometer. The purpose of the present work is to make a fresh attempt in the line with a closer analysis of the absorption curves of the homogeneous beta-rays on which, theoretically, the success of the whole

calculation should rest. The results derived are applied to the actual calculation of the RaE-β-ray energy spectrum which is compared with the observed spectrum. Some improvement is obtained over the previous position. Results are discussed in detail in the light of the mechanism of the absorption of β-rays.

2. REVIEW OF EARLIER WORKS

Let $\phi(x,R)$ denote the function representing the absorption of the homogeneous group of β-rays of the extra-polated range $R(\text{gm/cm}^2)$ at an absorber thickness $x(\text{gm/cm}^2)$ and $N(R)$ the number of the homogeneous β-rays of this group. Then the intensity of β-rays transmitted through the absorber thickness x and having a range lying between R and $R+dR$ is given by

$$dI = N(R)\,\phi(x,R)\,dR \qquad \ldots \quad (1)$$

Consider now a certain composite β-ray energy spectrum whose upper limit of range is x_0 gm/cm², i.e., at which absorber thickness the intensity of the β-rays falls off to zero. The total intensity of the β-rays of all different ranges which will be transmitted through the absorber thickness x is then given by

$$I(x) = \int_{R=x}^{R=x_0} N(R)\,\phi(x,R)\,dR \qquad \ldots \quad (2)$$

Thus knowing the function $\phi(x,R)$ and the absorption curve for the β-ray spectrum giving $I(x)$ as a function of x, the original intensity $N(x)$ of the homogenous β-rays of all ranges present at a given absorber thickness x can be determined by suitable solution of (2). The range distribution $N(x)$ would easily give us the energy distribution $N(E)$, provided the range energy relation is known.

Rutherford and others (1930) made extensive studies of the absorption of homogenous β-rays in aluminium and expressed the absorption function ϕ by the empirical relation.

$$\phi(x/R) = 1 + a\left(\frac{x}{R}\right) + b\left(\frac{x}{R}\right)^2 + c\left(\frac{x}{R}\right)^3 + \ldots \qquad \ldots \quad (3)$$

the constants, a, b, c, etc., are completely arbitrary and are to be adjusted to fit in the experimental curve. A careful study of the absorption of monochromatic β-rays of different energies was afterwards done by Marshall and Ward (1937) for the two β-emitters RaE and UX. It becomes clear from the curves they obtain that near the full absorption thickness R_{max}, some amount of straggling effect takes place making the actual range rather uncertain. However, the extrapolated range measured agrees quite well with the results of previous works done by Schonland (1925), Varder (1915), Madgwick (1927), and Eddy (1929). Because of the certainty of the extrapolated range, it is generally to be preferred to the maximum range in all range considerations. In what follows, R stands

for the extrapolated range. For Ra E the various absorption curves for the constituent homogeneous groups of β-rays obtained by Marshall and Wards, when replotted as a function of x/R, fit well into a single common curve a, figure 1.
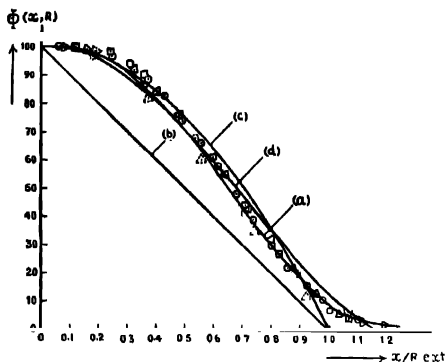

Fig. 1.—The transmission function ϕ (x/R) for beta-rays against x/R : (a) Replotted from Marshall and Wards' experimental absorption curves, (b) $\phi = 1-x/R$, (c) $\phi = 1-(x/R)^2$ and (d) $\phi = 1-1.25$ $(x/R)^2$ $+0.38$ $(x/R)^4$.

Dass Gupta and Choudhury (1948) were the first to attempt a calculation of the energy distribution of Ra E β-rays from the absorption data by using in the first approximation, a linear relation,

$$\phi(x/R) = 1 - \frac{x}{R} \qquad \ldots \quad (4)$$

Introducing this in eqn. (2) and solving under the limiting conditions $\phi = 1$ at $x = 0$ and $\phi = 0$ at $x = R$, one obtains the range distribution

$$N(x) = x\,I''(x) \qquad \ldots \quad (5)$$

where I'' denotes the second derivative of I with respect to the absorber thickness x.

