Sensitivity Bond Graphs

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Abstract

A sensitivity bond graph, of the same structure as the system bond graph, is shown to provide a simple and effective method of generating sensitivity functions of use in optimisation. The approach is illustrated in the context of partially-known system parameter and state estimation.

Key words: Bond graphs; sensitivity; optimisation; parameter estimation; partially-knowm systems

1 Introduction

Optimisation [1] has always been a fundamental technique in control and parameter estimation of dynamic systems. Such methods rely on a *model* of the corresponding physical system. In some application areas (for example process control) physical, or first principles, models are hard to come by and so empirical models tend to be used. However in other application areas, in particular mechatronics, physical models are more readily available. There are a number of techniques for building control-relevant models but the bond graph [2,3,4,5] approach is well established and has been suggested as a basis for control design [6,7,8].

However, optimisation is eased by having a *sensitivity* model in addition to the model itself [1]. This paper develops the idea of *sensitivity bond graphs* [9,10] to provide a basis for model-based optimisation for the identification of systems for which a bond graph model is available. *Sensitivity* theory of dynamic systems and its application is well established and summarised in the textbooks [11,12]. There are many applications of sensitivity methods to systems and control problems including: system optimisation [9], controller tuning [13,14,15] and parameter estimation [16]. The particular class of dynamic systems described by electrical networks has its own techniques [17] based on the *adjoint circuit* approach.

A basic theoretical result of sensitivity theory [11,12] is that, given the time-invariant ordinary differential equation with state $X \in \Re^n$, initial state X_0, input u and constant parameter vector $\theta \in \Re^{n_\theta}$ of the form:

$$\begin{cases} \dot{x} &= f(x, u, \theta) \\ y &= g(x, \theta) \\ x(0) &= x_0 \end{cases} \tag{1}$$

then the state sensitivity functions $x^j = \frac{\partial^j x}{\partial \theta_j}$ and the output sensitivity functions $y^j_i = \frac{\partial^j y}{\partial \theta_j}$ are the solutions of the n_θ *linear time-varying* ordinary differential equations:

$$\begin{cases} \dot{x^j} &= A(x, u)x^j + U(x, u) \\ y^j &= C(x, u)x^j \\ x^j(0) &= x^j_0 \end{cases} \tag{2}$$

Except in special cases [18], it is necessary to use n_θ systems of the form of Equation 2, together with original system of Equation 1, to compute the n_θ sensitivity functions.

In contrast to standard sensitivity theory [11,12], which operates at the system ordinary differential equation level, this paper (in common with [9,10]) devel-

2

ops a component-based approach which, given the sensitivity models of each component, builds the system sensitivity ordinary differential equation from that of the system components. *Linearisation* theory is a special case of sensitivity theory. In the context of linearisation, Karnopp [19] has stated: "Rather than treating linearized systems as abstract sets of equations, we here look for structural analogies between non-linear components relating total system variables and linearized models relating incremental system variables"; with "linearized" replaced by "sensitivity" this statement summarises the aim of this paper. In particular, we follow Karnopp [19] in using a bond graph [2,3,4,5] approach to sensitivity theory.

Cabanellas, Felez and Vera [9] have established a bond graph interpretation of sensitivity and applied it to system optimisation. This approach has the advantage of having a clear *physical* interpretation and focuses on the class of parameters associated with the constitutive relationships of bond graph components. The same ideas have also been presented (apparently independently) by Roe and Thoma [10].

Cabanellas, Felez and Vera [9] also demonstrate the advantages of algebraic generation of sensitivity functions as opposed to numerical approximations. Indeed, symbolic methods for nonlinear systems modelling, analysis and optimisation are currently strong research areas [20] driven by the ready availability of symbolic computational tools. In particular, the bond graph approach [2,5] has been used to generate models applicable to control design [8].

In this paper the work of Cabanellas, Felez and Vera [9] is extended. In particular, it is shown that by making use of vector bonds and vector junctions, the sensitivity bond graph is structurally identical to the system bond graph. This is useful because the bond graph of the sensitivity system can be derived directly (and hence automatically) from the system bond graph. This sensitivity bond graph can be used not only as a way to derive the system sensitivity equations, but also as a means of directly analysing the system sensitivity properties on the bond graph itself.

To illustrate the application of the sensitivity bond graph theory of this paper to a concrete optimisation example, the identification of partially-known systems is considered. Many engineering systems of interest to the control engineer are *partially known* in the sense that the system structure, together with some system parameters are known, but some system parameters are unknown. This gives rise to a problem of *parameter estimation* when values for the unknown parameters are to be determined from experimental data comprising measurements of system inputs and outputs. There is a considerable literature in the area including [21,22,23,24,25]. Although in special cases such identification may be *linear*-in-the parameters [21] or *polynomial*-in-the parameters [24,25] in general the problem is *nonlinear*-in-the parameters and

gives rise to a *nonlinear least-squares* problem [1]. This means that, in general, the resultant optimisation problem is not quadratic or polynomial, and may even be non-convex; such nonlinear least-squares optimisation problems are conveniently solved using Gauss-Newton or Levenberg-Marquardt algorithms [1]. In such cases, the optimisation task is eased by knowing a symbolic expression for (rather than deducing numerically) the derivative of the error function with respect to the unknown system parameters. It therefore follows that partially known system identification is one application of optimisation using sensitivity bond graphs. This is used as an example application in this paper.

The outline of the paper follows. Section 2 discusses the sensitivity of systems described by bond graphs and shows that a new type of bond graph, the sensitivity bond graph, represents the system sensitivity whilst having the same structure and causality as the system bond graph. Section 3 reviews the Gauss-Newton optimisation technique [1] and shows how the sensitivity bond graph can be used to generate the corresponding gradient information. Section 4 illustrates the approach via combined state and parameter estimation of a nonlinear system. Section 5 makes some concluding remarks.

