Optimal Control of Nonlinear Systems: A Predictive Control Approach

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Abstract

A new nonlinear predictive control law for a class of multivariable nonlinear systems is presented in this paper. It is shown that the closed-loop dynamics under this nonlinear predictive controller explicitly depend on design parameters (prediction time and control order). The main features of this result are that an explicitly analytical form of the optimal predictive controller is given, on-line optimisation is not required, stability of the closed-loop system is guaranteed, the whole design procedure is transparent to designers and the resultant controller is easy to implement. By establishing the relationship between the design parameters and time-domain transient, it is shown that the design of an optimal generalised predictive controller to achieve desired time-domain specifications for nonlinear systems can be performed by looking up tables. The design procedure is illustrated by designing an autopilot for a missile.

Key words: Optimal control, nonlinear systems, generalised predictive control, feedback linearisation, time-domain transient

1 Introduction

Optimal control of nonlinear systems is one of the most active subjects in control theory. One of the main difficulties with classic optimal control theory is that, to determine optimal control for a nonlinear system, the Hamilton-Jacobi-Bellman (HJB) partial differential equations (PDEs) have to be solved (Bryson and Ho, 1975). There is rarely an analytical solution although several numerical computation approaches have been proposed; for example, see Polak (1997). The same difficulty also occurs in recently developed "nonlinear" H^∞ control theory where Hamilton-Jacobi-Issacs PDEs have to be solved (for example see Ball *et al.* (1993)).

As a practical alternative approach, model based predictive control (MPC) has received a great deal of attention and is considered by many to be one of the most promising methods in control engineering (Garcia *et al.*, 1989). The core of all model based predictive algorithms is to use "open loop optimal control" instead of "closed-loop optimal control" within a moving horizon. Among them, long range Generalised Predictive Control (GPC) is one of the most promising algorithms (Clarke, 1994). Following the successes with linear systems, much effort has been taken to extend GPC to nonlinear systems (for state-of-the-art of nonlinear predictive control see Allgöwer and Zheng (1998)). Various nonlinear GPC methods for discrete-time systems have been developed; for example, see Bequette (1991), Biegler and Rawlings (1991), Mayne (1996), and Allgöwer and Zheng (1998). The main shortcoming of these methods is that on-line dynamic optimisation is required, which, in general, is non-convex. As pointed by Chen *et al.* (2000), heavy on-line computational burden is the main obstacle in the application of GPC in nonlinear engineering systems. This causes two main problems. One problem is a large computational delay and the other problem is that global minimum may not be achieved, or even worse a local minimum cannot be achieved due to time limitation in each

optimisation cycle.

To avoid the online computational issue, one way is to develop a closed form optimal GPC. To this end, Lu (1995), Soroush and Soroush (1997) and Siller-Alcala (1998) limit the control order to be zero, that is, to limit the control effort to be a constant in the predictive interval. Then the closed form optimal GPC laws are given. However, it is difficult to predict the system output over a long horizon since the output order is limited to be the relative degree of a nonlinear system in this approach. Moreover, as shown in this paper, however small the predictive horizon is chosen, the closed-loop system is unstable for plants with large relative degree, i.e., $\rho > 4$. To obtain adequate performance, the control order should be chosen to be reasonably large. When this approach is used to deal with the control order larger than zero by augmenting the derivatives of the control as additional state, the control law derived depends on the derivatives of the control that are unknown and thus is impossible to implement. Alternatively, it is shown by Gawthrop *et al.* (1998) that the special case of zero prediction horizon also leads to an analytic solution related to those obtained by the geometric approach (Isidori, 1995).

In a similar vein, this paper looks at another special case of the nonlinear GPC of Gawthrop *et al.* (1998) where the degree of the output prediction is constrained in terms of both relative degree and control order. The approach gives an analytic solution for a class of multivariable nonlinear systems in terms of a generalised predictive control performance index. The result is based on four concepts: prediction via Taylor series expansion, receding horizon control, control constraints (within the moving horizon time frame) and optimisation. In order to avoid the numerical computation difficulties in optimisation, an analytical solution to a set of nonlinear equations arising in optimisation is derived. As a result, an optimal generalised predictive control law is presented in a closed form, which turns out to be a time invariant nonlinear state feedback control law. By showing that the closed-loop system is linear, the stability of

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the closed-loop system is established. Moreover the design parameters in this nonlinear control design method can be directly chosen according to desired time-domain transient and thus a trade-off between performance specifications and control effort is possible.