A plot of the above linear relation for ϕ (curve b, Fig. 1) against x/R shows, however, a wide divergence from the experimental curve for $\phi(x/R)$ cited above (curve a, Fig. 1). The evaluation of the second derivative $I''(x)$ from the experimental curve for $I(x)$ also involves considerable uncertainty. The energy distribution of the β-rays calculated by these authors is, therefore, not expected to yield good results.

H. D. Evans (1949) suggested an improvement in the evaluation of the derivatives $I'(x)$ and $I''(x)$ by introducing a logarithmic function $L(x) = \log_e I(x)$, so that (5) takes the form

$$N(x) = xI(x)[L''(x)+\{L'(x)\}^2] \qquad \ldots \quad (6)$$

Since $L''(x)$, as found from the experimental data, is generally small (about 10% to 20%) compared to $\{L'(x)\}^2$, the main contribution in (6) comes from $L'(x)$ which can be determined much more accurately than $L''(x)$. Although the derivative of $I(x)$ was thus quite accurately evaluated by Evans, the energy distribution of β-rays derived by him did not show any substantial agreement with the β-ray spectrometric results. Only improvement found by Evans seems to be that the distribution curve showed a maximum which was not present in Dass Gupta's result.

Hughes *et al.* (1949) have obtained very similar results with ^{64}Cu, ^{187}Au, ^{185}W and radium E and compared the energy distributions derived from absorption curves with magnetic spectrograph observations. They found that the end point energy agreed in the two methods for all cases, but the maxima of the calculated curves occurred at about 100 kev higher energy than the observed values.

H. Schopper (1951) suggested a parabolic relation for $\phi(x/R)$ as

$$\phi(x/R) = 1 - \left(\frac{x}{R} \right)^2 \qquad \ldots \quad (7)$$

This relation (curve c, Fig. 1) agrees much better with the experimental curve a, as compared to the linear relation (4), except for the values of x/R approaching 1.

Using the relation (7) and solving the differential equation obtained, he arrived at the expression

$$2N(x) = xI''(x) - I'(x) \qquad \ldots \quad (8)$$

Also by putting $L(x) = \log_e I(x)$ according to Evans, the above equation (8) becomes

$$2N(x) = I(x)\{x[L'^2(x) + L''(x)] - L'(x)\} \qquad \ldots \quad (9)$$

In a previous work Schopper showed that the intensity $I(x)$ of the transmitted β-rays at x can be represented empirically by the relation

$$I(x) = \text{const } \{(x_0 - x)^2 + 0.22(x_0 - x)\}^{n/2} \qquad \ldots \quad (10)$$

where x_0 is the maximum range obtained from the absorption curve and n is an exponent obtainable from a plot of the observed values of $\log I(x)$ against $\log \{(x_0 - x)^2 + 0.22(x_0 - x)\}$. By introducing (10) in eqn. (9) one obtains

$$N(x). \text{ const} = n\{(x_0 - x)^2 + 0.22(x_0 - x)\}^{n/2 - 1}(x_0 + 0.11)$$
$$+ n(n-2)\{(x_0 - x)^2 + 0.22(x_0 - x)\}^{n/2 - 2}(x_0 - x + 0.11)^2 x \quad \ldots \quad (11)$$

He further points out that the equation (11) would not be valid in the low energy region, since the analytical expression (10) used here suppresses the finer details

of the absorption curve at small absorber thicknesses. In the low energy region, therefore, the relation (9) is directly utilised for the calculation.

3. COMPARISON WITH EXPERIMENT

The β-ray energy spectrum of RaE has been accurately measured by the magnetic spectrometer by several authors (Neary, 1940; Flammerrfeld, 1939). The most satisfactory determination is due to Neary, the results of which are shown by the curve a, figure 2. The relation (11) derived by Schopper has been

Fig. 2.—The energy distribution curve for Ra E β-rays : (a) The spectrometric measurements of Neary, (b) recalculated from H. Schopper's formula (8) and (9) using (12a), (c) calculated from relations (25) and (26).