2 Sensitivity Bond Graphs

A bond graph component is associated with a *constitutive relationship* (or CR) which relates the time-varying *signals* within the component and the time-invariant *parameters* associated with the system within which the component lies. Thus the CR for the ith component (associated with n_{v_i} signals and n_θ parameters) of a system can be written as:

$$\Phi_i(v_i(t), \theta) = 0 \tag{3}$$

where $v_i \in \Re^{n_{v_i}}$ contains the *component* signals and $\theta \in \Re^{n_\theta}$ contains the *system* parameters.

In the special case that the CR is linear; it can be written as:

$$A_i(\theta)^T v_i(t) = 0 \tag{4}$$

where the time-invariant vector $A_i(\theta) \in \Re^{n_{v_i}}$.

In this paper, the *sensitivity* $^j v_i \in \Re^{n_{v_i}}$ of the signal vector v_i with respect to the jth component θ_j of the parameter vector θ is of interest. In particular, $^j v_i$ is defined as:

$$^j v_i = \frac{\partial v_i}{\partial \theta_j} \tag{5}$$

4

It follows from Equation 3 that $\frac{d\Phi_i(v_i(t),\theta)}{d\theta} = 0$ and so the *sensitivity* CR
${}^j\phi_i({}^jv_i(t), v_i(t), \theta)$ becomes:

$$
\begin{aligned}
{}^j\phi_i({}^jv_i(t), v_i(t), \theta) &= \frac{\partial\Phi_i(v_i(t),\theta)}{\partial v_i}^T {}^jv_i(t) \\
&+ \frac{\partial\Phi_i(v_i(t),\theta)}{\partial\theta_j} = 0
\end{aligned}
$$

Equation 6 is the sensitivity CR with respect to the jth parameter. It has the following important properties:

(1) The first term of the right-hand side of Equation 6 represents the *linearised* (about $v_i(t)$) CR relating the sensitivity functions ${}^jv_i(t)$. In other words, it is a *linear* CR modulated by the variables $v_i(t)$ associated with the system itself. It can be written as the summation:

$$
\frac{\partial\Phi_i(v_i(t),\theta)}{\partial v_i}^T {}^jv_i(t) = \sum_{k=1}^{n_{v_i}} \frac{\partial\Phi_i(v_i(t),\theta)}{\partial v_{ik}}^T {}^jv_{ik}(t) \tag{6}
$$

where k indexes the n_{v_i} elements of v_i and ${}^jv_i$.

(2) The second term of the right-hand side of Equation 6 represents an additional input to the sensitivity CR dependent on the variables $v_i(t)$ associated with the system itself.

(3) The sensitivity CR of Equation 6 is *local* to the the component in the sense that the only variables appearing in Equation 6 are $v_i(t)$ and ${}^jv_i(t)$.

(4) The jth sensitivity CR of Equation 6 does *not* depend on ${}^lv(t)$ for $l \neq j$.

(5) If the ith CR $\Phi_i(v_i(t), \theta)$ does not depend on θ_j then the second term of Equation 6 is zero and there is no *explicit* coupling between the actual and sensitivity systems (though there will be *implicitly* if $\Phi_i(v_i(t), \theta)$ is nonlinear.

(6) If $\Phi_i(v_i(t), \theta)$ is *linear* in v_i (Equation 4, then Equation 6 becomes:

$$
A_i^T(\theta) {}^jv_i(t) + \frac{\partial A_i^T(\theta)}{\partial\theta_j} v_i(t) = 0 \tag{7}
$$

(7) If the conditions of both items 5 and 6 hold; then the CRs of the actual and sensitivity components are identical and uncoupled.

For these reasons, it is possible to encapsulate two CRs: the system CR of Equation 3 and the jth sensitivity CR of Equation 6 within in a single component containing $2n_{v_i}$ variables: those contained in v_i and ${}^jv_i$. If the original component had N ports, the new sensitivity component (s-component) therefore has $2N$ ports. More conveniently, each port on the original component is replaced by a sensitivity port (or s-port) which carries not only the effort/flow pair e and f but also the corresponding sensitivity pair ${}^je$ and ${}^jf$. Such ports

may be considered to be connected by a *sensitivity bond* (s-bond) which encapsulates the energy bond carrying e and f with the pseudo-bond carrying ^je and ^jf.

The encapsulated CR (Equations 3 and 6) refer to only the jth of n_θ parameters. To obtain the bond graph for all n_θ parameters there are two possible approaches:

(1) Derive the n_θ CRs of Equation 6 and encapsulate n_θ CRs together with the component CR in each component and extend each port and s-bond to correspond to $n_\theta + 1$ bonds.

(2) Define the scalar pseudo effort and flow variables $^\lambda e$ and $^\lambda f$ as

$$
\begin{aligned}
^\lambda e &= \sum_{j=1}^{n_\theta} \Lambda_j \, ^je \\
^\lambda f &= \sum_{j=1}^{n_\theta} \Lambda_j \, ^jf
\end{aligned}
\tag{8}
$$

where $\Lambda \in \Re^{n_{v_i}}$ is a vector of *weighting coefficients*. In this case, define:

$$
\phi_i(\,^\lambda v_i(t), v_i(t), \theta) = \sum_{j=1}^{n_\theta} \lambda_j^j \phi_i(\,^jv_i(t), v_i(t), \theta)
\tag{9}
$$

where λ_j is the jth component of Λ.

The latter approach has the advantage of giving a simpler bond graph where all s-bonds encapsulate two bonds: the energy bond carrying e and f with the pseudo-bond carrying $^\lambda e$ and $^\lambda f$. It is parameterised by Λ and any particular sensitivity function can be extracted by setting $\Lambda = \Lambda_j$, the unit vector with the jth element unity and the rest zero. This is the approach used in the remainder of this paper.

Having created the sensitivity bond graph using this approach, the resultant dynamic system will n_y outputs corresponding to the underlying bond graph together with a further n_y outputs corresponding to the sensitivity outputs denoted by $^\lambda y \in \Re^{n_y}$.

The output sensitivity function y_θ can be constructed by solving the ordinary differential equation corresponding to the sensitivity system n_θ times with $\Lambda = \Lambda_j$ for $0 \le j \le n_\theta$.