2 Predictive control for nonlinear systems

2.1 Nonlinear Generalised Predictive Control (NGPC)

Consider the nonlinear system

$$
\begin{cases}
\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t) \\
y_i(t) = h_i\big(x(t)\big), \qquad i = 1, \ldots, m,
\end{cases}
\tag{1}
$$

where $x \in R^n$, $u \in R^m$ and $y = [y_1, y_2, \ldots, y_m]^T \in R^m$ are the state, control and output vectors respectively. In general, an optimal tracking problem can be stated as follows: design a controller such that the closed-loop system is asymptotically stable and the output, $y(t)$, of the nonlinear system (1) optimally tracks a prescribed reference, $w(t)$, in terms of a given performance index.

To avoid the difficulties in solving PDEs in classic optimal control theory, the moving-horizon control concept is adopted in this paper (Mayne and Michalska, 1990; Demircioglu and Gawthrop, 1991). Briefly, the basic idea is, at any time t, to design "open-loop control" within a moving time frame located at time t regarding $x(t)$ as the initial condition of a state trajectory $\hat{x}(t + \tau)$ driven by an input $\hat{u}(t + \tau)$ together with associated predicted $\hat{y}(t + \tau)$. To distinguish them from the real variables, the hatted variables are defined as the variables in the *moving* time frame.

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The receding-horizon performance index adopted in this paper is given by

$$J = \frac{1}{2} \int_0^T \left(\hat{y}(t+\tau) - \hat{w}(t+\tau) \right)^T \left(\hat{y}(t+\tau) - \hat{w}(t+\tau) \right) d\tau \qquad (2)$$

where T is the predictive period. Following Gawthrop *et al.* (1998), Demircioglu (1989) and Chen *et al.* (1999a), the control weighting term is not included in the performance index (2). How to limit the control effort in NGPC will be discussed in Section 3. Similar to other receding control strategies, the actual control input, $u(t)$, is given by the initial value of the optimal control input $\hat{u}(t+\tau)$, $0 \leq \tau \leq T$, which minimises the performance index (2); that is,

$$u(t+\tau) = \hat{u}(t+\tau) \text{ when } \tau = 0. \qquad (3)$$

Hence the hatted variables are different from the actual variables except when $\tau = 0$

In this paper the following assumptions are imposed on the nonlinear system (1):

$A1$: The zero dynamics are stable (Isidori, 1995);

$A2$: All states are available;

$A3$: Each of the system output $y(t)$ has the same well-defined relative degree ρ (Isidori, 1995);

$A4$: The output $y(t)$ and the reference signal $w(\mathrm{t})$ are sufficiently many times continuously differentiable with respect to t.

Assumption $A3$ is not necessary: the result and method in this paper can be extended to a system with different relative degrees. This assumption is imposed to emphasise the main contribution of this paper. Assumption $A4$ is made in most of nonlinear control theory (Nijmeijer and van der Schaft, 1990).

Definition 1: The control order in continuous time predictive control is said

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to be r if the control signal $\hat{u}(t + \tau)$ satisfies

$$
\begin{cases}
\hat{u}^{[r]}(t + \tau) \neq 0 & \text{for some } \tau \in [0, T]; \\
\hat{u}^{[k]}(t + \tau) = 0 & \text{for all } k > r \text{ and } \tau \in [0, T].
\end{cases}
$$

where $\hat{u}^{[r]}(t + \tau)$ denotes the rth derivative of $\hat{u}(t + \tau)$ with respect to τ. The specification of the control order determines the allowable set, $\mathcal{U}$, of the optimal control input $\hat{u}(t+\tau)$ in the moving horizon time frame, hence imposes the constraints on $\hat{u}(t + \tau)$. In the NGPC algorithm presented in this paper, similar to the linear system case (Demircioglu and Gawthrop, 1991; Gawthrop et al., 1998), the control constraints in the moving horizon time frame are imposed by the choice of the control order r. For example, if the control order is chosen as 0, then the control input $\hat{u}(t + \tau)$ to be optimised in the moving time frame is constant. There is no limitation on the control order and it can be an arbitrary positive integer in this paper. It should also be noted that the control constraints are only imposed on the control signal in the *moving* horizon time frame, $\hat{u}$, rather than the actual control signal u.

2.2 Output prediction

The output in the *moving* time frame is predicted by Taylor series expansion. Repeated differentiation up to ρ times of the output $\hat{y}$ with respect to time, together with repeated substitution of the system (1) gives

$$
\dot{\hat{y}}(t) = L_f h(x) \tag{4}
$$

$$
\vdots
$$

$$
\hat{y}^{[\rho-1]}(t) = L_f^{\rho-1} h(x) \tag{5}
$$

$$
\hat{y}^{[\rho]}(t) = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x) \hat{u}(t) \tag{6}
$$

where

$$
h(x) = \Big[h_1(x), \ldots, h_m(x) \Big]^T. \tag{7}
$$

The standard Lie notation is used in this paper (Isidori, 1995) and the only reason for using it is to simplify the notation.