used by him to calculate the energy distribution of β-rays from RaE, ^{32}P, UX$_1$ +UX$_2$ and ^{90}Sr+^{90}Y. For RaE, in the region of high energies, Flammersfeld's range-energy relation

$$E(x) = 1.92 \sqrt{(x_0-x)^2+0.22(x_0-x)} \qquad \ldots \quad (12)$$

for β-rays has been used to obtain the necessary $\dfrac{dx}{dE}$ values. It is, however, clear that the last relation cannot be true at low energies, since according to this, $\dfrac{dx}{dE}$ tends to zero at x approaching zero, whereas experiment shows that $N(E) = N(x)\dfrac{dx}{dE}$ has a finite value at very low energies. In the low energy region Schopper, therefore, uses the relation (9) directly together with the values of $\dfrac{dx}{dE}$ obtained by himself earlier* and calculates the energy distribution. The latter values of $\dfrac{dx}{dE}$ near about the maximum of the distribution curve appear

* We are indebted to Dr. H. Schopper for his very friendly private communication this ubject.

to be somewhat higher than those obtained from Flammersfeld's relation. The RaE β-ray-spectrum calculated by Schopper by the above procedure shows considerable agreement with the experimental curve of Neary. It may be pointed out, however, that the $\dfrac{dx}{dE}$ values used by Schopper cannot be taken as quite certain unless they are confirmed by an independent relation which is known to be strictly valid throughout the entire energy range of the β-rays. Such a consistent range-energy relation for β-rays has recently been given by Katz and Penfold (1952) for aluminium absorbers. This is

$$R = 412\ E^n\ \text{(mg/cm}^2), \quad \text{where } n = 1.265 - 0.0954 \log_e E \quad \dots \quad (12a)$$

and fits well with the most reliable published values of practical ranges of mono-energetic electrons and the maximum ranges of nuclear β-rays in energy region $0.01 \leqslant E \leqslant 2.5$ Mev. It would, therefore, be logical to recalculate the distribution curve by employing Schopper's formulae and using the dx/dE values derived from Katz and Penfold's relation (12a). The result of such a recalculation is shown in curve b of figure 2. Clearly enough, this curve does not show a good agreement with the experimental results (curve a) in the medium and the high energy regions. The apparent good agreement of Schopper's original results with experiments thus appears to be somewhat exaggerated.

In the present work we propose to review the relation representing the absorption of the homogeneous β-rays more critically with a view to achieving better agreement with Marshall and Ward's absorption curves than hither to obtained by previous authors. It appears that the addition of a fourth power term to the parabolic relation (7) gives some improvement in the fit. The plot of the relation

$$\phi(x/R) = 1 - 1.25(x/R)^2 + 0.38(x/R)^4 \qquad \dots \quad (13)$$

is shown in figure 1, curve d, where we also reproduce for comparison the curve c representing the parabolic relation, curve b representing the linear relation and the curve a the experimental results of Marshall and Ward. It is clearly seen that the curve d follows the experimental curve more closely than the others, although there is some departure in the high energy region. Taking equation (13) as the basic relation for $\phi(x/R)$, we carry out the calculations following the equation (2) and solve the differential equation resulting from that. The corresponding relation giving the energy distribution of β-rays is obtained by utilising the range energy relation of Katz and Penfold. Unfortunately a complete analytical solution cannot be achieved due to mathematical difficulties, but a method of successive approximations has been used as a first approach. The results obtained are shown in curve c, of figure 2 which suggest a slight improvement over previous results. It, however, appears that considerable divergence

from experimentally observed β-ray-spectrum still exists. The significance of this departure will be discussed later.

4. THE MODIFIED INTEGRAL DIFFERENTIAL EQUATION AND ITS SOLUTION

As already mentioned in section 3, the empirical relation (13) or

$$\phi(x/R) = 1 + a(x/R)^2 + b(x/R)^4 \qquad \ldots \text{ (13a)}$$

with $a = -1.25$ and $b = +0.38$ is used as the basis of our calculation.