2.1 Some sensitivity components

It follows from the discussion of Section 2 that any Bond Graph component has an s-component equivalent. This section provides some examples, including those which will be used in the sequel.

2.1.1 A linear one-port $\mathbf{R}$ component

The standard linear bond graph $\mathbf{R}$ component has a single port with effort e_i and flow f_i covariables related by the CR

$$\Phi_i(v_i(t), \theta) = A_i(\theta)^T v_i(t) = e - rf = 0 \tag{10}$$

That is, in the notation of Equation 4

$$A_i(\theta) = \begin{pmatrix} 1 \\ -r \end{pmatrix} ; \; v_i = \begin{pmatrix} e_i \\ f_i \end{pmatrix} \tag{11}$$

$\frac{\partial A_i^T(\theta)}{\partial r} = \begin{pmatrix} 0 & -1 \end{pmatrix}^T$ and so the sensitivity CR of Equation 7 can be written as:

$$\begin{cases} \frac{\partial e_i}{\partial r} - r\frac{\partial f_i}{\partial r} - f_i = 0 & \text{if } \theta_j \equiv r \\ \frac{\partial e_i}{\partial r} - r\frac{\partial f_i}{\partial r} = 0 & \text{if } \theta_j \not\equiv r \end{cases} \tag{12}$$

or alternatively:

$$^\lambda e_i - r \; ^\lambda f_i - \lambda_j f_i = 0 \tag{13}$$

The corresponding sensitivity component is a linear two port $\mathbf{R}$ with CR given by Equations 10 and 13. The upper part of Figure 1 shows this two port $\mathbf{R}$; the lower part of Figure 1 shows the encapsulated version.

2.1.2 A nonlinear one-port $\mathbf{R}$ component

Consider a nonlinear $\mathbf{R}$ component with a single port with effort e and flow f covariables related by the CR

$$\Phi_i(v_i(t), \theta) = e_i - \beta f_i^\alpha = 0 \tag{14}$$

This corresponds to a typical hydraulic resistance with unidirectional flow $(f_i > 0)$.

Using v_i as defined in 11, it follows that

$$\frac{\partial \Phi_i(v_i(t), \theta)}{\partial v_i} = \begin{pmatrix} 1 \\ -\beta \alpha f_i^{\alpha-1} \end{pmatrix} \tag{15}$$

Defining

$$\theta = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{16}$$

It follows that

$$\frac{\partial \Phi_i(v_i(t), \theta)}{\partial \theta} = \begin{pmatrix} -\beta \ln f \\ -1 \end{pmatrix} f_i^{\alpha} \tag{17}$$

Hence

$$\phi_i(\,^{\lambda}v_i(t), v_i(t), \theta) = \,^{\lambda}e_i - \alpha f_i^{\alpha-1}\,^{\lambda}f_i - \lambda_1 \beta f_i^{\alpha} \ln f_i - \lambda_2 f_i^{\alpha} = 0 \tag{18}$$

The CR represented by Equations 14 and 18 is convenient for computing e_i and $\,^{\lambda}e_i$ in terms of f_i and $\,^{\lambda}f_i$, but only has an implicit solution for f_i and $\,^{\lambda}f_i$ in terms of e_i and $\,^{\lambda}e_i$. However an explicit expression for f_i and $\,^{\lambda}f_i$ may be obtained by rewriting Equation 14 to give the alternative CR ($\Phi_i^a(v_i(t), \theta)$)

$$\Phi_i^a(v_i(t), \theta) = \left(\frac{e_i}{\beta}\right)^{\frac{1}{\alpha}} - f_i = 0 \tag{19}$$

Equation 18 may then be replaced by the alternative sensitivity CR ($\phi_i^a(\,^{\lambda}v_i(t), v_i(t), \theta)$)

$$\begin{aligned}
\phi_i^a(\,^{\lambda}v_i(t), v_i(t), \theta) = \,&^{\lambda}f_i \\
&- \left(\frac{e_i}{\beta}\right)^{\frac{1}{\alpha}} \left[+\frac{1}{\alpha e_i}\,^{\lambda}e_i - \lambda_1 \frac{1}{\alpha^2} \ln\left(\frac{e_i}{\beta}\right) - \lambda_2 \frac{1}{\alpha\beta} \right] \\
&= 0
\end{aligned}$$

The corresponding sensitivity component is a nonlinear two port **R** with CR given by Equations 14 and 18 or by Equations 19 and 20. The upper part of Figure 1 shows this two port **R**; the lower part of Figure 1 shows the encapsulated version.

2.1.3 A linear one-port **C** component

The standard linear bond graph **R** component has a single port with effort e_i and integrated flow q_i covariables related by the CR

$$\Phi_i(v_i(t), \theta) = A_i(\theta)^T v_i(t) = e_i - a(q_i + q_{i0}) = 0 \tag{20}$$

where a is a constant parameter and q_{i0} is the initial state.

If the parameter vector θ is taken as

$$\theta = \begin{pmatrix} a \\ q_{i0} \end{pmatrix} \qquad (21)$$

then the sensitivity CR of Equation 7 can be written as:

$$\phi_i(\,^\lambda v_i(t), v_i(t), \theta) = \,^\lambda e_i - a\,^\lambda q_i - \lambda_1(q_i + q_{i0}) - \lambda_2 a \qquad (22)$$

The corresponding sensitivity component is a linear two port $\mathbf{C}$ with CR given by Equations 20 and 22.