When the control order is chosen to be r, the order of the Taylor expansion of the output $\hat{y}(t + \tau)$ must be at least $\rho + r$. to make the rth derivative of the control signal appear in the prediction. Differentiating Equation (6) with respect to time yields

$$\hat{y}^{[\rho+1]}(t) = L_f^{\rho+1} h(x) + p_{11}\big(\hat{u}(t), x(t)\big) + L_g L_f^{\rho-1} h(x)\dot{\hat{u}}(t) \tag{8}$$

where

$$p_{11}\big(\hat{u}(t), x(t)\big) = L_g L_f^{\rho} h(x)\hat{u}(t) + \frac{dL_g L_f^{\rho-1} h(x)}{dt}\hat{u}(t) \tag{9}$$

Note that p_{11} is nonlinear in both $\hat{u}(t)$ and $x(t)$.

Similarly the higher derivatives of the output, $\hat{y}(t)$, can be derived and by summarising them with (4)–(8), one has

$$\hat{\bar{Y}}(t) = \begin{bmatrix} \hat{y}^{[0]} \\ \hat{y}^{[1]} \\ \vdots \\ \hat{y}^{[\rho]} \\ \hat{y}^{[\rho+1]} \\ \vdots \\ \hat{y}^{[\rho+r]} \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f^1 h(x) \\ \vdots \\ L_f^{\rho} h(x) \\ L_f^{\rho+1} h(x) \\ \vdots \\ L_f^{\rho+r} h(x) \end{bmatrix} + \begin{bmatrix} 0_{m\times1} \\ \vdots \\ 0_{m\times1} \\ H(\hat{\bar{u}}) \end{bmatrix} \tag{10}$$

where $H(\hat{\bar{u}}) \in R^{m(r+1)}$ is a matrix valued function of $\hat{u}(t), \dot{\hat{u}}(t), \ldots, \hat{u}^{[r]}(t)$,

given by

$$H(\hat{\bar{u}}) = \begin{pmatrix} L_g L_f^{\rho-1} h(x)\hat{u}(t) \\[2ex] p_{11}\big(\hat{u}(t), x(t)\big) + L_g L_f^{\rho-1} h(x)\dot{\hat{u}}(t) \\[2ex] \vdots \\[2ex] p_{r1}\big(\hat{u}(t), x(t)\big) + \cdots + \\[2ex] p_{rr}\big(\hat{u}(t), \ldots, \hat{u}^{[r-1]}(t), x(t)\big) + \\[2ex] L_g L_f^{\rho-1} h(x)\hat{u}^{[r]}(t) \end{pmatrix} \tag{11}$$

and

$$\hat{\bar{u}} = \begin{bmatrix} \hat{u}(t)^T & \dot{\hat{u}}(t)^T & \ddot{\hat{u}}(t)^T & \ldots & \hat{u}^{[r]}(t)^T \end{bmatrix} \tag{12}$$

Within the moving time frame, the output $\hat{y}(t + \tau)$ at the time τ is approximately predicted by

$$\hat{y}(t + \tau) \doteq \mathcal{T}(\tau)\hat{\bar{Y}}(t) \tag{13}$$

The $m \times m$ matrix $\bar{\tau}$ and the $m \times m(\rho + r + 1)$ matrix $\mathcal{T}(\tau)$ are given by

$$\bar{\tau} = \mathrm{diag}\{\tau, \ldots, \tau\} \tag{14}$$

$$\mathcal{T}(\tau) = \begin{bmatrix} I & \bar{\tau} & \ldots & \frac{\bar{\tau}^{(\rho+r)}}{(\rho+r)!} \end{bmatrix} \tag{15}$$

In the moving time frame, the reference $w(t+\tau)$ at the time τ is approximated by the Taylor expansion of $w(t)$ at the time t up to $(\rho + r)$th order, given by

$$\hat{w}(t + \tau) = \mathcal{T}(\tau)\bar{W}(t) \tag{16}$$

where

$$\bar{W}(t) = \begin{bmatrix} w(t)^T & \dot{w}(t)^T & \cdots & w^{[\rho+r]}(t)^T \end{bmatrix}^T \tag{17}$$

The optimal tracking problem now can be reformulated as, at any time t, to find optimal $\hat{\bar{u}}(t) \in R^{(\rho+r+1)m}$ such that the performance index (2) is minimised.

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It should be noted that $p_{11}, p_{21}, p_{22}, \ldots, p_{r1}, \ldots, p_{rr}$ are complicated nonlinear functions of the elements in the optimised vector $\hat{\bar{u}}(t)$.