Proceeding as in equation (2), we have

$$I(x) = \int_{x}^{R_0} \phi(x/R)N(R)dR \qquad \ldots \text{ (14)}$$

$$I'(x) = -(1+a+b)N(x) + \frac{1}{x}\int_{x}^{R_0} \left[\frac{2ax^2}{R^2} + \frac{4bx^4}{R^4} \right] N(R)dR \qquad \ldots \text{ (15)}$$

$$I''(x) = -(1+a+b)N'(x) - \frac{1}{x}[2a+4b]N(x)$$

$$+ \frac{1}{x^2}\int_{x}^{R_0} \left(2a\frac{x^2}{R^2} + 12b\frac{x^4}{R^4} \right)N(R)dR \qquad \ldots \text{ (16)}$$

If we write

$$f(x) = \int_{x}^{R_0} \left(\frac{x}{R} \right)^2 N(R)dR$$

$$\psi(x) = \int_{x}^{R_0} \left(\frac{x}{R} \right)^4 N(R)dR \qquad \qquad \ldots \text{ (i)}$$

$$K(x) = a f(x) + b\psi(x)$$

we obtain

$$I(x) = \int_{x}^{R_0} N(R)dR + a f(x) + b\psi(x) \qquad \ldots \text{ (17)}$$

$$I'(x) = -(1+a+b)N(x) + \frac{2a}{x}f(x) + \frac{4b}{x}\psi(x) \qquad \ldots \text{ (18)}$$

$$I''(x) = -(1+a+b)N'(x) - \frac{1}{x}(2a+4b)N(x)$$

$$+ \frac{2a}{x^2}f(x) + \frac{12b}{x^2}\psi(x) \qquad \dots \quad (19)$$

Also from the last equation of (i) we get

$$\psi(x) = \frac{K(x) - a f(x)}{b} \qquad \dots \quad (20)$$

Substituting this in (18), we obtain

$$f(x) = -\left[I'(x) + (1+a+b)N(x) - 4\frac{K(x)}{x}\right]\frac{x}{2a} \qquad \dots \quad (21)$$

Substituting (21) in (20), results

$$\psi(x) = -\frac{K(x)}{b} + \frac{x}{2b}\left[I'(x)+(1+a+b)N(x)\right] \qquad \dots \quad (22)$$

Also from (19) using (21) and (22) follows :

$$I''(x) = -(1+a+b)N'(x) + \frac{1}{x}(5+3a+b)N(x) - \frac{8K(x)}{x^2} + \frac{5}{x}I'(x)$$

or $\qquad I''(x) - \frac{5}{x}I'(x) + \frac{8K(x)}{x^2} = -(1+a+b)N'(x) + \frac{1}{x}(5+3a+b)N(x)$

Substituting for $K(x)$ from (17),

$$I''(x) - \frac{5}{x}I'(x) + \frac{8}{x^2}I(x) - \frac{8}{x^2}\int_x^{R_o} N(R)dR$$

$$= -(1+a+b)N'(x) + \frac{1}{x}(5+3a+b)N(x)$$

$$= AN'(x) + \frac{B}{x}N(x) \qquad \dots \quad (23)$$

where $A = -(1+a+b)$ and $B = (5+3a+b)$.

If we put $a = -1$ and $b = 0$, equation (23) should reduce to Schopper's equation. In this case equation (23) will become as

$$I''(x) - \frac{5}{x}I'(x) + \frac{8}{x^2}I(x) - \frac{8}{x^2}\int_-^{R_o} N(R)dR = \frac{2}{x}N(x) \qquad \dots \quad (23a)$$

and equation (14) will give,

$$I(x) = \int\limits^{R_o} N(R)dR - \int\limits^{R_o} (x/R)^2 N(R)dR,$$

so that

$$\int\limits_x^{R_o} N(R)dR = I(x) + \int\limits_x^{R_o} \left(\frac{x}{R} \right)^2 N(R)dR \qquad (23b)$$

Substituting this in equation (23a), we obtain

$$I''(x) - \frac{5}{x} \; I'(x) - \frac{8}{x^2} \int\limits_x^{R_o} \left(\frac{x}{R} \right)^2 N(R)dR = \frac{2}{x} \; N(x) \qquad \left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} \quad \cdots \;(24)$$

Also from equation (15) we have in this particular case,

$$I'(x) = -\frac{2}{x} \int\limits_x^{R_o} \left(\frac{x}{R} \right)^2 N(R)dR \qquad \cdots \;(24a)$$

Substituting this value in (24) we get,

$$2N(x) = xI''(x) - I'(x) \;\;(Eq.8)$$

which is exactly the same solution (8) obtained by Schopper, here appearing as a particular case of the solution (23).

Now our object will be to solve the equation (23). As is evident from the form of the equation, it will not be possible to get a simple solution in a very rigorous manner. Our attempt here will be to assume the solution (8) to be true to a first approximation and then by substituting in (23) to get a solution to the next higher approximation.