2.1.4 A nonlinear one-port $\mathbf{C}$ component

An open right-conical tank containing of incompressible liquid can be regarded as a nonlinear $\mathbf{C}$ component with CR:

$$\Phi_i(v_i(t), \theta) = e_i - \rho g \left(\frac{12}{\pi}(q_i + q_{i0}) \right)^{\frac{1}{3}} = 0 \qquad (23)$$

where q is the integrated flow variable, ρ is the fluid density, g the gravitational constant and q_{i0} the initial volume of liquid. If the parameter vector θ is taken as

$$\theta = \begin{pmatrix} \rho \\ q_{i0} \end{pmatrix} \qquad (24)$$

then the sensitivity CR of Equation 7 can be written as:

$$\phi_i(\,^\lambda v_i(t), v_i(t), \theta) = \,^\lambda e_i$$
$$- \left(\frac{12}{\pi}(q_i + q_{i0}) \right)^{\frac{1}{3}} \left[(\,^\lambda q_i + \lambda_2)\rho g \frac{1}{3(q_i + q_{i0})} + \lambda_1 g \right]$$

The corresponding sensitivity component is a nonlinear two port $\mathbf{C}$ with CR given by Equations 23 and 25. The upper part of Figure 1 (with $\mathbf{R}$ replaced by $\mathbf{C}$) shows this two port $\mathbf{C}$; the lower part of Figure 1 (with $\mathbf{R}$ replaced by $\mathbf{C}$) shows the encapsulated version.

3 Optimisation

As discussed by Cabanellas, Felez and Vera [9], one application of sensitivity methods is optimisation. This paper considers optimisation problems of the form:

$$\min_{\theta} J(\theta) \tag{25}$$

where $\theta \in \Re^{n_\theta}$ is the *parameter vector* and the *cost function* $J(\theta)$ is

$$J(\theta) = \frac{1}{2} \int_0^T e^T(t)Q(t)e(t)dt \tag{26}$$

where $Q(t) \in \Re^{n_y \times n_y}$ is a positive semi-definite weighting function and the error $e(t) \in \Re^{n_y}$ is

$$e(t) = [y(t,\theta) - z(t)] \tag{27}$$

$y(t) \in \Re^{n_y}$ the system output and $z(t) \in \Re^{n_y}$ a function of time. In general, $y(t)$ is *not* linear in θ and so $J(\theta)$ is *not* in general quadratic in θ.

In the context of real-time control and estimation and so optimisation speed is of the essence. Therefore, rapid convergence combined with simplicity is desirable. As, using the sensitivity bond graph approach, gradient information is cheaply available, this suggests the use of methods which make use of gradient information. For these reasons, out of the plethora of methods available (see for example [1,26]), we choose the Gauss-Newton method. Further research may yield alternative choices, but our experience so far has been good.

Differentiation of Equations 25 and 27 with respect to θ relate the gradient J_θ of J (with respect to θ) to the corresponding gradient $y_\theta(t)$ and the output $y(t)$ as:

$$J_\theta = \int_0^T [y(t,\theta) - z(t)]^T Q(t)y_\theta(t)dt \tag{28}$$

The Gauss-Newton approach approximates the second derivative $J_{\theta\theta}$ of the cost function by:

$$J_{\theta\theta} \approx \hat{J}_{\theta\theta} = \int_0^T y_\theta(t)Q(t)y_\theta^T(t)dt \tag{29}$$

Thus both J_θ and $\hat{J}_{\theta\theta}$ can be derived from the outputs $y(t)$ and $y_\theta(t)$ of the sensitivity bond graph system described in Section 2.

The optimisation algorithm is then to repeatedly compute:

$$\theta := \theta - \Delta\theta \tag{30}$$

until some convergence criterion is satisfied where $\Delta\theta$ is the solution of the set of linear equations:

$$\hat{J}_{\theta\theta}\Delta\theta = J_\theta \tag{31}$$

Two simple modifications of this method give additional robustness in the face of difficult optimisation problems [1,26]:

(1) Equation 31 is solved via a singular value decomposition based pseudo inverse and
(2) A check is made that the cost function decreases at each step; if it doesn't, the step length is halved (whilst retaining the step direction) until it does.

This approach has proved effective for this example. However related non-linear least-squares algorithms (such as the Levenberg-Martquardt algorithm [1]) may be equally or more effective.

4 Example

This example has appeared previously [27] but is extended here to include state estimation and is solved using the novel methods of this paper rather than the preliminary approach given in reference [27].

Figure 2 shows two identical right-conical tanks of liquid and two identical pipes; Figure 3 gives the corresponding bond graph. It is assumed that the pipes can be modelled using the CRs of Section 2.1.2 (Equations 19 and 20) (and therefore that the flow is unidirectional) and the tanks by the CRs of Section 2.1.4 (Equations 23 and 25). The system input $u(t)$ is the inflow, and the system output $y(t)$ the pressure at the base of the right-hand tank.

Figure 3 gives the *sensitivity* bond graph of the system. The **sR** and **sC** components are as discussed in Section 2.1.2 and 2.1.4 respectively. The **Sf** component imposes a flow $u(t)$ on the flow component of the vector bond and zero (corresponding to the fact that $u(t)$ is independent of θ) on the sensitivity flow variable. The **sDe** component measures the output effort $y(t)$ together with the corresponding sensitivity signal.

As discussed in Section 2, the bonds (indicated by the bold lines) are vector bonds carrying *sensitivity* efforts $^\lambda e$ and flows $^\lambda f$ in addition to efforts e and flows f.

For the purposes of this example, it was assumed that $\rho g = 1$ but that the initial volumes of liquid in each tank $q_{10} = V_1$ and $q_{20} = V_2$ are unknown. It is further assumed that the pipe parameters r and α are unknown. Thus the

11

parameter vector θ is chosen as:

$$\theta = \begin{pmatrix} V_1 \\ V_2 \\ \alpha \\ \beta \end{pmatrix} \tag{32}$$

The system equations were deduced from the bond graph of Figure 3 using the MTT software [28] using symbolic algebra methods [29] to deduce the sensitivity CRs of Equations 20 and 25. These equations, together with Euler integration code were automatically translated into C and compiled.

The system was simulated using Euler integration with a step size of 0.01 up to time $T = 10$ and using parameters:

$$\theta = \begin{pmatrix} V_1 \\ V_2 \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.25 \\ 1.5 \\ 1.0 \end{pmatrix} \tag{33}$$

The tank inflow was $u = 1.0$.

Figure 5 shows the system output as a function of time. This data was used as the actual system output corresponding to $z(t)$ in Equation 27.