2.3 Nonlinear optimal controllers

One of our main results is given in Theorem 1. The proof of Theorem 1 is given in Appendix A

Theorem 1 *Consider a multivariable continuous-time nonlinear system satisfying Assumptions $A1 - A4$ and suppose that the output in the predictive interval is predicted by Taylor expansion up to $\rho + r$ order where ρ is the relative degree. Then for a given control order $r \geq 0$, the optimal nonlinear control law which minimises the receding horizon performance index (2) is given by*

$$u(t) = -\left(L_g L_f^{\rho-1} h(x)\right)^{-1} \left(K M_\rho + L_f^\rho h(x) - w^{[\rho]}(t)\right) \tag{18}$$

where $M_\rho \in R^{m\rho}$ is given by

$$M_\rho = \begin{pmatrix} h(x) - w(t) \\ \\ L_f^1 h(x) - w^{[1]}(t) \\ \\ \ldots \\ \\ L_f^{\rho-1} h(x) - w^{[\rho-1]}(t) \end{pmatrix} \tag{19}$$

and $K \in R^{m \times m\rho}$ is the first m columns of the matrix $\bar{\mathcal{T}}_{rr}^{-1} \bar{\mathcal{T}}_{\rho r}^T$, given by

$$\bar{\mathcal{T}}_{rr} = \begin{bmatrix} \bar{\mathcal{T}}_{(\rho+1,\rho+1)} & \cdots & \bar{\mathcal{T}}_{(\rho+1,\rho+r+1)} \\ \vdots & \ddots & \vdots \\ \bar{\mathcal{T}}_{(\rho+r+1,\rho+1)} & \cdots & \bar{\mathcal{T}}_{(\rho+r+1,\rho+r+1)} \end{bmatrix}, \tag{20}$$

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$$\bar{\mathcal{T}}_{\rho r} = \begin{bmatrix} \bar{\mathcal{T}}_{(1,\rho+1)} & \cdots & \bar{\mathcal{T}}_{(1,\rho+r+1)} \\ \vdots & \ddots & \vdots \\ \bar{\mathcal{T}}_{(\rho,\rho+1)} & \cdots & \bar{\mathcal{T}}_{(\rho,\rho+r+1)} \end{bmatrix}, \tag{21}$$

$$\bar{\mathcal{T}}_{(i,j)} = \frac{\bar{T}^{i+j-1}}{(i-1)!(j-1)!(i+j-1)}, \qquad i,j = 1,\ldots,\rho+r+1, \tag{22}$$

and

$$\bar{T} = \mathrm{diag}\{T,\ldots,T\} \in R^{m \times m}. \tag{23}$$

Remark 1: The optimal control law (18) is a nonlinear time invariant state feedback law. The matrix K in the control law (18) is constant. It only depends on the predictive time, T, the control order, r, and the relative degree of the system, ρ. Moreover, the expressions of the Lie derivatives of the output can be obtained by manual computation or using symbolic computation software, for example, Reduce (Hearn, 1995). It can be seen that they only depend on the system states.

Remark 2: In many continuous time GPC algorithms, the control order r and the output order N_y, which is defined as the highest order derivative of the output used in predicting the output in the moving time frame, are two free parameters to be determined (Demircioglu and Gawthrop, 1991; Demircioglu and Gawthrop, 1992; Gawthrop *et al.*, 1998). The NGPC developed in this paper only has one free parameter, i.e., the control order r. The output order is determined by the control order plus the relative degree of the system. As pointed by Demircioglu and Clarke (1992), this is reasonable and natural since the rth derivative of the control first appears in the expression of the $\rho + r$th derivative of the output and higher derivatives of the control u will appear in the expression of the derivatives of the output with the order larger than $\rho + r$. The control order is also chosen as the only design parameter in Lu (1995), Soroush and Soroush (1997) and Siller-Alcala (1998) where the control order r is zero. The choice of the control order and the output order as

two independent free parameters does provide the ability for GPC to handle nonlinear systems with unstable zero dynamics and ill-defined relative degree (Demircioglu, 1989; Siller-Alcala, 1998; Gawthrop *et al.*, 1998), where the derivatives of the control signal in the moving time frame with the order larger than $N_y - \rho - r$ are set to be zero. However, as shown in above and later, there are several significant advantages in choosing only the control order as the design parameter, including having the nonlinear GPC law in a closed-form.

Remark 3: Another advantage of the NGPC method derived in this paper is that the highest order of the derivatives of the output $y(t)$ necessary for computing the control law (18) is equal to the relative degree of the nonlinear system, ρ, no matter what control order, r, is prescribed. This is an important property in computation. It is not easy to compute the high order derivatives for a nonlinear system manually, however it can be done by a computer algebra program. Experience has shown that even for a simple nonlinear system such as a two link robotic manipulator, to predict the output of the systems with the fifth order derivative, an equation with more than one hundred terms arises.