Thus we use for I(x) the expression from (23b) and for I'(x) from (24a) and rewrite the general relation (23).

Equation (23) then becomes

$$xI''(x) - I'(x) = Ax \; N'(x) + B \; N(x)$$

Again using (8),

$$2N'(x) = I''(x) + xI'''(x) - I''(x) = xI'''(x)$$

Substituting this above we get

$$xI''(x) - I'(x) = A\frac{x^2}{2} I'''(x) + B N(x)$$

or
$$BN(x) = -A\frac{x^2}{2} I'''(x) + xI''(x) - I'(x)$$

But $A = -(1+a+b) = -0.13$, $B = (5+3a+b) = 1.63$, because $a = -1.25$ and $b = 0.38$. Thus, we get finally

$$1.63N(x) = 0.065x^2I'''(x) + xI''(x) - I'(x) \qquad \dots \quad (25)$$

and putting $L = \log_e I(x)$,

$$1.63N(x) = I(x)[0.065x^2\{L'^3 + 3L'L'' + L'''\}$$
$$+ x\{L'^2 + L''\} - L'] \qquad \dots \quad (26)$$

Evidently, this final solution (25) differs from Schopper's solution by an extra term $0.065x^2I'''(x)$.

For calculating the energy distribution curve we have used for the medium and high energy regions relation (25), obtaining, I', I'' and I''' from Schopper's annalytical equation (10). As pointed out by Schopper, at low energies the finer variations of the absorption curve are suppressed if relation (10) is used. Therefore, for the low energy region, we have used equation (26) in which L', L'' and L''' are obtained from the plot of experimental values of L versus x given by Evans. Finally for obtaining $N(E)$ as a function of E, we have made use of Katz and Penfold's range energy relation, (12a). The results obtained are shown by the plot c in figure 2.

5. DISCUSSION

The energy spectrum of β-rays calculated according to the above procedure is shown in figure 2, curve c. The spectrometric measurements due to Neary are plotted in curve a, and the energy distribution recalculated from Schopper's formula by using Katz and Penfold's range-energy relation is also shown by the curve b for comparison. The five experimental points at $E \leqslant 0.05$ Mev energy are due to Pniewski and Danysz (1953). As already emphasized in section 3, the recalculated curve b shows considerable divergence from the experimental distribution, particularly at the high energy side beyond the maximum. In fact the recalculated maximum comes out to be very flat (inherent defect of

the procedure) and falls near about the position of the experimental maximum. The curve *c* obtained from our calculation shows a good agreement with the experimental curve towards the high energy region of the distribution curve, but the agreement in the intermediate region shows only a slight improvement. It is unlikely that the mathematical approximation involved in our calculation or the inaccuracies of the graphical estimates of L', L'' and L''' can account for the observed disagreement. It also seems clear that the introduction of still higher order terms in the expression (13) for $\phi(x/R)$ will not improve matters, since the fourth power term, added by us already, makes the agreement of $\phi(x/R)$ with its experimental values quite good. The reason for the disagreement seems to be more deep seated. Probably the analytical expression (10) for $I(x)$ in terms of the absorber thickness needs modification. It would be possible to modify it if the detailed mechanism of absorption of β-rays in absorber thickness could be exactly formulated theoretically. This possibility is not yet in sight. The other alternative is to look for uncertainty, if any, in the spectrometric determination of the β-ray energy distribution curve for RaE in the region of intermediate energies. The general accuracy of Neary's data is certainly better than 2%. It may, however, be pointed out that the results calculated from a set of theoretical formulae, as done by us, ultimately depend on the accuracy of the absorption measurements of the continuous β-rays. This accuracy is probably not very high (variable between 6.3% and 1% in Evan's measurements). For a strict comparison the conditions for the spectrometric and the absorption measurements should be identical. Probably not enough attention has yet been paid to this point.

When the main problem of theoretically calculating a single continuous β-ray energy distribution curve from the absorption data has been satisfactorily solved, one may probably attempt a possible extension of the theory in other directions. For example, one would like to ask : is it possible to apply the theory to the cases, where more than one continuous β-spectra with different end points are superimposed on each other, or when line spectra are superposed on a continuous β-spectrum ? Beta emitters of these complicated types are, of course, quite numerous. But it is too early to consider these questions now before the simplest problem outlined above has been completely solved.

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