The shape of the graph is explained as follows. The initial output (the pressure at the base of the second tank) is given by Equation 23 as:

$$y(0) = \rho g \left(\frac{12}{\pi} V_2 \right)^{\frac{1}{3}} \approx 0.98 \tag{34}$$

The final output is governed by the pressure drop across the second pipe at the steady-state flow rate $f_\infty = 1$ corresponding to the unit input flow. Using Equation 14

$$y(\infty) = \beta f_\infty^\alpha = 1 \tag{35}$$

$y(t)$ in Figure 5 is asymptotic to this value.

Figure 6 shows the corresponding four sensitivity functions plotted against time. Of the four parameters, the initial output $y(0)$ is dependent only on V_2, thus the sensitivity with respect to V_1, α and β is initially zero. From Equation 25

$$y_{V_1}(0) = \left(\frac{12}{\pi} V_1 \right)^{\frac{1}{3}} \left[\rho g \frac{1}{3V_1} \right] \approx 1.3 \tag{36}$$

12

Thus three of the sensitivity function start at zero, whilst that with respect to V_2 starts at about 1.3. In the steady-state, $y(t)$ only depends on the flow $f_\infty = 1$ though the second pipe, thus the sensitivity with respect to V_1 and V_2 is zero. From Equation 18, the sensitivity of the steady-state output $y(t)$ with respect to α is

$$y_\alpha(\infty) = \beta f_\infty^\alpha \ln f_\infty = 0 \tag{37}$$

and the sensitivity of the steady-state output $y(t)$ with respect to β is

$$y_\beta(\infty) = f_\infty^\alpha = 1 \tag{38}$$

Thus three of the sensitivity functions are asymptotic to zero, and that with respect to β is asymptotic to unity.

Using different parameters $\hat{\theta}$, the same code was used to generate the model output $y(t, \hat{\theta})$ together with the four sensitivity functions contained in the vector $y_\theta(t, \hat{\theta})$ The optimisation method of Section 3 was used to determine the parameter estimate $\hat{\theta}$. The starting value of $\hat{\theta}$ was

$$\hat{\theta} = \begin{pmatrix} V_1 \\ V_2 \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 2.0 \\ 1.0 \\ 1.0 \\ 2.0 \end{pmatrix} \tag{39}$$

Figure 7 shows the parameter estimates $\hat{\theta}$ plotted against iteration number. The initial estimates correspond to Equation 39, and the estimates converge to those given in Equation 33.

Figure 8 shows the estimated system output plotted against time corresponding to the initial value of $\hat{\theta}$ and the estimates at iterations 1, 5 and 10 together with the system response. That corresponding to iteration 10 is almost identical to the system response.

Figures 7 and 8 demonstrate that the method is effective in this case.

5   Conclusion

A sensitivity bond graph, of the same structure as the system bond graph, is shown to provide a simple and effective method of generating sensitivity functions of use in optimisation.

One application of such optimisation in the context of partially-known system

parameter and state estimation is given and illustrated using a highly non-linear system. This sensitivity-based optimisation approach is also relevant to model-based predictive control; this aspect has also been investigated [30].