2.4 Stability of closed-loop systems

One of the main issues in optimal control is whether or not the closed loop system under the derived optimal control law is stable. In this section, stability of the optimal nonlinear control law (18) is analysed. Before giving the stability result, the following assumption is imposed on the system (1).

$A5$: (Isidori, 1995) The internal zero dynamics of the system (1) driven by w are defined for all $t \geq 0$, bounded and uniformly asymptotically stable.

Let the matrix K in (18) be partitioned as

$$K = \begin{bmatrix} k_0 & k_1 & \cdots & k_{\rho-1} \end{bmatrix} \tag{24}$$

where $k_i \in R^{m \times m}, i = 0, \ldots, \rho - 1$. Substituting the control law (18) and (19) into Equation (6) yields

$$y^{[\rho]}(t) = L_f^\rho h(x) - \sum_{i=0}^{\rho-1} k_i \left(L_f^i h(x) - w^{[i]}(t) \right) - L_f^\rho h(x) + w^{[\rho]}(t), \tag{25}$$

Following Equations (4)–(6) and defining the tracking error $e(t)$ as

$$e(t) = w(t) - y(t), \tag{26}$$

Equation (25), which defines the closed-loop system consisting of the plant (1) and the control (18), becomes

$$e^{[\rho]}(t) + k_{\rho-1}e^{[\rho-1]}(t) + \cdots + k_0 e(t) = 0 \tag{27}$$

Based on this result, the stability result for the NGPC (18) is given in Theorem 2. The proof of Theorem 2 is given in Appendix B.

Theorem 2 *The closed loop system for a plant (1) satisfying Assumptions A1–A5 under the NGPC (18) is linear. The stability of the overall closed-loop system only depends on the relative degree ρ and the control order, r. The closed-loop stability result under the NGPC (18) is given in Table 1 where "+" and "−" denote that the closed-loop system is stable and unstable respectively.*

Remark 4: The stability of the linear closed-loop system (27) is independent of the predictive time T since it can be shown that the choice of T does not affect the sign of the roots of the polynomials (B.5). It only depends on the relative degree order, ρ, of the nonlinear plant and the control order r.

Table 1 lists the stability result of a nonlinear system with a relative degree less than 10 where r and ρ are the control order and the relative degree respectively. It can be shown from Table 1 that when a nonlinear system has a low relative

Table 1

Stability of optimal nonlinear Generalised Predictive Control

ρ	1	2	3	4	5	6	7	8	9	10
$r = 0$	+	+	+	+	−	−	−	−	−	−
$r = 1$	+	+	+	+	+	−	−	−	−	−
$r = 2$	+	+	+	+	+	+	+	−	−	−
$r = 3$	+	+	+	+	+	+	+	+	−	−
$r = 4$	+	+	+	+	+	+	+	+	+	−
$r = 5$	+	+	+	+	+	+	+	+	+	+
$r = 6$	+	+	+	+	+	+	+	+	+	+
$r = 7$	+	+	+	+	+	+	+	+	+	+

degree (less or equal to 4), the closed-loop system is always asymptotically stable no matter what control order r is chosen. However when the relative degree of a nonlinear system is high, the system is unstable if the control order is too low. It can be explained by the fact that a high relative degree means a large delay in control effect and therefore the nonlinear system is difficult to control. Table 1 shows that the system is always stable when the difference between the relative degree and the control order is less than 4.

3   Design procedure based on time-domain specifications

In the nonlinear predictive control design method developed above there are two design parameters: the control order, r, and the predictive time, T. How to choose these parameters according to time-domain specifications is discussed in this section.

Table 2

Relative degree $\rho = 2$

r	σ (%)	$T_{5\%}(T)$	$T_{2\%}(T)$
0	5.23	2.536	3.288
1	2.13	0.839	1.384
2	1.30	0.532	0.593
3	0.97	0.368	0.413
4	0.79	0.270	0.305
5	0.70	0.206	0.234
6	0.63	0.163	0.185
7	0.59	0.132	0.150

In many cases the time-domain specification of a system is given in terms of its step response. There are two important indices, overshoot and settling time. Since the error equation for a nonlinear system under the nonlinear GPC (18) is given by (27), it is easy to calculate the step response when the matrix K is known, which is determined by the control order, r, and the predictive time, T. For the nonlinear systems with relative degree 2, the relationships between the time-domain specifications and the design parameters is shown in Tables 2, where σ is the overshoot, $T_{5\%}$ and $T_{2\%}$ are the settling time for within 5% or 2% steady state error respectively. The settling time is given in terms of the predictive time T. For a nonlinear system with other relative degree, the similar tables have been worked out but due to the limitation of the space, they are not reported here.

One interesting aspect of the NGPC (18) is that although control weighting does not appear in the performance index (2) explicitly, the control effort does not approach infinity. Here we give an informal discussion of the effects

of the choice of the predictive time and the control order on the control effort. Firstly the control effort can be reduced by increasing the predictive time in the performance index (2). It can be explained from (B.1) and (18). When the predictive time T increases, (B.1) indicates that the elements in the matrix K in (18) are reduced. Consequently, the control effort is, in general, reduced under the same tracking error since the gain matrix in the control law (18) decreases. This has been shown with a two-link robotic manipulator in (Chen *et al.*, 1999*a*). Similarly it can be argued that the control effort can be reduced by choosing a lower control order.

Therefore while the time domain specifications are satisfied, the control order should be selected as low as possible and the predictive time should as large as possible. This gives a small control effort and input energy requirement while the time-domain transient is achieved. If the time-domain specifications conflicts with the limitation of the control effort, one has to relax the time-domain specifications or the limitation on the control effort.

4   Example: Predictive control of a missile

The method of this paper is demonstrated by the design of an autopilot for a high angle of attack missile. The model of the longitudinal dynamics of a missile is taken from Reichert (1990), given by

$$\dot{\alpha} = f_1(\alpha) + q + b_1(\alpha)\delta \tag{28}$$
$$\dot{q} = f_2(\alpha) + b_2\delta \tag{29}$$

where α is the angle of attack (deg), q the pitch rate (deg/s), and δ the tail fin deflection (deg). The nonlinear functions $f_1(\alpha)$, $f_1(\alpha)$, $b_1(\alpha)$ and b_2 are determined by the aerodynamic coefficients (Reichert, 1990).

The tail fin actuator dynamics are approximated by a first-order lag

$$\dot{\delta} = \frac{1}{t_1}(-\delta + u) \tag{30}$$

where u is the commanded fin deflection(deg). In addition the fin deflection is limited by

$$|\delta| \leq 30 \text{ deg} \tag{31}$$

The output of this system is the angle of attack. The design objective is to develop a controller such that the missile can track a given angle of attack command. The tracking performance specifications are given in terms of its step response. More specifically, under all angle of attack command $|w(t)| \leq 20$ deg, the overshoot is less than 3% and the settling time (within 5% steady state error) is less than 0.2 second.

It can be shown that the longitudinal dynamics of the missile have well-defined relative degree, $\rho = 2$. According to the overshoot specification, from Table 2, the controller order should be equal to or greater than 1. To make the control effort is as small as possible, the control order should be chosen as small as possible and hence is chosen $r = 1$ in this paper; using this control order, the corresponding overshoot is 2.13%. Since the settling time specification is less than 0.2 sec, following from Table 2, the predictive time should satisfies

$$T \leq \frac{0.2}{0.839} = 0.2384 \text{ sec}$$

So T is chosen as 0.238 sec. The design parameters in the NGPC are determined and then the optimal GPC can be obtained by Theorem 1

The controller design is tested by simulation. The angle of attack (α), in response to a square wave between 0 deg and 15 deg, is plotted in Figure 1. It can be seen that all time-domain specifications are achieved: the overshoot is less than 3% and the settling time is less than 0.2 sec. It exhibits excellent tracking performance.
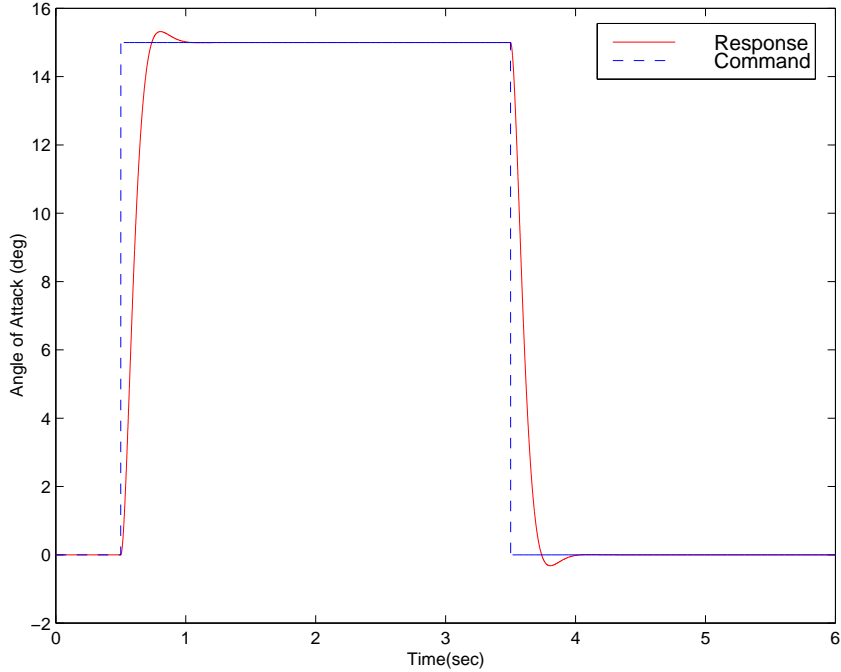
Fig. 1. The tracking performance under the nonlinear GPC