6   Acknowledgements

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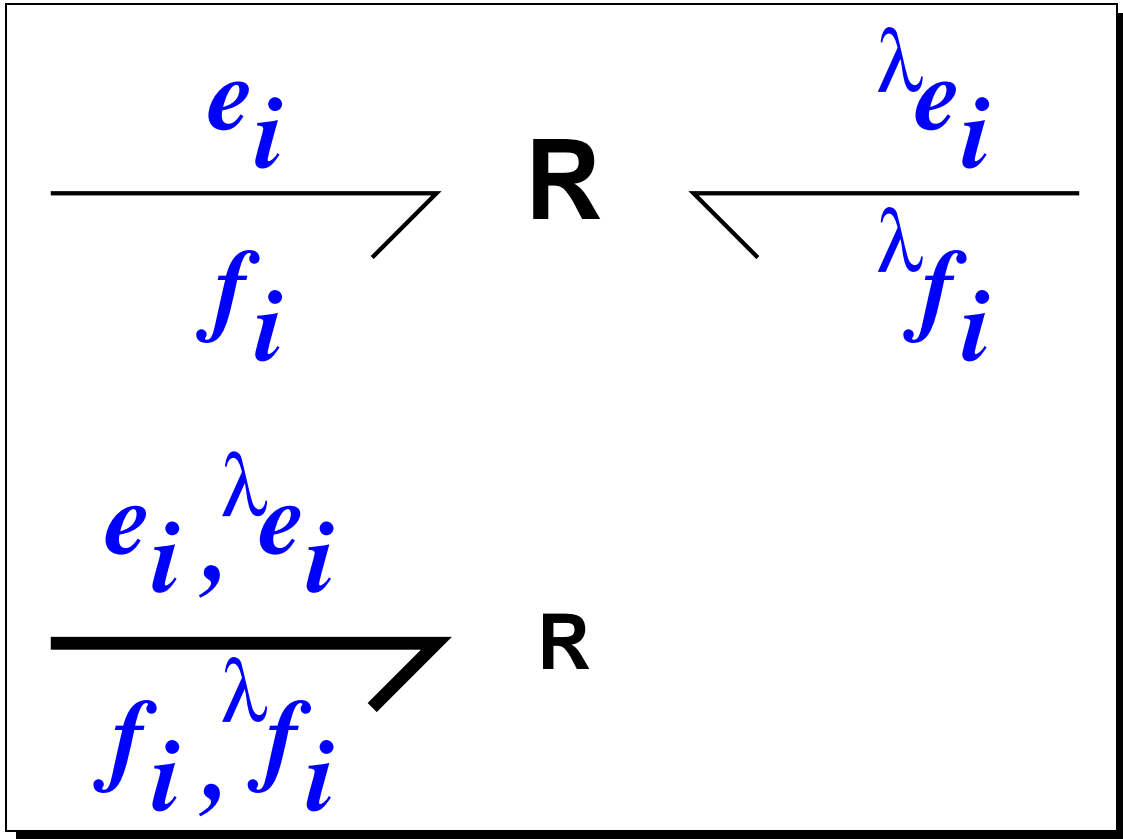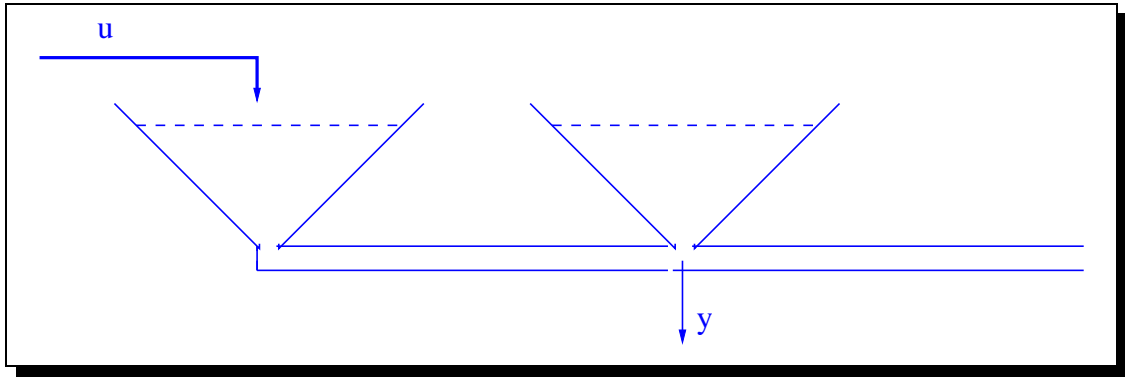
Fig. 1. The sensitivity **R** component **sR**
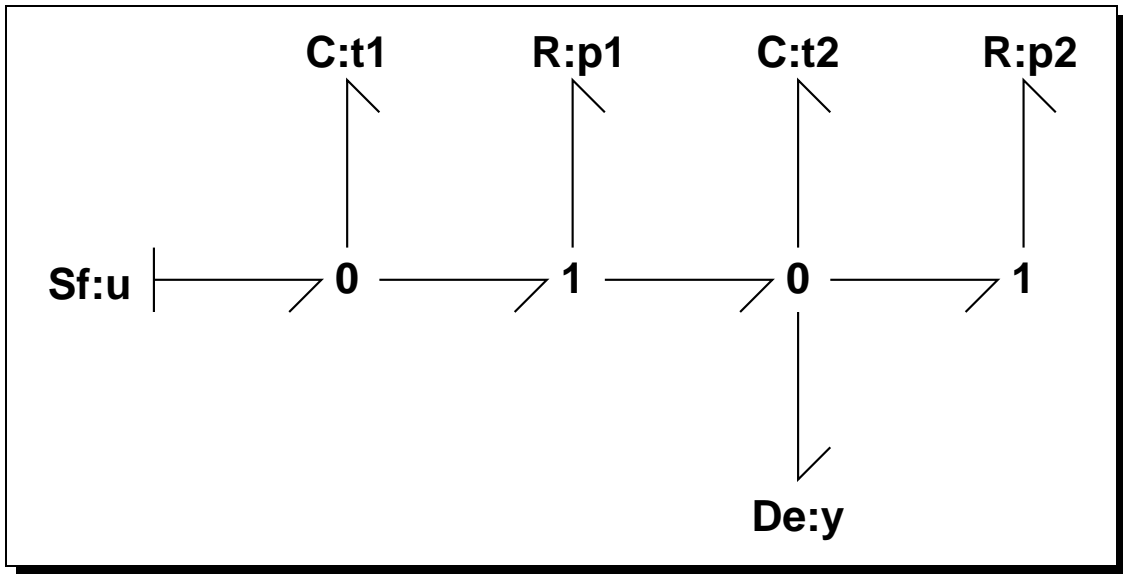
Fig. 2. Two tank system: Schematic

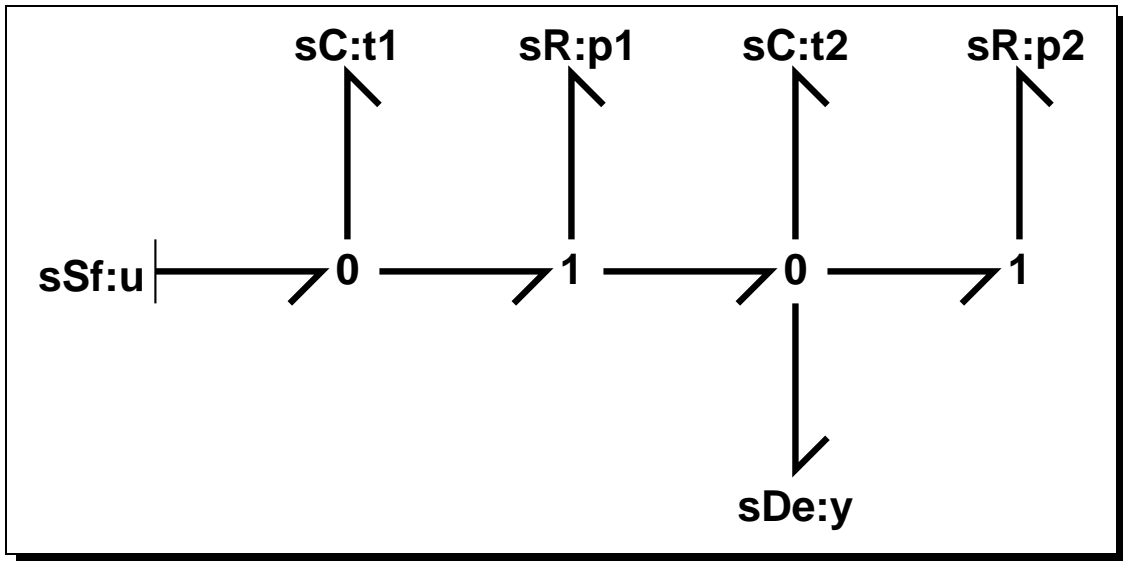Fig. 3. Two tank system: Bond Graph
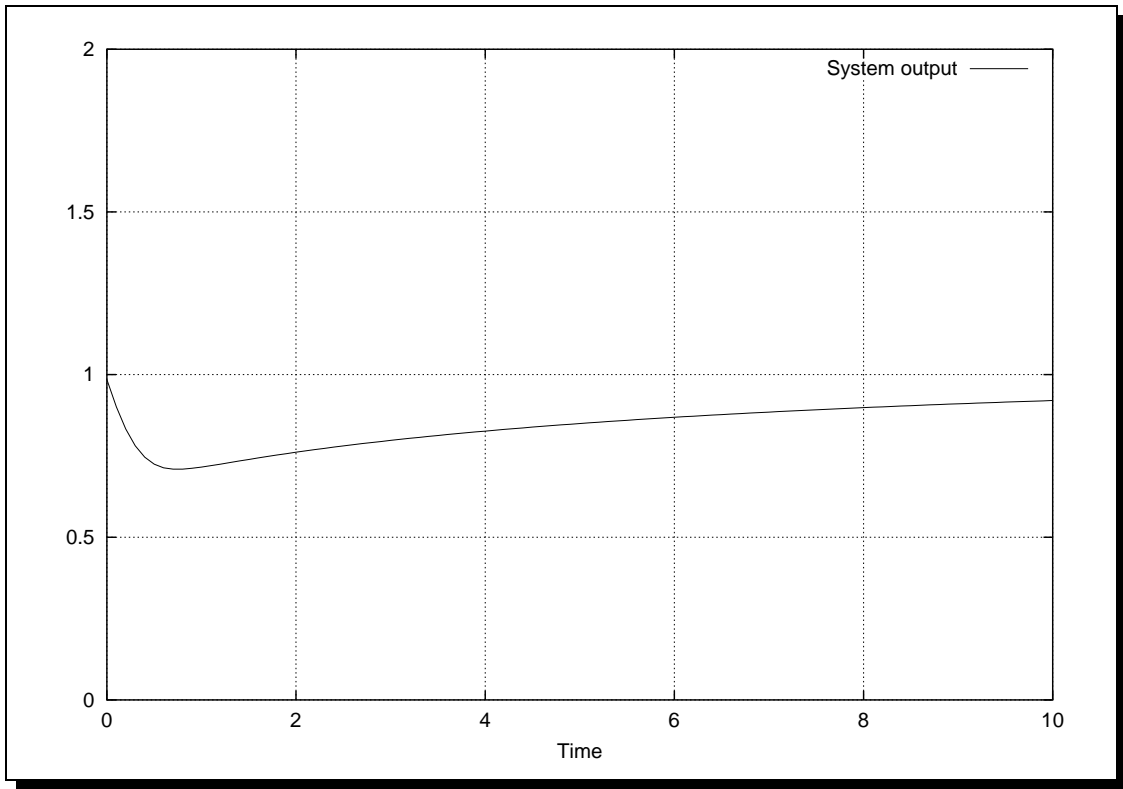
Fig. 4. Two tank system: Sensitivity Bond Graph
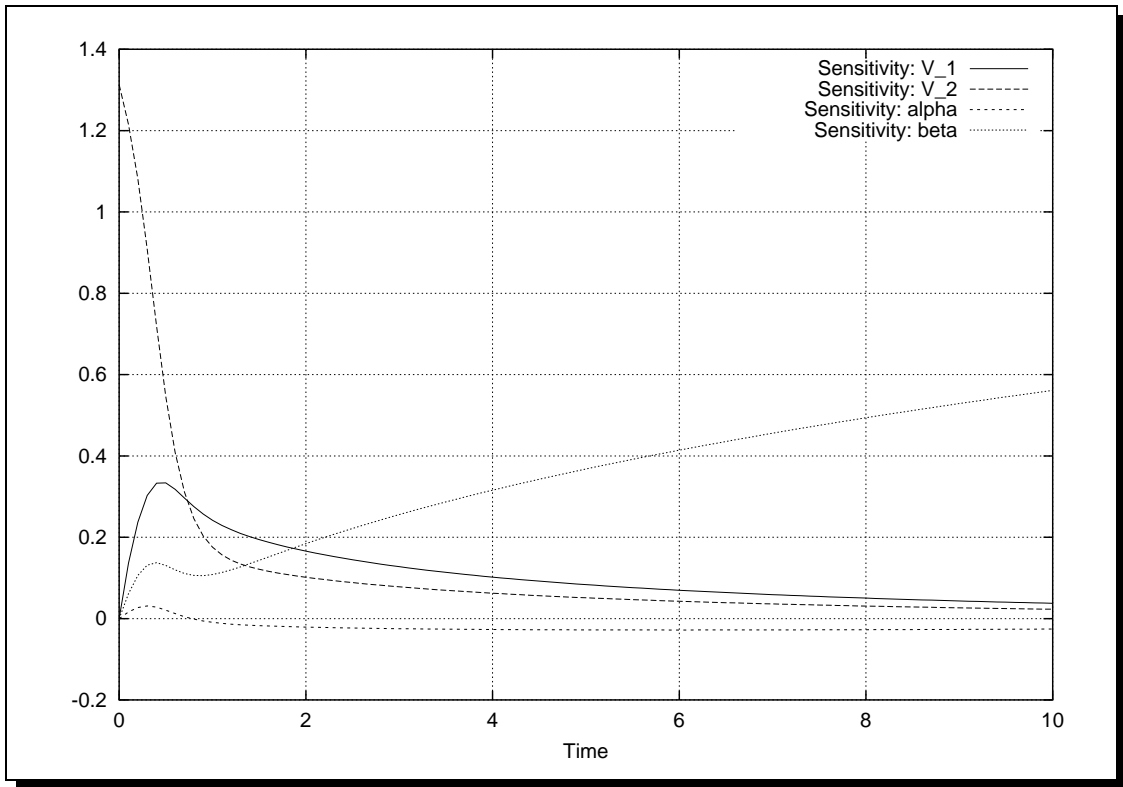
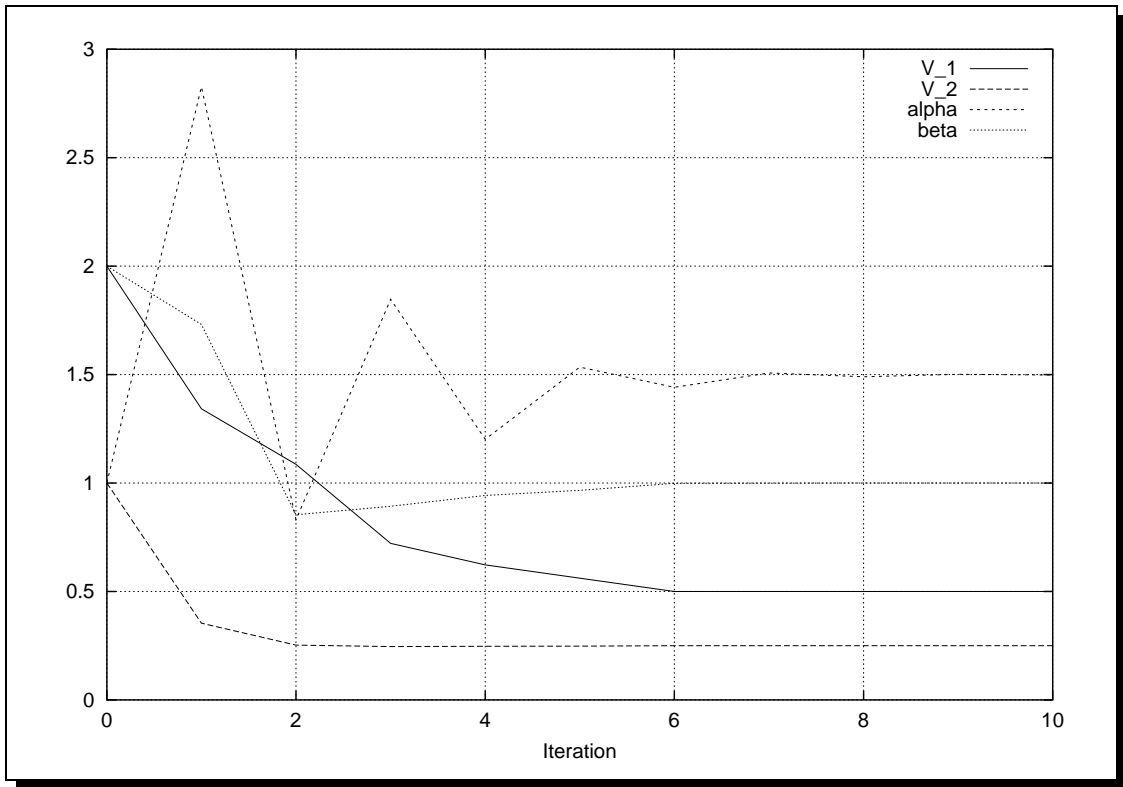Fig. 5. System output