5   Conclusion

This paper presents a systematic method for designing an optimal controller to achieve prescribed time-domain specifications for a nonlinear system that satisfies Assumptions A1–A4. By defining an optimal tracking problem in terms of a generalised predictive control performance index, an optimal control law for a continuous-time nonlinear plant has been derived.

The significant features of this new optimal predictive control law are:

- the optimal control is given in a closed-form, which only depends on the states of a nonlinear system;
- on-line optimisation is not necessary;
- the design parameters, the control order and the predictive time, can be determined according to overshoot and settling time specifications directly by looking up tables;
- the stability of the closed-loop system is guaranteed for a nonlinear system

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with a low relative degree(≤ 4);

- the stability is achieved for a nonlinear system with a high relative degree by increasing the control order.

The NGPC developed in this paper is easy to design and implement. The design parameters are transparent to the designer. The whole design procedure should be understandable by, and acceptable to, engineers. A method to ensure integral action in NGPC is discussed in Chen *et al.* (1999b).

Appendix

A   Proof of Theorem 1

Using (13) and (16), the performance index (2) can be written as

$$
\begin{aligned}
J &= \frac{1}{2} \int_0^T \left(\hat{\bar{Y}}(t) - \bar{W}(t) \right)^T \mathcal{T}(\tau)^T \mathcal{T}(\tau) \left(\hat{\bar{Y}}(t) - \bar{W}(t) \right) d\tau \\
&= \frac{1}{2} \left(\hat{\bar{Y}}(t) - \bar{W}(t) \right)^T \bar{\mathcal{T}} \left(\hat{\bar{Y}}(t) - \bar{W}(t) \right)
\end{aligned} \tag{A.1}
$$

where $\bar{\mathcal{T}}$ is an $m(\rho + r + 1) \times m(\rho + r + 1)$ matrix, defined by

$$
\bar{\mathcal{T}} = \int_0^T \mathcal{T}(\tau)^T \mathcal{T}(\tau) d\tau \tag{A.2}
$$

Let

$$
\bar{\mathcal{T}} = \begin{bmatrix} \bar{\mathcal{T}}_{(1,1)} & \cdots & \bar{\mathcal{T}}_{(1,\rho+r+1)} \\ \vdots & \vdots & \vdots \\ \bar{\mathcal{T}}_{(\rho+r+1,1)} & \cdots & \bar{\mathcal{T}}_{(\rho+r+1,\rho+r+1)} \end{bmatrix}. \tag{A.3}
$$

From (15) and (A.2), the ijth (matrix) element in the matrix $\bar{\mathcal{T}}$ is given by (22) where $\bar{T}$ is obtained by replacing τ with T in (14).

It follows from (10) that

$$\hat{\bar{Y}} - \bar{W} = M + \begin{bmatrix} 0_{m\rho \times 1} \\ \\ H(\hat{\bar{u}}) \end{bmatrix}. \tag{A.4}$$

where

$$M = \begin{bmatrix} L_f^0 h(x) \\ L_f^1 h(x) \\ \vdots \\ L_f^{\rho+r} h(x) \end{bmatrix} - \begin{bmatrix} w(t) \\ \dot{w}(t) \\ \vdots \\ w^{[\rho+r]}(t) \end{bmatrix} \tag{A.5}$$

Differentiating H in (11) with respect to $\hat{\bar{u}}$ gives

$$\frac{\partial H(\hat{\bar{u}})}{\partial \hat{\bar{u}}}$$

$$= \begin{bmatrix} L_g L_f^{\rho-1} h(x) & 0_{m\times m} & 0_{m\times m} & \cdots & 0_{m\times m} \\ \times_{m\times m} & L_g L_f^{\rho-1} h(x) & 0_{m\times m} & \cdots & 0_{m\times m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \times_{m\times m} & \times_{m\times m} & \cdots & \times_{m\times m} & L_g L_f^{\rho-1} h(x) \end{bmatrix} \tag{A.6}$$

where '$\times_{m\times m}$' denotes the nonzero $m \times m$ block element in the matrix $\frac{\partial H(\hat{\bar{u}})}{\partial \hat{\bar{u}}}$.