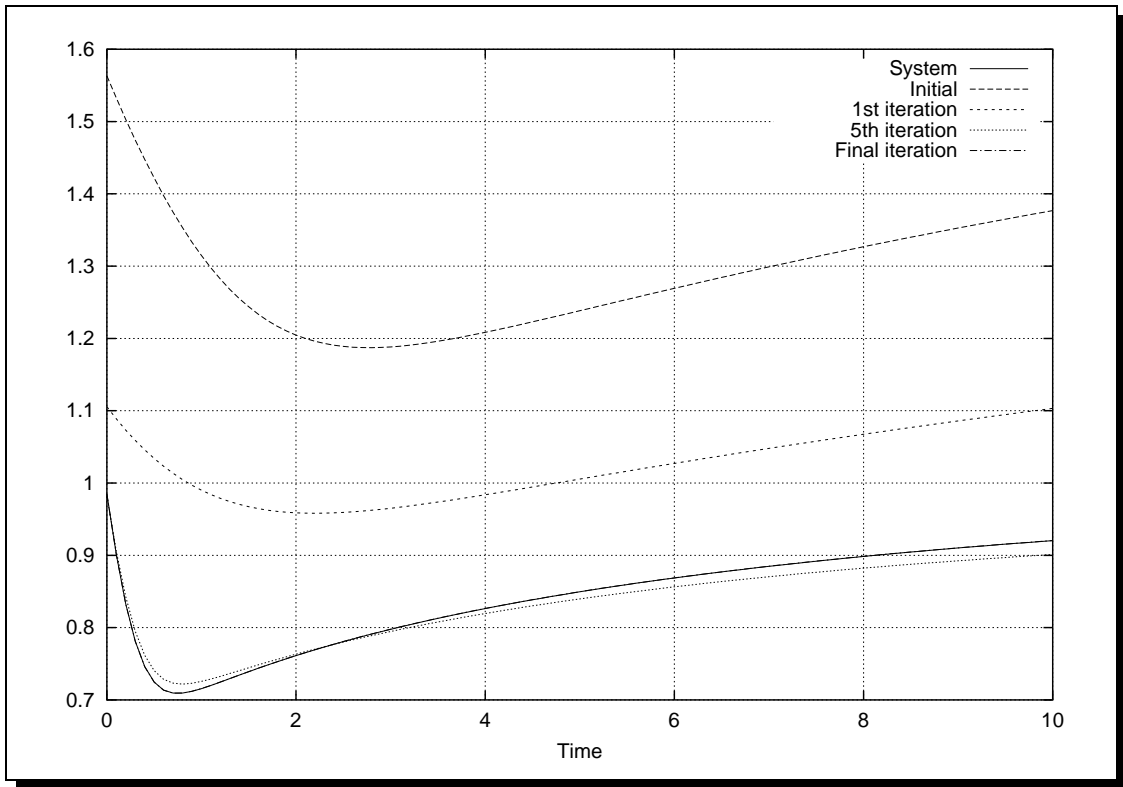Fig. 6. Model sensitivity functions

Fig. 7. Estimated parameters

Fig. 8. Estimated output at iterations 0, 1, 5 and 10