The necessary condition for the optimal control $\hat{\bar{u}}$ is given by

$$\frac{\partial J}{\partial \hat{\bar{u}}} = 0 \tag{A.7}$$

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Partition the matrix $\bar{\mathcal{T}}$ into the submatrices

$$\bar{\mathcal{T}} = \begin{bmatrix} \bar{\mathcal{T}}_{\rho\rho} & \bar{\mathcal{T}}_{\rho r} \\ \bar{\mathcal{T}}_{\rho r}^T & \bar{\mathcal{T}}_{rr} \end{bmatrix} \tag{A.8}$$

where

$$\bar{\mathcal{T}}_{\rho\rho} \in R^{m\rho \times m\rho}, \bar{\mathcal{T}}_{\rho r} \in R^{m\rho \times m(r+1)}, \bar{\mathcal{T}}_{rr} \in R^{m(r+1) \times m(r+1)}$$

It can be shown that the necessary condition (A.7) can be written as

$$\left(\frac{\partial H(\hat{\hat{u}})}{\partial \hat{\hat{u}}} \right)^T \begin{bmatrix} \bar{\mathcal{T}}_{\rho r}^T & \bar{\mathcal{T}}_{rr} \end{bmatrix} M + \left(\frac{\partial H(\hat{\hat{u}})}{\partial \hat{\hat{u}}} \right)^T \bar{\mathcal{T}}_{rr} H(\hat{\hat{u}}) = 0 \tag{A.9}$$

Since $L_g L_f^{\rho-1} h(x)$ is invertible according to the relative degree definition, then $\frac{\partial H(\hat{\hat{u}})}{\partial \hat{\hat{u}}}$ in (A.6) is also invertible. Note that the matrix $\bar{\mathcal{T}}_{rr}$ is positive definite. Equation (A.9) implies

$$H(\hat{\hat{u}}) = - \begin{bmatrix} \bar{\mathcal{T}}_{rr}^{-1} \bar{\mathcal{T}}_{\rho r}^T & I_{m(r+1) \times m(r+1)} \end{bmatrix} M \tag{A.10}$$

Noting equation (11), the first m equations in (A.10) can be written as

$$L_g L_f^{\rho-1} h(x) \hat{u}(t) + K M_\rho + L_f^\rho h(x) - w(t)^{[\rho]} = 0 \tag{A.11}$$

where K denotes the first m columns of the matrix $\bar{\mathcal{T}}_{rr}^{-1} \bar{\mathcal{T}}_{\rho r}^T$ and, following (A.5), M_ρ is given by

$$M_\rho = \left[\left(h(x) - w(t) \right)^T \left(L_f^1 h(x) - w^{[1]}(t) \right)^T \dots \right.$$
$$\left. \left(L_f^{\rho-1} h(x) - w^{[\rho-1]}(t) \right)^T \right]^T \tag{A.12}$$

The optimal control $\hat{u}(t)$ can be uniquely determined by (18) since this is the only solution to Equation(A.10) and thus optimal condition (A.7).

B Proof of Theorem 2

Since K is the first m columns of the matrix $\bar{\mathcal{T}}_{rr}^{-1}\bar{\mathcal{T}}_{pr}^{T}$, it can be shown that

$$k_i = z_i T^{-\rho+i}\frac{\rho!}{i!}, i = 0, \ldots, \rho - 1 \tag{B.1}$$

where $z_i \in R^{m \times m}$, $i = 0, \ldots, \rho - 1$, are the submatrices of the first m columns of the matrix $M_1^{-1}M_2$ and

$$M_1 = \left\{ \frac{1}{i + j + 2\rho - 1} I_{m \times m} \right\}_{i,j=1,\ldots,r+1} \tag{B.2}$$

$$M_2 = \left\{ \frac{1}{i + j + \rho - 1} I_{m \times m} \right\}_{i=1,\ldots,r+1, j=1,\ldots,\rho} \tag{B.3}$$

It follows from (27) that the characteristic polynomial matrix equation of the closed-loop system is given by

$$I_{m \times m}s^{\rho} + k_{\rho-1}s^{\rho-1} + \cdots + k_0 = 0 \tag{B.4}$$

Substituting (B.1) into (B.5) yields

$$\frac{\bar{T}^{\rho}}{\rho!}s^{\rho} + z_{\rho-1}\frac{\bar{T}^{\rho-1}}{(\rho-1)!}s^{\rho-1} + \cdots + z_0 = 0 \tag{B.5}$$

The roots of the above polynomial matrix can be calculated for different control orders and relative degrees and hence the stability of the linear closed loop dynamics can be determined as in Table 1. Furthermore, according to standard result in geometric nonlinear control theory (Proposition 4.5.1 in Isidori (1995)), the stability for the linear closed-loop dynamics implies that of the overall closed-loop system under the NGPC since Assumption A5 is satisfied.

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